Topology Proceedings

Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings Department of Mathematics & Statistics Auburn University, Alabama 36849, USA
E-mail:	topolog@auburn.edu
ISSN:	0146-4124

Topology Proceedings
Vol 18, 1993

DISCONTINUOUS CLOSED DARBOUX FUNCTIONS

HARVEY ROSEN

ABSTRACT. A function $f : X \to Y$ is *Darboux* (*closed*) if $f(C)$ is connected (closed) for each connected (closed) subset C of X. Closed Darboux functions that are discontinuous are constructed. One such function is from the unit interval $[0,1]$ onto a continuum Y.

Although a closed Darboux function $f : X \to Y$ does not have to be continuous, it does if X and Y are Euclidean spaces. This is due to H. Pawlak's result in [1] that a closed function $f : R^n \to R^m$ is continuous if and only if the image of each segment is connected. It follows that each closed Darboux function $f : I \to I$ is continuous, where I denotes the topological space $[0,1]$ with the usual topology T_0. In [2], H. Pawlak and R. J. Pawlak give three ways to construct discontinuous closed Darboux functions $f : X \to Y$. We answer some of their open problems here.

A topological space (X, T) is *paracompact* if each open cover of X has a locally finite open refinement. A function $f : X \to Y$ is *nowhere constant* at x if $f(U)$ is nondegenerate for each open neighborhood U of x. Let H denote the Hashimoto-type topology on [0,1] generated by the base $B = \{U - A : U \in T_0$ and A is countable$\}$. H is finer than T_0, but $([0,1], H)$ is not paracompact. According to Theorem 1 of [2], every closed Darboux (and therefore continuous) function $f : I \to I$ considered as a function $f : I \to ([0,1], H)$ is closed Darboux and discontinuous at each point where it is nowhere constant. It is asked if versions of this result still hold when H is replaced by any topology T finer than T_0 or "close to compact" like paracompact.

Theorem 1. *Given any nonconstant closed Darboux function $f : I \to I$, there exists a topology T on $[0,1]$ finer than T_0 such that $([0,1], T)$ is a normal connected paracompact space and $f : I \to ([0,1], T)$ is a closed Darboux discontinuous function.*

Proof: Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in $[0,1]$ T_0-converging to a point p such that $\{f(x_n)\}_{n=1}^{\infty}$ T_0-converges to $f(p)$ and for $m \neq n, f(x_m) \neq f(x_n) \neq f(p)$. Let $\{B_n\}_{n=1}^{\infty}$ be a sequence of disjoint closed intervals not containing $f(p)$ such that B_n has center $f(x_n)$ and radius b_n. $\boldsymbol{C}$ denotes the collection of all sequences $\{C_n\}_{n=1}^{\infty}$ of closed intervals such that C_n has center $f(x_n)$ and radius c_n with $0 < c_n < b_n$. Define T to be the topology with base $\boldsymbol{B}$ consisting of T_0 along with all sets of the form $U - \cup_{n=1}^{\infty} C_n$, where $f(p) \in U \in T_0$ and $\{C_n\}_{n=1}^{\infty} \in \boldsymbol{C}$.

Since $f : I \to I$ is closed and T is finer than T_0, then $f : I \to ([0,1], T)$ is closed and $([0,1], T)$ is Hausdorff. Let J be any subinterval of $[0,1]$. Then $f(J)$ is a point or an interval because $f : I \to I$ is Darboux . Suppose $f(J)$ is an interval. If $f(p) \notin f(J)$, then $f(J)$ is T-connected. If $f(p) \in f(J)$, then $f(J) - \{f(p)\}$ is either (1) an interval K_1 or (2) the union of disjoint intervals K_1 and K_2. For case (1), K_1 is T-connected because relative T-open sets in K_1 are relative T_0-open sets in K_1, too. Since $f(p)$ is in the T-closure of K_1, $f(J) = K_1 \cup \{p\}$ is T-connected. For case (2), $f(p)$ is in the T-closure of the T-connected sets K_1 and K_2, and so $f(J) = K_1 \cup \{p\} \cup K_2$ is T-connected. This shows $f : I \to ([0,1], T)$ is a Darboux function. By construction, it is discontinuous at p.

