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NORMALITY AND COVERING PROPERTIES OF OPEN SETS OF UNCOUNTABLE PRODUCTS

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ABSTRACT. Let X be a product of paracompact σ-spaces X_α. Then the following conditions for an open set U of X are equivalent:

(a) U is normal (countably paracompact, collectionwise Hausdorff, orthocompact, θ-refinable, paracompact);

(b) U is an F_σ-set of X and X_α is compact metrizable for all but countably many α's.

0. INTRODUCTION

Nagami [N] proved that the normaliy and the countably paracompactness are equivalent for a product of paracompact Σ-spaces. He asked: Let ω_1 be the first uncountable cardinal, and I^{ω_1} the product of ω_1 many copies of the unit interval. Then is an open set of I^{ω_1} normal if it is countably paracompact? The purpose of this paper is to answer Nagami's question positively. More generally, we show the following theorem in section 2:

Theorem. *Let $X = \prod_{\alpha<\lambda} X_\alpha$ be a product of paracompact σ-spaces. Then the following conditions for an open set U of X are equivalent:*

(a) *U is normal (countably paracompact, collectionwise Hausdorff, orthocompact, θ-refinable, paracompact);*

[1] This work was done while the first author was visiting Auburn University.

(b) *U is an F_σ-set of X and X_α is compact metrizable for all but countably many α's.*

In section 1, we prepare lemmas which will be used in the proof of our theorem. The key to the proof is the fact that if U doesn't satisfy (b), then U contains a closed copy of $2^{\omega_1} - \{\mathbf{0}\}$, which has none of the covering properties in (a).

Let us briefly review σ-spaces and semi-stratifiable spaces. The reader is referred to [O] and [G] for these spaces. A space is a *σ-space* if it has a σ-discrete network. A space X is *semi-stratifiable* if there is a function G which assigns to each $n \in \omega$ and closed set $H \subset X$, an open set $G(n, H)$ containing H such that

(i) $H = \cap_n G(n, H)$;
(ii) $G(n, H) \subset G(n, K)$ if $H \subset K$.

Every σ-space is semi-stratifiable. A semi-stratifiable space is compact metrizable if it is countably compact. Every countable product of (paracompact) σ-spaces is a (paracompact) σ-space. Every countable product of semi-stratifiable space is semi-stratifiable. The product $\prod_{n<\omega} X_n$ is paracompact (Lindelöf) if each X_n is semi-stratifiable and each finite subproduct is paracompact (Lindelöf). This follows from the fact that the product $\prod_{n<\omega} X_n$ is paracompact (Lindelöf) if each finite subproduct is perfectly normal and pracompact (Lindelöf) (see [O] for the paracompact case).

All spaces are assumed to be completely regular T_2. κ and λ are cardinal numbers. ω is the smallest infinite ordinal and cardinal, and ω_1 is the smallest uncountable ordinal and cardinal. A cardinal number is the set of all ordinals which precede it. A product $X = \prod_{\alpha<\lambda} X_\alpha$ of spaces is the product with the Tychonoff topology. If each X_α is identical to a fixed space Y, we write Y^λ for $\prod_{\alpha<\lambda} X_\alpha$. For $S \subset \lambda$, denote by π_S the natural projection $\pi_S : X \to \prod_{\alpha \in S} X_\alpha$. The space 2^λ is the product of λ many copies of the two point discrete space $2 = \{0, 1\}$. Define $\mathbf{0}$, $\mathbf{1} \in 2^\kappa$ by $\mathbf{0}(\alpha) = 0$, $\mathbf{1}(\alpha) = 1$ for any $\alpha < \kappa$.

1. Preliminary Lemmas

Let (X, τ) be a space and κ an uncountable cardinal. A subset of X which is an intersection of less than κ many open sets is called a $G_\delta^{<\kappa}$-*set*. A $G_\delta^{<\omega_1}$-set is simply called a G_δ-*set*. Let τ_κ be the topology on the same underlying set X with the base consisting of all $G_\delta^{<\kappa}$-sets. The space (X, τ_κ) is called the κ-*modification* of (X, τ). We need the following three lemmas:

Lemma 1 [Y]. *Let X be a product of semi-stratifiable spaces. Then a closed set of X is a G_δ-set if and only if it is a union of G_δ-sets.*

The following Lemma ensures an embedding of $2^{\omega_1} - \{\mathbf{0}\}$ in an open set of a product space.

