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INTERPLAY BETWEEN INFINITE-DIMENSIONAL TOPOLOGY AND FUNCTIONAL ANALYSIS. MAPPINGS DEFINED BY EXPLICIT FORMULAS AND THEIR APPLICATIONS

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ABSTRACT. We recall some explicit formulas of analytic character which were invented during the process of formation of infinite-dimensional topology, and present some applications of them. The following topics are covered:

A. Radial homeomorphisms and retractions of convex bodies; analogues of gauge functionals and radial retractions for Banach lattices. Applications: Lipschitz retraction onto c_0 (Lindenstrauss) and lack of fixed points for Lipschitz self-maps of non compact convex sets (Lin - Sternfeld).

B. Non-complete-norm deleting homeomorphisms and diffeomorphisms with applications (Garay) to ordinary differential equations. An analogy with West's theorem on fixed point sets of transformation groups.

C. The coordinate switching technique: a "simultaneous" proof of West's theorem and the Ribe–Aharoni–Lindenstrauss example of uniformly homeomorphic and not Lipschitz homeomorphic separable Banach spaces.

Introduction

The aim of the article is to recall some explicit formulas of analytic character which were invented during the process of formation of infinite-dimensional topology, and to present some old and recent applications of them.

The approaches based on these formulas have been later replaced by more efficient and more general topological techniques: Z-sets, the limiting process for back-and-forth homeomorphisms and absorbers, Bing shrinking criterion, etc. (cf. [vM]), and partially forgotten by specialists. After several years some of them were re-discovered (perhaps, others are going to be re-discovered) by mathematicians working in other areas.

A digression: when I read the recent paper of Simon and Watson [SW], it was a sort of shock for me to learn that the back-and-forth technique in the topological context, which is the main tool for absorbers, was known to Cantor already in the 19-th century.

An advantage of explicit formulas over other constructions of maps is that they allow one to examine the maps for some additional properties and, often, the formulas can readily be modified to get the required additional properties of resulting maps, cf. sect. 1 and 2.

The words "infinite-dimensional topology" in the title are understood in the narrow sense: the topology of manifolds modeled on infinite-dimensional Banach spaces, including classification and identification (recognition) problems. The topics connected with global analysis such that as applications of fixed-point theorems and compact fields, cf. [DG], The Leray-Schauder Theory [LS], the infinite-dimensional Morse theory, the Atiyah–Singer index theory, etc. will be completely omitted.

1. Geometric constructions versus analytic formulas. Radial maps. Ultra-smoothness

Let X be a normed space. By a *convex body* in X we mean a closed convex subset of X with nonempty interior; a *starlike body* in X is a closed subset S of X such that $0 \in S$ and $ts \in int\, S$ for every $x \in S$ and for every $0 \leq t \leq 1$. We shall denote by

$$cb(X), \quad sb(X)$$

the class of all convex bodies U with $0 \in int\, U$ and the class of all starlike bodies in X, respectively. Clearly $cb(X) \subset sb(X)$. For $U \in sb\,(X)$ the *gauge functional* $g_U : X \to \mathbb{R}^+$ and the *characteristic cone* of U are defined by

$$g_U(x) = \inf\{t > 0 \,:\, x \in tU\} \quad \text{and} \quad cc\, U = g_U^{-1}(0).$$

In other words $cc\, U$ is the union of all the rays emanating from 0 which are contained in U or the singleton $\{0\}$ if no such rays exist. It is easy to prove that, for $U \in sb\,(X)$ [for $U \in cb\,(X)$], the functional g_U is continuous [Lipschitz].

EXERCISE 1. Suppose $V_1, V_2 \in cb\,(X)$ are radially bounded. Construct an autohomeomorphism h of X carrying V_1 onto V_2.

Our finite-dimensional intuition suggests the following naive approach: For each ray L emanating from 0, let $\{v_L\} = L \cap \partial V_1$ and $\{w_L\} = L \cap \partial V_2$. Take the required h so that, for each L, the restriction $h|L$ is affine, and $h(v_L) = w_L$.

Unfortunately, in the infinite-dimensional case so defined maps are not always continuous. However the construction can easily be corrected: Consider a third closed convex body U such that $0 \in U \subset int\, V_1 \cap int\, V_2$. For any ray L emanating from 0, let u_L, v_L and w_L be the three points of the intersection of L with the boundaries of the bodies. Now define h to be the identity on U, affine on each ray $L \setminus [0; u_L]$ and such that $\mathrm{h}(v_L) = w_L$.

