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FRACTIONAL TOPOLOGICAL DIMENSION FUNCTION

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ABSTRACT. We shall study the relationship between the usual topological covering dimension function and real-valued dimension functions satisfying Menger's axioms defined on some subcollections of separable metric spaces. In particular, the existence of topological dimension functions taking every non-negative real value is shown for locally compact spaces.

1. MENGER'S DIMENSION FUNCTIONS

We start from *what dimension functions are.* In 1929 K. Menger proposed the following axioms (A.1) - (A.5), and showed that dim is the unique dimension function which satisfies them for the class $\mathcal{F} = \mathcal{E}_2$, where $\mathcal{E}_n$ denotes all subsets of n-dimensional Euclidian space and put $\mathcal{E} = \cup_{n\geq 1} \mathcal{E}_n$.
Let $\mathcal{F} \subset \mathcal{E}$, and define $\Lambda_{\mathcal{F}} = \{n \in N : I^n \in \mathcal{F}\}$. A real-valued function d is said to be a *Menger dimension function* with respect to $\mathcal{F}$ provided that d satisfies the following five conditions (see also [11]):
(A.1) $d\ \emptyset = -1, d\ \{0\} = 0$, and $d\ I^n = n$ for every $n \in \Lambda_{\mathcal{F}}$ (Regularity).
(A.2) $d\ X \leq d\ Y$ for any $X, Y \in \mathcal{F}$ with $X \subset Y$ (Monotonicity).
(A.3) $d\ X = \sup d\ X_i$ for $X = \cup_{i=1}^{\infty} X_i \in \mathcal{F}$, where each $X_i \in \mathcal{F}$ is closed in X (Countable stability).
(A.4) Any $X \in \mathcal{F}$ can be embedded in some compact $K \in \mathcal{F}$ with $d\ X = d\ K$ (Compactification).

(A.5) $d\,X = d\,Y$ for any $X, Y \in \mathcal{F}$ with $X \approx Y$ (Topological invariance).

Let $\mathcal{M}(\mathcal{F})$ denote the set of all Menger's dimension functions with respect to $\mathcal{F}$. Menger asked the problem whether or not usual topological (covering) dimension dim is the only Menger function for every $\mathcal{F} = \mathcal{E}_n$, or $\mathcal{E}$. Though it is still open that whether dim satisfies (A.4) for $\mathcal{E}_n$ [12, Problem 406], the Menger's problem itself was solved *negatively*, since every cohomological dimension $\dim_G$ with respect to a finitely generated abelian group G is a member of $\mathcal{M}(\mathcal{E})$ [3] (note that dim satisfies (A.4) for $\mathcal{F} = \mathcal{E}$).

All the known Menger's dimension functions, however, are *integer* valued so that the following question naturally arises:

Question 1. *Are there any fractional topological dimension functions?*

In other words, is every Menger dimension function *forced* to be discrete-valued? In the following section we will give some partial answer to this problem. In this paper all spaces are assumed to be separable metric, and see [4] for undefined terminology.

Remark 1. There are many *fractional* dimension functions, which are defined for every separable metric space [5]. In particular, the Hausdorff dimension function $\dim_H$ is a famous tool to investigate complexities of the so-called *fractals* in real world [10]. Unfortunately, its exact calculation is, sometimes, difficult and it needs quite different techniques from that of $\dim X$ even when the spaces X are compact [9]. We believe that one of their reasons is that none of them are topological invariant. See [6] for some more information about topological properties of fractals.

It is known that the inequality $\dim X \le \dim_H X$ holds for every $X \in \mathcal{E}$ [8]. On the contrary we have:

Remark 2. The following inequalities are known for any $d \in \mathcal{M}(\mathcal{E})$ [7, Theorem 4.1 and Lemma 5.2]:

$$\min\{1, \dim\} \leq d \leq \dim.$$

From these inequalities we have

(a) $d\,X \geq 1$ if $\dim\ X \geq 1$, and hence $Range(d) \cap (0,1) = \emptyset$ for any $d \in \mathcal{M}(\mathcal{E})$.

(b) Assume that there exists a $d \in \mathcal{M}(\mathcal{E})$, which is *different* from dim. Then, there exists a *compact* space X such that $\dim X \neq dX$.

2. Realization of functions for locally compact spaces

In this section we show an example of dimension function having the property announced in the abstract for the collection $\mathcal{F} = \mathcal{L}$, consisting of all finite dimensional locally compact spaces. In the construction of such functions, the following Facts 1 and 2 due to H. Cook [2] and K. Borsuk [1], respectively, play an essential rôle. In other words, we can say that we found unexpected applications of these interesting spaces.