Lemma 2. *Let κ be a regular uncountable cardinal, and $\lambda \geq \kappa$ a cardinal. Suppose that $(X, \tau) = \prod_{\alpha<\lambda} X_\alpha$ is a product of spaces such that every point of X_α is a $G_\delta^{<\kappa}$ in X_α for any $\alpha < \lambda$. Let $A \in \tau_\kappa$, where (X, τ_κ) is the κ-modification of (X, τ). Then the following conditions are equivalent:*

(a) $X - A \notin \tau_\kappa$;

(b) *There is a subspace C in (X, τ) such that C is homeomorphic to 2^κ and $C - A$ is a singleton.*

Proof: (b) $\Rightarrow$ (a): Let $C - A = \{p\}$. Then every $G_\delta^{<\kappa}$-set containing p meets $C - \{p\} = C \cap A$, because no point of 2^κ is a $G_\delta^{<\kappa}$-set of 2^κ. Hence $X - A \notin \tau_\kappa$.

(a) $\Rightarrow$ (b): Assume (a). Then there is a point $p \in \mathrm{cl}_{\tau_\kappa} A - A$. Define

$$\operatorname{supp} q = \{\alpha < \lambda : q(\alpha) \neq p(\alpha)\} \text{ for each } q \in X.$$

By induction on $\xi < \kappa$, we take a sequence $\langle p_\xi : \xi < \kappa \rangle$ of points of A and an increasing sequence $\langle S_\xi : \xi < \kappa \rangle$ of subsets of λ of cardinality less than κ satisfying the following:

$$(1) \qquad \operatorname{supp} p_\xi \subset S_\xi - \bigcup\{S_\eta : \eta < \xi\}; \text{ and}$$

(2) $$\{q \in X : q|S_\xi = p_\xi|S_\xi\} \subset A \text{ for each } \xi < \kappa.$$

Assume that $\langle p_\eta : \eta < \xi \rangle$ and $\langle S_\eta : \eta < \xi \rangle$ are defined for $\xi < \kappa$. Define

$$T_\xi = \cup\{S_\eta : \eta < \xi\}, \text{ and}$$
$$Z_\xi = \{q \in X : q(\alpha) = p(\alpha) \text{ for any } \alpha \in T_\xi\}.$$

Since $|T_\xi| < \kappa$ and every point of each X_α is a $G_\delta^{<\kappa}$-set of X_α, Z_ξ is a $G_\delta^{<\kappa}$-set of X. Hence there is a point $p' \in Z_\xi \cap A \in \tau_\kappa$ because $p \in Z_\xi$ and $p \in \mathrm{cl}_{\tau_\kappa} A$. Take a subset S_ξ of λ of size less than κ such that $T_\xi \subset S_\xi$, and $\pi_{S_\xi}^{-1}\pi_{S_\xi}(p') \subset Z_\xi \cap A$. Define $p_\xi(\alpha) \in X$ by

$$p_\xi(\alpha) = \begin{cases} p'(\alpha) & \text{if } \alpha \in S_\xi \\ p(\alpha) & \text{if } \alpha \in \lambda - S_\xi. \end{cases}$$

Then p_ξ satisfies (1) and (2), which completes the induction step.