The best way to see that now h and (by the symmetry of the assumptions) h^{-1} are continuous is to write h in terms of the gauge functionals g_U, g_1, g_2 of the bodies U, V_1, V_2:

$$(1)\quad h(x) = \varphi(g_U(x), g_U(x)/g_1(x), g_U(x)/g_2(x)) \cdot (x/g_U(x)),$$

where

$$\varphi(t) = \begin{cases} t, & \text{if } t \leq 1; \\ 1 + (t-1)(\alpha - 1)/(\beta - 1), & \text{otherwise.} \end{cases}$$

The following is evident

Theorem 1.1. *Let $U, V_1, V_2 \in sb\,(X)$ be such that $U \subset int\, V_1 \cap int\, V_2$ and $cc\, U = cc\, V_1 = cc\, V_2$; then the formula (1) defines an autohomeomorphism of X which carries V_1 onto U_2. If $U, V_1 \in cb\,(X)$, then h is locally Lipschitz.*

There are many possibilities of modifying the last theorem. One of them:

Proposition 1.1. *Let $(X, \|\cdot\|)$ be a normed space whose norm $\|\cdot\|$ is of class C^r (except 0), let ω be another norm of class C^r such that $\|x\| > 2\omega(x)$ for $x \neq 0$. Let*

$$U(x) = \{x : \|x\| \leq 1\}, \qquad V = \{x : \omega(x) \leq 1\}.$$

Then there exists a C^r autodiffeomorphism h of X preserving the rays emanating from 0 and such that $h|U = id$, $h(2U) = V$.
The required h can be defined by the formula

$$(2)\qquad h(x) = f(\|x\|, \omega(x)/\|x\|) \cdot (x/\omega(x)),$$

where $f : (0; +\infty) \times (0; 1/2) \cup \{(0,0)\} \to [0; +\infty)$ is of class C^r, $f(t,u)$ increasing with respect to t (for fixed u),

$$f(t,u) = ut, \quad \textit{if} \quad t \leq 1, \qquad f(t,u) = t/2, \quad \textit{if} \quad t \geq 2.$$

Here is another example of a map defined in terms of gauge functionals:

Theorem 1.2. *Let $U \in sb(X)$ [$U \in cb(X)$]. The formula*

$$r(x) = (\max[1, g_B(x)])^{-1} \cdot x. \tag{3}$$

defines a continuous [locally Lipschitz] retraction of X onto U with the property $r(X \setminus \operatorname{int} U) = \partial U$.

DEFINITION. Let $p : X \to \mathbb{R}^+$ be a seminorm. Let Y be a subset of X. A map $h : Y \to X$ is said to be *ultra-continuous* with respect to p if it is continuous, when regarded as

$$h : (Y, q) \to (X, q)$$

for an arbitrary (not necessarily continuous) norm q dominating p. Analogously one defines *ultra-Lipschitz* [*locally ultra-Lipschitz*] and differentiable *ultra-C^r* maps. Maps of all these kinds will be referred to as *ultra-smooth.*

Remark. The maps h and r of Theorems 1.1 and 1.2 are locally ultra-Lipschitz with respect to $p_U = g_{-U \cap U}$. The map h of Proposition 1.1 is an ultra-C^r diffeomorphism with respect to $\|\cdot\|$.

2. Lipschitz retractions onto order starlike sets in function spaces. The Sternfeld–Lin membrane

Assume that X is a Banach lattice consisting of real functions with the supremum norm and with the natural order relation $\leq$.

$$X^+ = \{x \in X : 0 \leq x\}$$

is the *positive cone* of X. Let $e \in X$ be the constant function identically equal to 1.