Fact 1 ([2] ***Theorem 9***). *There exists a 1-dimensional hereditarily indecomposable continuum no two of whose non-degenerate subcontinua are homeomorphic.*

Since the continuum, satisfying Fact 1, has the cardinality of continuum many composants, let $\mathcal{C}_0 = \{C_r : r \in (0,1)\}$ be a collection of non-degenerated subcontinua, each of them lies in different composants. Then, we shall show that $\mathcal{C}_0$ satisfies the following property (P).

(P) *For any two of its distinct elements, it holds that no open subset of one can be topologically embedded into the other.*

Indeed, suppose that U is a non-empty open subset of C_r and that it is a subset of C_s for some $s \neq r$. Take a non-degenerate subcontinuum C in U. Then, it contradicts Fact 1, since there exist two copies of C in two different composants.

Fact 2 ([1] ***Theorem 6.1 in Ch. VI***). *For every $n \geq 1$ there exists a family $\mathcal{C}_n \subset \mathcal{E}_{n+1}$ of the cardinality of continuum,* *consisting of n-dimensional AR - sets, satisfying the property (P).*

For each collection $\mathcal{C}_n$ given by Fact 2 we can name it by

$$\mathcal{C}_n = \{C_r\}_{r\in(n,n+1)},$$ and we can assume that $I^n \notin \mathcal{C}_n$. Note that by (P) it holds that $M_n \notin \mathcal{C}_n$, where M_n is the n-dimensional Menger universal space. It also holds that $\mathcal{C}_0 \cap \mathcal{C}_1 = \emptyset$, since each element of $\mathcal{C}_1$ is locally connected. Put also

$$C_n = I^n \text{ for each } n \geq 1.$$

Example 1. *There exists a $d \in \mathcal{M}(\mathcal{L})$ such that $Range(d) = \{-1\} \cup [0, +\infty)$.*

Proof: Fix a space $X \in \mathcal{L}$ and a positive number $r \in (0, +\infty)$. Then, an *r-d sequence* $\{r_i\}_{i\geq 0}$ is provided when $\sup\{r_i\} = r$ and there exists a sequence $\{X_i\}$ of compact sets such that

$$X = \cup_{i=0}^{\infty} X_i, \text{ where } X_i \text{ embeds in } C_{r_i} \text{ for each i.}$$

For every $X \in \mathcal{L}$ let $n = \dim X$. Then, using a sequence of collections $\{\mathcal{C}_i : i \geq 0\}$, we shall define dX as follows.

$$dX = \begin{cases} \dim X & \text{when } \ n \leq 0, \\ \inf\{r : & \text{there exist } r\text{-}d \text{ sequences}\}. \end{cases}$$

Our definition is well-defined, since X can be covered by countably many compacta which embed in the $2n+1$-dimensional cube $I^{2n+1} = C_{2n+1}$, so letting $r_i = 2n+1$, one gets an r-d sequence for X. Hence, $dX \leq 2n+1$. We shall show that $Range(d) = \{-1\} \cup [0, +\infty)$. It suffices to show that $d\,C_r = r$ for each $r \in (0, +\infty)$. Note that $d\,I = 1$, since each C_r, where $r < 1$, is hereditarily indecomposable, and hence does not contain any arc. We have $d\,I^n = n$ when $n \geq 2$, since each C_r satisfies that $\dim C_r < n$ if $r < n$ and that dim satisfies (A.3). For $r \notin N$ let $\{r_i\}$ be the constant sequence $r_i = r$. Then, it is an r-d sequence by letting $X_i = C_r$ for each $i \geq 0$. Hence $d\,C_r \leq r$. Suppose that there exists an r'-d sequence for some $r' < r$. Then

$$C_r = \cup X_i, \text{ where } X_i \subset C_{r_i} \text{ for some } r_i \leq r'.$$

Since the Baire category theorem is valid in the space C_r, there exists an i such that there is a non-empty open subset G of C_r

satisfying that

$$G \subset X_i \subset C_{r_i}.$$

This contradicts the property (P).

It is evidently true that d satisfies (A.3) and (A.5). It also satisfies (A.2), since if $X, Y \in \mathcal{L}, X \subset Y$ then X is open in its closure in Y so that there exists a sequence $\{X_i\}$ of closed subsets of Y such that $X = \cup_{i=0}^{\infty} X_i$.

It satisfies (A.4), since its one-point compactification K satisfies $dK = dX$. □

Remark 3. It is hopeless to extend our function d to *non-locally compact* spaces, since it does not satisfy the property in Remark 2 (a). On the other hand, for any given $d \in \mathcal{M}(\mathcal{E})$, we can restrict it to compact spaces in order to confirm that d coincides with dim for all $X \in \mathcal{E}$ by Remark 2 (b).

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