Now define

$$\begin{aligned} R_\xi &= S_\xi - \cup\{S_\eta : \eta < \xi\}; \\ R &= \lambda - \cup\{S_\xi : \xi < \kappa\} \\ &= \lambda - \cup\{R_\xi : \xi < \kappa\}; \text{ and} \\ C &= \{q \in X : q|R = p|R, \text{ and} \\ &\quad \text{for any } \xi < \kappa, \text{either } q|R_\xi = p|R_\xi \text{ or } q|R_\xi = p_\xi|R_\xi\}. \end{aligned}$$

It is easy to check that the subspace C is homeomorphic to 2^κ. It remains to show that $C - \{p\} \subset A$. To see this, let $q \in C - \{p\}$. Define

$$\xi_0 = \min\{\xi < \kappa : q|R_\xi \neq p|R_\xi\}.$$

Then by the definition of C,

(3) $$q|R_{\xi_0} = p_{\xi_0}|R_{\xi_0}.$$

By the minimality of ξ_0, we have

(4) $$q|R_\xi = p|R_\xi \text{ for any } \xi < \xi_0.$$

It follows from (1) that supp $p_{\xi_0} \subset R_{\xi_0} \subset \lambda - \bigcup_{\xi<\xi_0} R_\xi$. Hence

$$(5) \qquad p_{\xi_0}|R_\xi = p|R_\xi \text{ for any } \xi < \xi_0.$$

Combining (3), (4) and (5), we have $q|S_{\xi_0} = p_{\xi_0}|S_{\xi_0}$. Hence by (2), we have $q \in A$. □

The space $2^{\omega_1} - \{\mathbf{0}\}$ has none of the covering properties mentioned in our theorem.

Lemma 3. *Let κ be an uncountable cardinal. Then $E = 2^\kappa - \{\mathbf{0}\}$ has the following properties:*

(a) *E contains a closed copy of the space $(\lambda_0 + 1) \times (\lambda_1 + 1) - \{(\lambda_0, \lambda_1)\}$ for any ordinals λ_0, λ_1 of cardinality $\leq \kappa$, where $\lambda_i + 1$, $i < 2$ are spaces with the order topology;*
(b) *E contains a closed discrete set of cardinality κ;*
(c) *E is not normal;*
(d) *E is not countably paracompact;*
(e) *E is not collectionwise Hausdorff; and*
(f) *E is not orthocompact.*

Proof: (a), (b): Let D be a discrete space of size κ and $D \cup \{\infty\}$ be its one point compactification. Since the weights of compact zero-dimensional spaces $(\lambda_0+1)\times(\lambda_1+1)$ and $D\cup\{\infty\}$ are less than or equal to κ, they can be embedded in 2^κ as closed sets. Hence $(\lambda_0 + 1) \times (\lambda_1 + 1) - \{(\lambda_0, \lambda_1)\}$ and D can be embedded in E as closed sets.

Before showing the properties from (c) to (d), note that $2^{\omega_1} \times \{\pi_{\kappa-\omega_1}(\mathbf{0})\} - \{\mathbf{0}\}$ is a closed subset of E homeomorphic to $2^{\omega_1} - \{\mathbf{0}\}$. Since normality, countable paracompactness, collectionwise Hausdorffness and orthocompactness are closed hereditary, it suffices to show that the space $2^{\omega_1} - \{\mathbf{0}\}$ doesn't have these properties. So from now on we assume that $\kappa = \omega_1$.

(c), (d): By (a), the Tychonoff plank $T = (\omega_1 + 1) \times (\omega + 1) - \{(\omega_1, \omega)\}$ can be embedded in E as a closed set. It is well known that T is neither normal nor countably paracompact.

(e): By (b), E contains an uncountable discrete closed set. Since E has the countable chain condition, it cannot be separated by open sets.

(f): By (b), E contains a discrete closed set $D = \{d_\xi : \xi < \omega_1\}$. For any $\xi < \omega_1$, take an open set U_ξ of E satisfying:

$$d_\xi \in U_\xi \subset E - \{d_\eta : \eta < \omega_1,\ \eta \neq \xi\}; \text{ and}$$

$$U_\xi \subset \pi_\xi^{-1}(\{d_\xi(\xi)\}). \tag{6}$$

Define $U = E - D$, and

$$\mathcal{U} = \{U\} \cup \{U_\xi : \xi < \omega_1\}.$$

Then $\mathcal{U}$ is an open cover of E. Suppose that there is an interior preserving refinement $\mathcal{V}$ of $\mathcal{U}$. For any $\xi < \omega_1$, take $V_\xi \in \mathcal{V}$ containing d_ξ. Since U_ξ is the only element of $\mathcal{U}$ which contains d_ξ, we have $V_\xi \subset U_\xi$. Since E is separable, there are a point $p \in E$ and an infinite subset A of ω_1 such that $p \in \bigcap\{V_\xi : \xi \in A\}$. But $\bigcap\{V_\xi : \xi \in A\} \subset \bigcap\{U_\xi : \xi \in A\}$, and by (6), $\bigcap\{U_\xi : \xi \in A\}$ has no interior, which contradicts the interior preserving property of $\mathcal{V}$. □