We show $([0,1],T)$ is paracompact. Let $\boldsymbol{V} = \{V_a : a \in A\}$ be a T-open cover of $[0,1]$. We may assume $\boldsymbol{V} \subset \boldsymbol{B}$. If $\boldsymbol{V} \subset T_0$, then $\boldsymbol{V}$ has a finite subcover of $[0,1]$. So suppose for some V in $\boldsymbol{V}$, $V = U - \cup_{n=1}^{\infty} C_n$, where $f(p) \in U \in T_0$ and $\{C_n\}_{n=1}^{\infty} \in \boldsymbol{C}$. Pick $\{D_n\}_{n=1}^{\infty} \in \boldsymbol{C}$ so D_n has radius d_n with $c_n < d_n < b_n$, and choose $W = U - \cup_{n=1}^{\infty} D_n$. Whenever $V_a \in \boldsymbol{V} \cap T_0$ for $a \in A$, we let $W_{an} = V_a \cap \operatorname{int}(D_n)$ for $n = 1,2,3,\ldots$. But whenever $V_a = U_a - \cup_{n=1}^{\infty} C_{an}$ for $a \in A$ where $f(p) \in U_a \in T_0$ and $\{C_{an}\}_{n=1}^{\infty} \in \boldsymbol{C}$, we let $W_{an} = V_a \cap \operatorname{int}(D_n)$ for $n = 1,2,3,\ldots$. Then for each n, $\{W_{an} : a \in A\}$ is a T_0-open cover of C_n having a finite subcover $\boldsymbol{W}_n$ because C_n is T_0-compact, and W misses each member of $\boldsymbol{W}_n$. Let $\boldsymbol{V}_1 = \{V_a : V_a \in \boldsymbol{V} \cap T_0$ and V_a meets $[0,1] - U\}$. $\boldsymbol{V}_1$ has a finite subcover $\boldsymbol{V}_2$ of $[0,1] - U$, which is T_0-compact. Therefore $\{V\} \cup \boldsymbol{V}_2 \cup (\cup_{n=1}^{\infty} \boldsymbol{W}_n)$ is a locally finite T-open refinement of $\boldsymbol{V}$. Finally, any paracompact Hausdorff space, like $([0,1],T)$, is normal.

If $f : I \to Y$ and the space Y is no longer an interval, then we can choose Y to be compact as the next example shows. We cannot choose it to be both compact and Hausdorff. For, if $f : I \to Y$ is closed Darboux and discontinuous and Y is compact and Hausdorff, then Urysohn's lemma would ensure there exists a continuous function $g : Y \to I$ such that $g \circ f : I \to I$ is discontinuous besides being closed and Darboux. According to [1], this is impossible.

Example 1. *There exists a closed Darboux discontinuous function $f : I \to Y$, where Y is a compact connected space.*

Proof: Let Q denote the set of rational numbers in $[0,1]$ with the relative topology T_1 from T_0, and for $p \notin Q$ let $Y = Q \cup \{p\}$ be the one-point compactification of Q [3]. Define $f : I \to Y$ by

$$f(x) = \begin{cases} x \text{ if } x \text{ is rational,} \\ p \text{ if } x \text{ is irrational.} \end{cases}$$ Then f is discontinuous because when $(a,b) \subset [0,1]$, $(a,b) \cap Q$ is open in Y but $f^{-1}((a,b) \cap$

$Q) = (a, b) \cap Q$ is not T_0-open in $[0, 1]$.

Suppose $C \subset [0,1]$. Then $C = (C \cap Q) \cup (C \cap ([0,1] - Q))$ and $f(C) =$

$$f(C \cap Q) \cup f(C \cap ([0,1] - Q)) = \begin{cases} C \text{ if } C \subset Q, \\ (C \cap Q) \cup \{p\} \text{ if } C \not\subset Q. \end{cases}$$

Suppose F is a T_0-closed subset of $[0,1]$. Then F is T_0-compact. Suppose $F \subset Q$. Then $f(F) = F$ and F is a T_1-compact and T_1-closed subset of Q. Consequently $Q - f(F)$ is open in Y, and so $f(F)$ is closed in Y. Now suppose $F \not\subset Q$. Then $f(F) = (F \cap Q) \cup \{p\}$, which is closed in Y because $Q - (F \cap Q)$ is a T_1-open subset of Q. This shows f is a closed function.