DEFINITION. A closed subset A of X is said to be *order starlike* if $0 \in A \subset X^+$ and for every $x \in A$ the order segment

$\{y \in X : 0 \leq y \leq x\}$ is contained in A. The order gauge functional of A is the function $g_A : X \to \mathbb{R}$ defined by

$$g_A(x) = \inf\{t \geq 0 : \max(0, x - te) \in A\}.$$

Proposition 2.1. *Let g_A be the order gauge functional of A and let $x, y \in X$. Then $0 \leq g_A(x) \leq \|x\|$; $\max[0, x - g_A(x)] \in A$; $g_A(x) = 0$ if and only if $\max[0, x] \in A$.*

$$\|g_A(x) - g_A(y)\| \leq \|x - y\|. \tag{2}$$

Proof of (2). Let $\|x - y\| = \epsilon$ and let $\max(0, x - ty) \in A$. Then $\max[0, x - (t + \epsilon e)] = \max[0, y + \epsilon e - te] \leq \max(0, x - te)$, whence $\max(0, x - te) \in A$, i.e., $g_A(y) \leq g_A(x) + \epsilon$ and, by symmetry, $g_A(x) \leq g_A(y) + \epsilon$.

Corollary 2.1. *Let A be an order starlike set in X. Then the formula*

$$r(x) = \max[0, x - g_A(x)e]$$

defines a retraction $r : X \to A$ such that $\|r(x) - r(y)\| \leq 2\|x - y\|$ for $x, y \in X$.

We shall present two applications of the last Corollary.

EXAMPLE 1. Consider the positive cone l_∞^+ in the space l_∞ of all bounded numerical sequences. By the *Lin–Sternfeld membrane* we shall mean the set M of all the points $x = (x_i) \in l_\infty^+$ with $\|x\| = \sup |x_i| \leq 1$ which have at most two non-zero coordinates.

The Lin–Sternfeld membrane is order starlike and has the following remarkable properties:

Proposition 2.2. *(i) M is a Lipschitz absolute retract;*

(ii) For every $\epsilon > 0$ there is a Lipschitz map $h : M \to M$ such that $\epsilon \leq \|x - f(x)\| \leq 2\epsilon$ for every $x \in M$.

(iii) Every noncompact closed convex set K in a Banach space contains a subset Lipschitz homeomorphic to M.

The statement (i) follows from the fact that the space l_∞ is a Lipschitz absolute retract and M is a Lipschitz retract of l_∞.

To prove (ii) observe that the membrane M is small-distance-Lipschitz homeomorphic to the half-strip

$$\mathbb{R}^+ \times [0;1).$$

Proof of (iii) . Applying the well-known Riesz construction we select a sequence (x_n), $x_n \in K$ such that, for all k, the distance of x_{k+1} from the plane spanned by $x_1, \dots, x_k$ is $\geq \epsilon$ for some $\epsilon > 0$. Let L be the union of all consecutive segments $[x_k; x_{k+1}]$. Then

$$M_1 = \bigcup_{y \in L} [0; y]$$

is the required Lipschitz copy of M.

An easy consequence of the last proposition is the following

Theorem 2.1 (Lin–Sternfeld [LiS].) *Every noncompact closed convex set K in a Banach space admits a Lipschitz map $f : K \to K$ with the property.*

$$\sup\{\|x - f(x)\| \, : \, x \in K\} > 0.$$

EXAMPLE 2 (Lindenstrauss). There is a Lipschitz retraction of l_∞ onto c_0.

In fact, the positive cone c_0^+ is an order starlike set in l_∞, therefore there is a Lipschitz $r : l_\infty \to c_0^+$. Taking the restriction $r_0 = r|l_\infty^+$ and next extending r_0 "by symmetry" to the whole l_∞, i.e, by letting $r(x_k) = (sgn\, x_k) \cdot (|x_k|)$, we obtain the required retraction of l_∞^+ onto c_0

The presentation of the material of this section had been influenced by Goebel and Kirk [GK]

3. More ultra-smooth maps: ncn deleting homeomorphisms. Garay's phenomena for ODE's in Banach spaces

The abbreviation ncn in the title of the section stands for "noncomplete norm".

EXERCISE 2. Let (X, ω) be a noncomplete normed space and let A be a subset of X which is complete in the metric induced by ω. Find a homeomorphism of $X \setminus A$ onto X.