Remark 1. The alternative proof of (f) is as follows: Let $S = \omega_1 \times (\omega_1 + 1)$. It is known that S is not orthocompact [S]. So it suffices to show that S can be embedded in E as a closed set. Let $S' = (\omega_1 + 1) \times (\omega_1 + 1)$ and S'' the space obtained from S' by identifying $\{\omega_1\} \times (\omega_1 + 1)$ to a single point called ∞. Since S'' is a compact zero-dimensional space of weight ω_1, S'' can be embedded in 2^{ω_1} as a closed set. Hence S, which is homeomorphic to $S'' - \{\infty\}$, can be embedded in E as a closed set.

2. Theorems

We state our main theorem in a general form:

Theorem 1. *Let $X = \prod_{\alpha<\lambda} X_\alpha$ be a product of paracompact semi-stratifiable spaces, each finite subproduct of which is paracompact. Suppose that $\mathcal{P}_0$ and $\mathcal{P}_1$ are closed hereditary classes* *of spaces containing all paracompact spaces such that* $2^{\omega_1} - \{\mathbf{0}\}$ *belongs to neither* $\mathcal{P}_0$ *nor* $\mathcal{P}_1$, *and* $\omega^{\omega_1} \notin \mathcal{P}_0$. *Then the following conditions for an open set* U *of* X *are equivalent:*

(a) $U \in \mathcal{P}_0$;

(b) $U \in \mathcal{P}_1$, *and* X_α *is compact for all but countably many* $\alpha < \lambda$;

(c) U *is an* F_σ *set of* X, *and* X_α *is compact for all but countably many* $\alpha < \lambda$.

Proof: (c) $\Rightarrow$ (a), (c) $\Rightarrow$ (b): Suppose the condition (c). Then X can be written as a product of a paracompact space and a compact space. Hence X is paracompact. So an F_σ-set U of X is paracompact. Thus we have (a) and (b).
(a) $\Rightarrow$ (c): Assume that $U \in \mathcal{P}_0$. First we show that X_α is compact for all but countably many $\alpha < \kappa$. Suppose not. Then uncountably many X_α's contain an infinite discrete closed set. Hence U contains a copy of the product space ω^{ω_1} as a closed subset, which contradicts the closed hereditariness of $\mathcal{P}_0$. Next we show that U is an F_σ-set. Suppose not. Note that each point of each X_α is a G_δ because X_α is semi-stratifiable. Hence by Lemma 1, $X - U$ is not a union of G_δ-sets. In other words, $X - U \notin \tau_{\omega_1}$. By Lemma 2, U contains a closed copy of $2^{\omega_1} - \{\mathbf{0}\}$, which again contradicts the closed hereditariness of $\mathcal{P}_0$.
(b) $\Rightarrow$ (c) : The proof is a part of the proof of (a) $\Rightarrow$ (c). □

Corollary 1. *Let* $X = \prod_{\alpha<\lambda} X_\alpha$ *be a product of paracompact semi-stratifiable spaces, each finite subproduct of which is paracompact. Then the following conditions for an open set* U *of* X *are equivalent:*

(a) U *is normal;*

(b) U *is countably paracompact;*

(c) U *is collectionwise Hausdorff;*

(d) U *is orthocompact;*

(e) U *is paracompact;*

(f) U *is* θ*-refinable;*

(g) U *is weakly* $\delta\theta$*-refinable, and* X_α *is compact for all but countably many* $\alpha < \lambda$.