Let K be a connected subset of $[0,1]$. We show $f(K)$ is connected. We may suppose K is an interval instead of a point. Since $K \not\subset Q, f(K) = (K \cap Q) \cup \{p\}$. Assume $(K \cap Q) \cup \{p\} = A \cup B$, a separation. A and B are disjoint sets open in $(K \cap Q) \cup \{p\}$ and suppose $p \in B$. There is an open set U in Y such that $B = U \cap ((K \cap Q) \cup \{p\})$. A is a subset of $Y - U$, which is a T_1-closed and T_1-compact subset of Q because $p \in U$. Therefore $Y - U$ and hence A is nowhere dense in Q. But $A = V \cap ((K \cap Q) \cup \{p\})$ for some open subset V of Y. Since $p \notin A$ implies $p \notin V$, V is a T_1-open subset of Q. Therefore $A = V \cap (K \cap Q)$ is somewhere dense in Q, a contradiction. This shows $f(K)$ is a connected set and f is a Darboux function.

A connected topological space X^* is said to have an *exploding point* a with respect to a point $x_0 \in X^*$ if $\{x_0\}$ is a component of $X^* - \{a\}$ and there exist disjoint open sets U and V with $x_0 \in U$ and $a \in V$. Theorem 2 of [2] states that if X^* has an exploding point a with respect to x_0 and $X = X^* - \{x_0\}$ is a dense compact connected subspace of X^*, then there exists a closed Darboux function $f : X^* \to I$ which is discontinuous at x_0. It is asked whether the compactness of the subspace X of the explosion set X^* can be weakened or when X^* can be a connected Alexandroff compactification of a connected locally compact space X. Figure 1 illustrates that both situations, minus local compactness, can occur as in the following theorem. The picture shows a fan consisting of line segments $L_1, L_2, L_3, \ldots$ emanating from the same endpoint a and limiting on a line segment L whose other endpoint is x_0. L_0 denotes the half of L which contains a. Then let $X = \cup_{n=0}^{\infty} L_n$ and $X^* = X \cup \{x_0\}$, the one-point compactification of X.

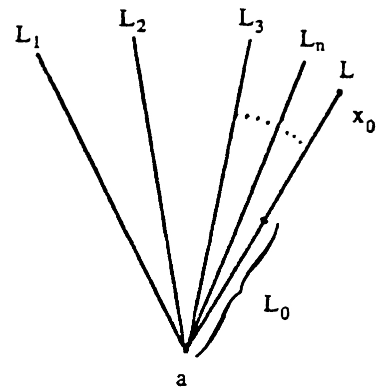

Figure 1.

Theorem 2. *Let X be a connected, completely regular, Frechet, noncompact space having the 1-point compactification $X^* = X \cup \{x_0\}$ such that X^* has an exploding point a with respect to x_0. Then there exists a closed Darboux function $f : X^* \to I$ which is discontinuous at x_0.*

Proof: Let U be an open neighborhood of x_0 in X^* such that $a \notin \text{cl}_{X^*}(U)$, and let $F = (\text{cl}_{X^*}(U)) - \{x_0\} = \text{cl}_{X^*}(U) \cap X$, which is closed in X. X is dense in X^* because X is not compact. Therefore X^* is connected and $U - \{x_0\} \neq \emptyset$. Since X is completely regular, there exists a continuous function $\eta : X \to I$ such that $\eta(a) = 0$ and $\eta(F) = 1$. Define

$$f : X^* \to I \text{ by } f = \begin{cases} \eta \text{ on } X \\ 0 \text{ at } x_0. \end{cases}$$ Then $f|X = \eta$ is continuous, but f is discontinuous at x_0 because $f(U - \{x_0\}) = \{1\} \not\subset [0, \frac{1}{2})$.

We claim f is a closed function. Suppose K is closed in X^*. Since $K - U$ is closed in $X^*, K - U$ is compact. If $x_0 \notin K$, then K is a compact subset of X, and so $f(K) = \eta(K)$ is compact and therefore closed. But if $x_0 \in K, f(K) = f(K - U) \cup f(K \cap (\text{cl}_X * (U))) = \eta(K - U) \cup \{0, 1\}$, which is compact and therefore closed.

For the sake of completeness, we show here that f is a Darboux function in the same fashion as in [2]. Suppose C is connected. If $x_0 \notin C$, then $C \subset X$ and so $f(C) = \eta(C)$ is connected. If $x_0 \in C$ and $C \neq \{x_0\}$, then $a \in C$ because a is an exploding point of X^* with respect to x_0. Therefore there exists $p \in C \cap \text{bd}_X * (U)$, and so $f(p) = 1$. We show $f(C) = [0, 1]$ to see it is connected. Assume there exists $\alpha \in (0, 1)$ such that $f^{-1}(\alpha) \cap C = \emptyset$. Let $A = \{x \in C - \{x_0\} : f(x) < \alpha\}$, $B_1 = \{x \in C - \{x_0\} : \; f(x) > \alpha\}$, and $B = B_1 \cup \{x_0\}$. Then $a \in A$ and $C = A \cup B$ is a separation, contrary to C being connected.