We shall present a solution given in [B1]. Denote by (Y, ω) the completion of X in the norm ω. (Notice that the extended norm for Y has been denoted by the same symbol ω.) and let $y_0 \in Y \setminus X$. Let $0 < \delta < 1$. Pick linearly independent vectors $x_1, x_2, \ldots \in X$ and $x_0 = 0$ so that

$$\omega(x_n - x_{n-1}) \to 0, \qquad \sum_{n=1}^{\infty} \omega(x_n - x_{n-1}) = \delta$$

and consider the arc

$$L = \bigcup_{n=1}^{\infty} [x_{n-1}; x_n].$$

Let $p : \mathbb{R}^+ \to L$ be (the only) continuous map such that

(4)
$$p(0) = y_0, \quad p(t) = 1 \text{ for } t \geq 0, \; \omega(p(u) - p(v)) \leq |u - v|.$$

Observe that the composed map: $x \to p(d_\omega(x, A))$ satisfies the Lipschitz condition with constant δ. Therefore $\bar{h} : Y \to Y$ defined by the formula

$$(ncn) \qquad \bar{h}(y) = y + p(d_\omega(y, A))$$

is a Lipschitz autohomeomorphism of Y, the Lipschitz constants for $\bar{h}$ and $\bar{h}^{-1}$ are $1 + \delta$ and $(1 - \delta)^{-1}$, respectively; moreover $\bar{h}(y) = y$ if $\omega(Y) \geq 1$.

The restricted map $h = \bar{h}|X \setminus A$ carries $X \setminus A$ onto X; both h and h^{-1} are locally ultra-Lipschitz with respect to ω.

Consequently, for an arbitrary norm $\|\cdot\|$ for X which dominates the norm ω (i.e. $\|x\| \geq c\,\omega(x)$ for some $c > 0$) the map

$$h : (X \setminus A, \|\cdot\|) \to (X, \|\cdot\|)$$

is a locally Lipschitz homeomorphism.

For an arbitrary infinite dimensional normed (in particular, Banach) space $(X, \|\cdot\|)$, the norm $\|\cdot\|$ dominates a certain non-complete norm ω. Therefore:

Proposition 3.1. *Let $(X, \|\cdot\|)$ be an infinite-dimensional normed space. If A is a compact subset of X then there is a homeomorphism h of $X \setminus A$ onto X. Moreover, if ω is an arbitrary continuous norm in X, the homeomorphism h can be so defined that $h(x) = x$ if $d_\omega(x, A) \geq 1$ and that both h and h^{-1} are locally Lipschitz (in every norm dominating the norm ω).*

In the case where the space $(X, \|\cdot\|)$ admits an additional non-complete differentiable norm, the argument above can be adapted to the diffeomorphic setting. To this end, replace the piece-wise linear parametrization of the arc L by a smooth trajectory of a particle moving with bounded velocity (measured by ω) which has velocity 0 at each vertex of L, and replace d_ω in formula (ncn) by a differentiable Urysohn function satisfying the Lipschitz condition with respect to ω with a sufficiently small constant.

Here are samples of this kind result:

Theorem 3.1. *Let X be an infinite-dimensional separable normed space and A a compact subset. Then there exists a C^∞ diffeomorphism of $X \setminus A$ onto X which is the identity on a complement of an "ellipsoid".*

By *ellipsoid* we mean a ball in a continuous inner product norm.

Theorem 3.2. *Let $(X, \|\cdot\|)$ be an infinite-dimensional separable normed linear space whose norm is of class C^r and let $A \subset \{x \in X : \|x\| \leq 1\}$ be compact. Then there exists a C^r diffeomorphism of $X \setminus A$ onto X such that $h(x) = x$ if $\|x\| \geq 2$.*

For more general results see Dobrowolski [Do] and [Do1].

We shall present an application of the above results to ordinary differential equations in normed spaces; we shall follow the argument of Garay [G].

In the rest of this section we assume that:

(*) $(X, \|\cdot\|)$ is an infinite-dimensional separable normed space, $B = \{x : \|x\| \leq 1\}$ the unit ball, and K is a compact subset of B.

From Theorems 3.1 and 3.2 we shall derive, respectively, the following two theorems:

Theorem 3.3. *There exists a C^∞ function $f : X \to X$ such that $f^{-1}(K) = 0$, $f(x) = x$ for x in the complement of a certain ellipsoid and such that, for every $(t_0, x_0) \in \mathbb{R} \times (X \setminus K)$, the differential equation*

$$y' = f(y) \tag{5}$$

has a unique solution passing through (t_0, x_0) and the solution is global and unbounded.

