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LOCALLY FINITE NEARNESS SPACES

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ABSTRACT. Locally finite nearness spaces and uniformly continuous maps are shown to form a bireflective subcategory of NEAR. The locally finite nearness structure generated by the collection of locally finite open covers of a symmetric topological space is studied. Under suitable conditions, the completion of such a space is shown to be the smallest paracompact subspace of the Wallman compactification containing the original space.

INTRODUCTION.

A nearness space is called a locally finite nearness space provided every uniform cover is refined by a locally finite uniform cover. These spaces are closely related to the interesting results obtained by Bentley [5] on paracompact nearness spaces and the material in this paper parallels the work by the author on metacompact nearness spaces [8].

It is shown that LFNEAR, the subcategory of NEAR consisting of the locally finite nearness spaces and uniformly continuous maps, is bireflective in NEAR. From this it follows that the product of a family of locally finite nearness spaces is a locally finite nearness space and a subspace of a locally finite nearness space is a locally finite nearness space.

A particular locally finite nearness structure, denoted by μ_{LF}, is studied in detail. It is defined to be the nearness structure on a symmetric topological space X generated by the family of all locally finite open covers on the space. It is shown that the full subcategory of NEAR consisting of objects of the form (X, μ_{LF}), where X is a symmetric topological space, is isomorphic to the category TOP, of symmetric topological spaces and continuous maps.

It is shown that if Y is a T_1 paracompact extension of X then $\mu_Y \subset \mu_{LF}$. Under suitable conditions, $(X*, \mu_{LF}*)$, the completion of the space (X, μ_{LF}), is the smallest paracompact subspace of wX, the Wallman Compactification of X, containing X.

1. PRELIMINARIES.

We will assume that the reader is basically familiar with the concept of a nearness space as defined by Herrlich in [7] and [8].

Definition 1.1 Let X be a set and μ a collection of covers of X, called uniform covers. Then (X, μ) is a nearness space provided:

N1. $\mathcal{H} \in \mu$ and $\mathcal{H}$ refines $\mathcal{L}$ implies $\mathcal{L} \in \mu$.

N2. $\{X\} \in \mu$ and $\emptyset \notin \mu$.

N3. If $\mathcal{H} \in \mu$ and $\mathcal{L} \in \mu$ then $\mathcal{H} \wedge \mathcal{L} = \{H \cap L : H \in \mathcal{H}$ and $L \in \mathcal{L}\} \in \mu$,

N4. $\mathcal{H} \in \mu$ implies $\{\text{int}(H) : H \in \mathcal{H}\} \in \mu$.
$(\text{int}(H) = \{x : \{X - \{x\}, H\} \in \mu\}.)$

For a given nearness space (X, μ) the collection of sets that are "near" is given by $\xi = \{\mathcal{H} \subset \mathcal{P}(X) : \{X - H : H \in \mathcal{H}\} \notin \mu\}$. The closure operator generated by a nearness space is given by $\text{cl}_\xi(A) = \{x : \{\{x\}, A\} \in \xi\}$. If we are primarily using these "near" collections we will denote the nearness space by (X, ξ). The underlying topology of a nearness space is always symmetric; that is, $x \in \text{cl}_\xi\{y\}$ implies $y \in \text{cl}_\xi\{x\}$. The following notation will be used: $\bar{\mathcal{H}} = \{\bar{H} : H \in \mathcal{H}\}$ and $X \backslash \bar{\mathcal{H}} = \{X \backslash \bar{\mathcal{H}} : H \in \mathcal{H}\}$ where $\bar{H} = \text{cl}_\xi(H)$.

Definition 1.2. Let (X, ξ) be a nearness space. The nearness space is called:

(1) topological provided $\mathcal{H} \in \xi$ implies $\cap\bar{\mathcal{H}} \neq \emptyset$.
(2) complete provided each ξ-cluster is fixed; that is, $\cap\bar{\mathcal{H}} \neq \emptyset$ for each maximal element $\mathcal{H}$ in ξ.
(3) concrete provided each near collection is contained in some ξ-cluster.
(4) contigual provided $\mathcal{H} \notin \xi$ implies there exists a finite $\mathcal{L} \subset \mathcal{H}$ such that $\mathcal{L} \notin \xi$.
(5) totally bounded provided $\mathcal{H} \notin \xi$ implies there exists a finite $\mathcal{L} \subset \mathcal{H}$ such that $\cap\mathcal{L} = \emptyset$.

Let (X, t) be a symmetric topological space. Set:

$$\xi_t = \{\mathcal{H} \subset \mathcal{P}(X) : \cap\bar{\mathcal{H}} \neq \emptyset\}$$

$$\xi_P = \{\mathcal{H} \subset \mathcal{P}(X) : \cap\bar{\mathcal{H}} \text{ has f.i.p.}\}$$

$$\xi_L = \{\mathcal{H} \subset \mathcal{P}(X) : \cap\bar{\mathcal{H}} \text{ has c.i.p.}\}$$

Each of these is a compatible nearness structure on X. They can be defined equivalently as follows:

$$\mu_t = \{\mathcal{L} \subset \mathcal{P}(X) : \ \mathcal{L} \text{ is refined by an open cover of } X\}$$

$$\mu_P = \{\mathcal{L} \subset \mathcal{P}(X) : \ \mathcal{L} \text{ is refined by a finite open cover of } X\}$$

$$\mu_L = \{\mathcal{L} \subset \mathcal{P}(X) : \ \mathcal{L} \text{ is refined by a countable open cover of } X\}$$

ξ_P is called the Pervin nearness structure on X and ξ_L the Lindelöf nearness structure on X. They are discussed in [5] and [4], respectively.