Proof: Apply Theorem 1 by noting the following facts. The properties of U in conditions from (a) to (g) are closed hereditary properties. ω^{ω_1} does not have properties from (a) to (f) (see [vD]). $2^{\omega_1} - \{0\}$ does not have properties from (a) to (e) by Lemma 3. $2^{\omega_1} - \{0\}$ is neither θ-refinable nor weakly $\delta\theta$-refinable because the space ω_1 with the order topology is neither θ-refinable nor weakly $\delta\theta$-refinable, and by Lemma 3 (a), ω_1 can be embedded in $2^{\omega_1} - \{0\}$ as a closed set. □

Remark 2 In Corollary 1, weak $\delta\theta$-refinability in (g) can be replaced by any other closed hereditary property $\mathcal{P}$ such that every paracompact space has the property $\mathcal{P}$ and ω_1 is not in the class $\mathcal{P}$. For such properties, see [L].

In case that each finite subproduct is Lindelöf, the properties in Corollary 1 are equivalent to Lindelöf property and $\aleph_1$-compactness.

Theorem 2. *Let $X = \prod_{\alpha<\lambda} X_\alpha$ be a product of semi-stratifiable spaces, each finite subproduct of which is Lindelöf. Suppose that $\mathcal{P}_0$ and $\mathcal{P}_1$ are closed hereditary classes of spaces containing all Lindelöf spaces such that $2^{\omega_1} - \{0\}$ belongs to neither $\mathcal{P}_0$ nor $\mathcal{P}_1$, and $\omega^{\omega_1} \notin \mathcal{P}_0$. Then the following conditions for an open set U of X are equivalent:*

(a) *$U \in \mathcal{P}_0$;*
(b) *$U \in \mathcal{P}_1$, and X_α is compact for all but countably many $\alpha < \lambda$;*
(c) *U is an F_σ set of X, and X_α is compact for all but countably many $\alpha < \lambda$;*
(d) *there is a countable subset S of λ such that $U = \pi_S^{-1}\pi_S(U)$, and X_α is compact for all but countably many $\alpha < \lambda$.*

Proof: (a) $\Rightarrow$ (c), (b) $\Rightarrow$ (c): Similar to the proof in Theorem 1. (c) $\Rightarrow$ (d): Since X is a Lindelöf space, the F_σ-set U is also Lindelöf. Hence U is a countable union of basic open sets, which implies (d). (d) $\Rightarrow$ (a), (d) $\Rightarrow$ (b): Suppose (d), then U can be written as a product of a Lindelöf space and a compact space, hence U is Lindelöf. Thus (a) and (b) hold. □

Corollary 2. *Let $X = \prod_{\alpha<\lambda} X_\alpha$ be a product of semi-stratifiable spaces, each finite subproduct of which is Lindelöf. Then the following conditions for an open set U of X are equivalent:*

(a) *U is Lindelöf;*
(b) *U has no subset of cardinality ω_1 which is closed and discrete in U i.e., U is $\aleph_1$-compact;*
(c) *U has any one of properties from (a) to (f) in Corollary 1;*
(d) *U has the property (g) in Corollary 1.*

Proof: Apply Theorem 2. We need only check that the $\aleph_1$-compactness satisfies the property of $\mathcal{P}_0$ in Theorem 2. Clearly $\aleph_1$-compactness is closed hereditary. ω^{ω_1} is not $\aleph_1$-compact by [M]. $2^{\omega_1} - \{0\}$ fails to be $\aleph_1$-compact by Lemma 3 (b). □

Remark 3. *For the realcompactness of a G_δ-set in a dyadic space, see* [PP]. *It is shown that for a G_δ-set G in a dyadic space, G is realcompact if and only if G is Lindelöf.*

We conclude with two questions.

Question 1. *Is the space $2^{\omega_1} - \{0\}$ countably metacompact?*

Kemoto and Yajima [KY] asked whether ω^{ω_1} is weakly θ-refinable. The negative answer to the following question implies that we can remove the compactness from the condition (g) in Corollary 1.

Question 2. *Is the space ω^{ω_1} weakly $\delta\theta$-refinable?*

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