Theorem 3.4. *Under the additional assumption, that the norm $\|\cdot\|$ is of class C^r, $r \geq 1$, there exists a C^{r-1} function $f : X \to X$ such that $f^{-1}(K) = 0$, $f(x) = x$ if $\|x\| \geq 2$ and such that, for every $(t_0, x_0) \in \mathbb{R} \times (X \setminus K)$, the differential equation*

$$y' = f(y) \tag{6}$$

has a unique solution passing through (t_0, x_0) and the solution is global and unbounded.

Proof of Theorem 3.3 (outline). Let h be a diffeomorphism of $X \setminus \{0\}$ onto $X \setminus K$ which is the identity outside an ellipsoid V. (h exists by Theorem 3.1). Consider the family of curves:

$$x = h(h^{-1}(x_0)e^t), \quad t \in \mathbb{R}, \quad x_0 \in X \setminus \{0\} \tag{7}$$

which are pair-wise disjoint and cover the set $x \setminus K$; they constitute the totality of the solutions of the differential equation:

$$x' = g(x) \tag{8}$$

where

$$g(x) = [\,[Dh^{-1}]\,(h(x)](x).$$

Let $f_1 : X \to X$ be the extension of g by letting $f_1|K = 0$. Now the differential equation:

$$x' = f_1(x) \tag{9}$$

almost does satisfy the assertion of Theorem 3.3, except that f_1 may be discontinuous at the points of K. We correct this failure by letting:

$$f(x) = \phi(x) f_1(x),$$

where $\phi : X \to [0;1]$ is a C^∞ Urysohn function such that $\phi^{-1}(0) = K$ and $\phi|(X \setminus V) = 1$.

The proof of Theorem 3.4 requires two modifications: 1^0 the homeomorphism h is now such that $h(x) = x$ if $\|x\| \geq 2$; 2^0 $\phi(x) = x$ for $\|x\| \geq 2$.

The following is a relatively easy consequence of Theorem 3.2, see [G]:

Corollary 3.1. *If X and A satisfy the assumptions of Theorem 3.4, then there is a C^{r-1} map $F : \mathbb{R} \times X \to X$ such that the Cauchy problem*

$$x' = F(t, x), \quad x(t_0) = x_0 \tag{10}$$

admits a unique (global) solution for each point $(t_0, x_0) \neq (0,0)$ *and the solutions for* $(0,0)$ *are not unique and given by:*

$$x(t) = 2^{-1}(t^2 + t|t|)a, \quad a \in K,$$

that means that at the time $t = 1$ *the solutions through (0,0) reach all the points of* K.

4. West's theorem on group actions. Coordinate switching technique

Roughly speaking Theorem 3.3 says that the additive group $\mathbb{R}$ can act as a group of diffeomorphisms on any separable infinite-dimensional normed space X so that a prescribed compact set A is the set of fixed points for all $\neq 0$ elements of the group $\mathbb{R}$. This should be compared with the following topological result of West [W], cf. [BP1], pp. 292–295.

Theorem 4.1. *Let* X *be a separable infinite dimensional Banach space and let* G *be a separable metric group. Then, for every closed subset* A *of* X, *the group* G *can act on* X *as a group of homeomorphisms so that* A *is the set of fixed points for each (but the identity* e*) element* $g \in G$.

The main point of the proof is the following

Lemma 4.1. X *is homeomorphic to the reduced cartesian product*

$$Z = (X \times X)_A = A \cup (X \setminus A) \times X$$

(the open sets in Z *are unions of open sets in* $(X \setminus A) \times X$ *and sets of the form* $V(U) = A \cap U \cup (U \setminus A) \times X$ *where* U *is an open subset of* X*).*

In fact, there is an action $g \to S_g$ of the group G on X such that $S_g(x) \neq S_g(y)$ for $x, y \in X$, $x \neq y$, $e \neq g \in G$ (cf. [BP1], p. 199). Represent X as Z and define the required action: $g \to T_g$ on Z by $T_g(x, y) = (x, S_g(y))$ for $x \in X \setminus A$ and $T_g(a) = S_g(a)$ for $a \in A$.

Now I shall try to present the main idea — *coordinate switching technique* — of the proof of the Lemma.