In Theorem 3 of [2], Pawlak and Pawlak extend a homeomorphism to a closed Darboux discontinuous function. They show that for a nondegenerate locally connected metrizable continuum X and $x_0 \in X$, there exist a locally connected continuum X_1 and a locally connected, connected paracompact space X_2 each having X as a subspace such that every homeomorphism $h : X \to X$ can be extended to a closed Darboux function $h^* : X_1 \to X_2$ discontinuous at x_0. They ask how close to compact can X_2 be chosen. We show X_2 can actually be compact.

Theorem 3. *Let X be a nondegenerate locally connected metrizable continuum and let $x_0 \in X$. Then there exist locally connected continua X_1 and X_2 such that X is a subspace of X_1 and X_2 and for each homeomorphism $h : X \to X$, there exists an extension $h^* : X_1 \to X_2$ of h such that h^* is a closed Darboux function discontinuous at x_0.*

Proof: Let $a \in X$ and let $\{x_n\}_{n=0}^{\infty}$ be a sequence of distinct elements of X different from a and converging to a. Define

$X_1 = X \cup (\{x_n : n \geq 0\} \times \{1\})$, and define a topology T_1 on X_1 generated by the neighborhood system $B_1(x) = \{K(x, \frac{1}{m}) \cup \bigcup_{x_n \in K(x,\frac{1}{m})}(\{x_n\} \times \{1\}) : m \in N\}$ for $x \in X$, where $K(x, \frac{1}{m})$ denotes the $\frac{1}{m}$-neighborhood of x in X. Define $X_2 = X \cup (\{h(x_n) : n \geq 0\} \times \{1\})$, and define a topology T_2 on X_2 generated by the following neighborhood system:

$$B_2(x) = \begin{cases} \{x\} \text{ if } x = (h(x_n), 1) \text{ for some } n \geq 0, \\ \{K(h(x), \frac{1}{m}) \cup \bigcup_{h(x_n) \in K(h(x),\frac{1}{m})}\{(h(x_n), 1)\} : \\ \qquad m \in N\} \text{ if } x \in X. \end{cases}$$

By construction X_1 and X_2 are locally connected, connected, and compact.

Define a function $h^* : X_1 \to X_2$ by

$$h^*(x) = \begin{cases} h(x) & \text{if } x \in X, \\ (h(x_n), 1) & \text{if } x = (x_n, 1) \text{ for some } n \geq 0. \end{cases}$$

Choose a sequence $\{a_n\}_{n=1}^{\infty}$ in X such that $a_n \to x_0$ in X. Then $a_n \to (x_0, 1)$ in X_1 and $h^*(a_n) = h(a_n) \not\to (h(x_0), 1)$ in X_2 because $\{(h(x_0), 1)\}$ is an open neighborhood of $(h(x_0), 1)$ containing no $h(a_n)$. Therefore h^* is discontinuous at x_0.

Let C be a connected subset of X_1. Then $x \in C \cap X$ whenever $(x, 1) \in C$. Therefore $\pi(C) = C \cap X$, where π denotes the projection $\pi : X_1 \to X$ defined by $\pi(x_n, 1) = x_n$ if $n \geq 0$ and $\pi(x) = x$ if $x \in X$. Since π is continuous, $C \cap X$ is connected. Then $h^*(C) = h(C \cap X) \cup \bigcup_{(x_n,1) \in C}\{(h(x_n), 1)\}$ is connected because each $\{h(x_n)\} \cup \{(h(x_n), 1)\}$ is a connected subset of X_2. This shows f is Darboux .

That h^* is closed follows from the facts that h^* is one-to-one and $\{h^*(U) : U \in T_1\} \subset T_2$.

References

1. H. Pawlak, *On some condition equivalent to the continuity of closed functions*, Dem. Math. **17** (1984), 723-732.

2. —— and R. J. Pawlak, *On some open problems connected with the discontinuity of closed and Darboux functions*, Topology Proceedings **18** (1993), 209–220.
3. L. A. Steen and J. A. Seebach, Jr., Counterexamples in Topology, Holt, Rinehart and Winston, N.Y., 1970.

University of Alabama
Tuscaloosa, AL 35487