By [K] and [A] X is homeomorphic to the Hilbert space $H = l_2$ and to its countable infinite power H^∞. So, instead of Z we may consider the reduced cartesian product

$$(H^\infty \times H)_A, \qquad A \subset H^\infty, \text{ closed.}$$

By definition of the product topology, each point $x = (x_1, x_2, \dots) \in H^\infty \setminus A$ has a neighborhood $U \in H^\infty \setminus A$ expressed in terms of finitely many coordinates. Define

$$\begin{aligned} \operatorname{ind}(x) = \min\{k \,:\, & x \text{ has a neighborhood } U \\ & \text{depending on } x_1, \dots x_k \text{ only } \}. \end{aligned}$$

The "first approximation" for the homeomorphism h of H^∞ onto $(H^\infty \times H)_A$ is the following

$$h(x) = ((x_1, \dots, x_{ind(x)}, x_{ind(x)+2}, \dots), x_{ind(x)+1}) \in (X^\infty \times X)_A$$

for $x \in X^\infty \setminus A$, $h(a) = a$ for $a \in A$.

Of course, the map h is not even continuous. This failure is corrected as follows:

1^o Instead of $ind(x)$ we use a continuous *control function* $\lambda : H^\infty \to \mathbb{R}^\infty \cup \{\infty\}$, $\lambda^{-1}(\infty) = A$, $\lambda > ind$, locally depending on finitely many (less than its value) coordinates.

2^o At the point x with $\lambda(x) = t \in [n+1; n+2]$ we factor out a hilbertian subspace H_t of $H_{n+1} \times H_{n+2}$ which interpolates between H_{n+1} (for $t = n+1$) and H_{n+2} (for $t = n+2$).

The interpolation is obtained by means of a, so called, *reflective isotopy* $f_t : H \times H \to H \times H$: $f_{n+1}(x,y) = (x,y)$, $f_{n+2}(x,y) = (y,x)$.

The easiest way to construct a reflective isotopy is to use the complex structure of the product space $H \times H$ for interpolating between the identity and the map $(x,y) \to (y,-x)$ by multiplying by the complex function e^{it} and next using the complex structure of the second Cartesian factor H (which is isomorphic to its square) to interpolate between $-x$ and x, cf. [Ku].

The final formula for the corrected homeomorphism h is

$$h(x) = f_{\lambda(x)}(x).$$

For details see [BP], p. 294.

It should be noted that the first explicit formulation of Lemma 4.1 is due to Anderson and Schori, [AS]; and it was the main tool in the proof of their celebrated theorem on topological stability of Hilbert manifolds: *M is homeomorphic to $M \times H$.*

In the eighties the coordinate switching method was rediscovered by functional analysts for the purpose of getting examples of non-isomorphic separable reflexive Banach spaces W and Z which are uniformly homeomorphic, see the next section.

Problem 1. Let A be a compact subset of the Hilbert space l_2 and let G be a locally compact Lie group. Does there exist a diffeomorphic action $g \to T_g$ of the group G on l_2 such that A is the fixed point set for every element $g \in G$ but the identity?

Problem 2. Let $A, B \subset l_2$, A compact, B closed and $A \cap B = \emptyset$. Let $g \to U_g$ be a free diffeomorphic action of a locally compact Lie group on l_2. Does there exist another action $g \to T_g$ such that A is the fixed point set for all $g \in G$ but the identity, and $T_g|B = U_g|B$ for all $g \in G$?

Problem 3. Find an elementary proof of the the Burghelea – Kuiper – Moulis – Eels – Elworthy (see [BK], [Mo], [EE]) theorem on diffeomorphic stability of C^∞ Hilbert manifolds.

5. Mazur homeomorphism. More coordinate switching: The Ribe – Aharoni – Lindenstrauss example

In 1929 Stanisław Mazur [M] proved that the space l_r is homeomorphic to l_p $(r, p \in [1; +\infty))$ under the map:

$$(x_n) \to (|x_n|^{p/r} sgn\, x_n).$$

By *norm-preserving Mazur homeomorphisms* we shall mean the maps: $M_{r,p} : l_r \to l_p$ defined by

(11)
$$M_{r,p}(x_n) = \|(x_n)\|^{1-(p/r)} (|x_n|^{p/r} sgn\, x_n), \quad r, p \in [1; +\infty).$$

Proposition 5.1. *The maps $M_{r,p}$ have the following properties:*

(i) $\|M_{r,p}(x)\| = \|x\|$, $M_{r,p} \circ M_{s,r} = M_{s,p}$ for each $s, r, p \in [1, +\infty)$ (here the same symbol $\|\cdot\|$ denotes the norm in l_r and the norm in l_p).

(ii) The family $\{M_{r,p} : r, p \in [1, +\infty)\}$, where each $M_{r,p}$ is restricted to the ball of radius $2^{1/|p-r|}$ in l_r, is equi-uniformly-continuous.

The properties (i) are obvious, for the proof of (ii) see [R].

We shall employ certain maps induced by $M_{r,p}$'s on product spaces.

Recall that, for Banach spaces $X, Y, X_1, X_2, \ldots$, the space $(X \times Y)_q$ is the Cartesian product $X \times Y$ equipped with the norm $\|(x, y)\| = (\|x\|^q + \|y\|^q)^{1/q}$;

$$(X_1 \times X_2 \times \cdots)_q$$

is the space of all sequences $x = (x_n)$ with $x_n \in X_n$ for $n = 1, 2, \ldots$ such that $\|x\| = (\sum \|x_n\|^p)^{1/p} < +\infty$. For every Banach space E, denote

$$S_t(E) = \{x \in E : \|x\| = t\},$$

the sphere of radius t centered at 0.

For fixed p, q and for each $r \in [1; +\infty)$, define the norm-preserving homeomorphism φ_r from the space l_r represented as $(l_r \times l_r)_r$ onto $(l_r \times l_p)_q$ by the formula:

$$\varphi_r(x, y) = (\|M_{r,p}(y)\|^p + \|x\|^p)^{-1/q} \|(x, y)\|(x, M_{r,p}(y)).$$

The following theorem is due to Aharoni and Lindenstrauss, see [Be], for the particular case $(p = 1)$, see also Ribe [R].

Theorem 5.1. *Let* $q, p, r(n) \in [1; +\infty)$ $(n \in \mathbb{N})$, $r(n) \to p$ *and moreover*

$$|r(n) - p| < 1/n. \tag{12}$$

Then there exists a uniform homeomorphism h from the space $W = (l_{r(1)} \times l_{r(2)} \times \cdots)_q$ onto $Z = (l_p \times W)_p$.

Note that when $p \notin \{q, r(1), r(2), \dots\}$ the space l_p hence also Z does not even Lipschitz embed into W, so W and Z cannot be isomorphic as Banach spaces. The spaces W and Z are separable and if $1 \notin \{p, q, r(1), r(2), \dots\}$, they are reflexive.

Obviously, the maps $G_k : W \to Z$, $k \in \mathbb{N}$, defined by the formulas

$$G_k((x_n) = (u, (y_n)),$$

where

$y_n = x_n$ for $n \leq k$, $(u, y_k) = \varphi_{r(k)}((M_{r(k+1),r(k)}(x_{k+1}), (x_k))$,
$y_n = M_{r(n+1),r(n)}(x_{n+1})$ for $n \geq k$.

are homeomorphisms. None of them is uniform. Hovewer, by the the assumption (12) and Proposition 5.1, the restrictions

$$F_{2^k} = G_k | S_{2^k}(W) : S_{2^k}(W) \to S_{2^k}(Z) \quad k \in \mathbb{N}$$

and their inverses $F_{2^k}^{-1}$ are equi-uniformly continuous.

Hence the map

$$f : \bigcup_{k=1}^{\infty} S_{2^k}(W) \to \bigcup_{k=1}^{\infty} S_{2^k}(Z)$$

such that $f|S_{2^k}(W) = F_{2^k} S_{2^k}(Z)$ for $k = 1, 2, \ldots$ is a uniform homeomorphism. To complete the proof of the theorem, apply reflexive isotopies, cf. sect. 4, to get equi-uniform homeomorphisms

$$F_t : S_t(W) \to S_t(Z), \quad t \in \mathbb{R}^+$$

interpolating the homeomorphisms F_{2^k} so that the map $h : W \to Z$ defined by $h|S_t(W) = F_t,$ for $t \in \mathbb{R}^+$, is the required homeomorphism.

6. Other formulas

I am going to mention some other constructions leading to explicit formulas which already have done their job in infinite-dimensional topology but very likely will find application in other areas of mathematics.

6.1 Relatives of Mazur homeomophisms

Let X and Y be infinite-dimensional complete metric lnear spaces with bases, represented as spaces of scalar sequences (x_n), (y_n) of the coefficients with respect to the bases. The following lemma can be easily proved, e.g. by the elementary "gliding hump" argument:

Lemma 6.1. *Let $f_n : \mathbb{R} \to \mathbb{R}$ be continuous functions with $f_n(0) = 0$ for $n = 1, 2, \ldots$ If*

$$(x_n) \in X \;\Rightarrow\; (y_n) \in Y$$

then the map $f : X \to Y$ defined by $f(x_n) = (f_n(x_n))$ is continuous. If f is a bijection it is already a homeomorphism, which will be referred to as a cordinate-wise homeomorphism.

The Lemma extends to Schauder bases of subspaces.

The original Mazur homeomorphisms (not the norm-preserving ones) are coordinate-wise.

If X is a countably-normed infinite-dimensional nuclear Fréchet space with a basis (the same as nuclear Köthe echelon space corresponding to a matrix of positive numbers) then there exists a coordinate-wise homeomorphism of X onto c_0, cf. [BP1]. In particular, if X is the space of Taylor coefficients of the space of all entire functions of one variable, then the homeomorphism is induced by the classical Cauchy–Hadamard formula:

$$f(x_n) = (\sqrt[n]{x_n} sgn\, x_n).$$

The original and the norm-preserving Mazur homeomorphisms do not change the supports of the sequences:

$$(y_n) = h(x_n) \quad \Rightarrow \quad \{n \,:\, x_n \neq 0\} = \{n \,:\, y_n \neq 0\}.$$

For a generalization of norm-preserving Mazur homeomorphisms to arbitrary Banach spaces with unconditional bases *not contaning l_∞ uniformly* and their applications, see [OS].

6.2 Kadec coordinates

We assume that X is an infinite dimensional separable Banach space, $\|\cdot\|$ is an equivalent strictly-convex norm on X which satisfies the condition

$$\text{(K)} \quad x_n \to x_0 \text{ weakly } \wedge \quad \|x_n\| \to \|x_0\| \quad \Rightarrow \quad \|x_n - x_0\| \to 0.$$

(such a norm exists in every separable Banach space). Let $X_0 = X \supset X_1 \supset X_2 \supset \cdots$ be a system of closed linear subspaces of X such that $dim\, X_{k-1}/X_k = 1$ for $k \in \mathbb{N}$ and $\bigcap X_k = \{0\}$. Let $x \in X$ be a point with the property that each metric projection $P_k(x)$ of x onto X_k exists.

The *Kadec coordinates* of $d_n(x)$ are defined by

$$d_n(x) = \epsilon_n \, dist\,(x, X_n), \quad n = 1, 2, \dots ,$$

where the signs ϵ_n depend on the position of $P_n(x)$ with respect to X_{n+1}.

The *Kadec map* attaches to x the point $y \in l_1$. If X is conjugate then the Kadec map is well-defined on the whole X and it maps X homeomorphically onto l_1, [K1], cf. [K12].

For an application to the *bounded Krein–Milman property*, see [BP].

Problem 4. Let X be an arbitrary separable infinite-dimensional Banach space. Does there exist a sigma Z-sets $A \subset X$ such that the Kadec map h is well-defined on $X \setminus A$, $h : X\setminus \to h(X \setminus A)$ is a homeomorphism, and $l_1 \setminus h(X \setminus A)$ is a sigma Z-set in l_1? Try to prove the existence of A without using Kadec's theorem [K2] on topological equivalence of separable infinite-dimensional Banach spaces.

6.3 Henderson's formula: applications to factoring AR's

Let X be an abelian group, X^∞ the countable infinite product of X. Let A be an arbitrary non-empty subset of X, $r : X \to X$ an idempotent map with $r(X) = A$. The following formula is due to Henderson [H2]:

$$f(a, x) = (a + x_1, a + x_2 - r(a + x_1), a + x_3 - r(a + x_2), \dots)$$

for $a \in A \quad x = (x_n) \in X^\infty$.

For applications to factoring AR's, see Henderson [H2] and Toruńczyk [T1].

* * *

Problem. Find new applications for the formulas presented in this article.

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