Definition 1.3. Let $\mathcal{F}$ be a closed filter in a topological space (X, t). Set $\mathcal{A}(\mathcal{F}) = \{A : \bar{A} \in \mathcal{F}\}$. If $\mathcal{F}$ is a prime closed filter, set $\mathcal{O}(\mathcal{F}) = \{O \in t : \ X - O \notin \mathcal{F}\}$.

If $\mathcal{F}$ is a prime closed filter then $\mathcal{O}(\mathcal{F})$ is a prime open filter and if $\mathcal{F}$ is a closed ultrafilter then $\mathcal{O}(\mathcal{F})$ is a minimal prime open filter and in this case $\mathcal{O}(\mathcal{F}) = \{O \in t : \ \text{there exists } F \in \mathcal{F} \text{ with } O \supset F\}$. That is; if $\mathcal{F}$ is a closed ultrafilter then the open envelope of $\mathcal{F}$ is a minimal prime open filter [6].

2. LOCALLY FINITE NEARNESS SPACES

Definition 2.1. A nearness space (X, μ) is called a locally finite nearness space if for each $\mathcal{H} \in \mu$ there exists a $\mathcal{L} \in \mu$ such that $\mathcal{L}$ refines $\mathcal{H}$ and $\mathcal{L}$ is locally finite. If (X, t) is a symmetric topological space, set $\mu_{LF} = \{\mathcal{L} \subset \mathcal{P}(X)$: There exists a locally finite open cover of X that refines $\mathcal{L}\}$.

Theorem 2.2. *Let (X, t) be a symmetric topological space. Then:*

(1) μ_{LF} *is a compatible locally finite nearness structure on* X.

(2) $\xi_{LF} = \{\mathcal{H} \subset \mathcal{P}(X) : X \backslash \bar{\mathcal{H}}$ *is not refined by a locally finite open cover of* $X\}$.

(3) *If μ is a compatible locally finite nearness structure on X then $\mu \subset \mu_{LF}$.*

Theorem 2.3. *Let (X, t) be a symmetric topological space. Then:*

(1) $\xi_t \subset \xi_{LF} \subset \xi_P$

(2) $\mu_P \subset \mu_{LF} \subset \mu_t$.

The proof of the following theorem parallels the proof of theorem 2.4 in Carlson [8].

Theorem 2.4. (1) *If $\mathcal{H} \in \xi_{LF}$ then $\bar{\mathcal{H}}$ has the finite intersection property.*

(2) *If $\mathcal{H} \in \xi_{LF}$ then the closed filter $\mathcal{F}$ generated by $\bar{\mathcal{H}}$ belongs to ξ_{LF}.*

(3) *If $\mathcal{H}$ is a ξ_{LF}-cluster and $\mathcal{F}$ is the closed filter generated by $\bar{\mathcal{H}}$ then $\mathcal{F}$ is a prime closed filter.*

Theorem 2.5. *let X be a symmetric topological space. Then:*

(1) $\mu_t = \mu_{LF}$ *iff X is paracompact.*

(2) $\mu_L \subset \mu_{LF}$ *iff X is countably paracompact.*

(3) $\mu_{LF} \subset \mu_L$ *iff every locally finite open cover of X has a countable subcover.*

Theorem 2.6. *Let X be a symmetric topological space. The following statements are equivalent.*

(1) $\mu_P = \mu_{LF}$
(2) *Every locally finite open cover of X has a finite subcover.*
(3) μ_{LF} *is totally bounded.*
(4) μ_{LF} *is contigual.*

Definition 2.7. A nearness space (X, μ) is called regular if for each $\mathcal{U} \in \mu$ there exists $\mathcal{V} \in \mu$ such that $\mathcal{V}$ strictly refines $\mathcal{U}$. (Recall: $\mathcal{V}$ strictly refines $\mathcal{U}$ if for each $V \in \mathcal{V}$ there exists $U \in \mathcal{U}$ such that $V < U$: that is, $\{U, X \backslash V\} \in \mu$).

Theorem 2.8. *Let (X, t) be a normal topological space. Then (X, μ_{LF}) is a regular nearness space.*

Proof: Easily $A <_{LF} \ B$ iff $\{B, X\backslash A\} \in \mu_{LF}$ iff $\bar{A} \subset \text{int}B$. Let $\mathcal{U} \in \mu_{LF}$. Then there exists a locally finite open cover $\mathcal{O} = \{O_\alpha : \alpha \in \Lambda\}$ that refines $\mathcal{U}$. Since X is normal and $\mathcal{O}$ is point finite it is shrinkable (Willard [11], Theorem 15.10). Thus, there exists an open cover $\mathcal{S} = \{S_\alpha : \alpha \in \Lambda\}$ such that $\bar{S}_\alpha \subset$int O_α for each $\alpha \in \Lambda$ and hence $\mathcal{S}$ is a strict refinement of $\mathcal{O}$. Easily $\mathcal{S}$ is locally finite and thus $\mathcal{S} \in \mu_{LF}$ and (X, μ_{LF}) is regular.

Locally fine nearness spaces are studied in [4] and used extensively by Bentley in [5].

Definition 2.9. A nearness space is called locally fine if it is a uniform covering of X and $(\mathcal{B}_A)_{A \in \mathcal{A}}$ is a family of uniform covers of X then $\{A \cap B : A \in \mathcal{A}$ and $B \in \mathcal{B}_A\}$ is a uniform cover of X.

Theorem 2.10. *Let (X, t) be a symmetric topological space. Then (X, μ_{LF}) is as locally fine nearness space.*

Proof: Let $\mathcal{A} \in \mu_{LF}$ and for each $A \in \mathcal{A}$ let $\mathcal{B}_A \in \mu_{LF}$. Now there exists a locally finite open cover $\mathcal{O}$ that refines $\mathcal{A}$.

Let $O \in \mathcal{O}$, then there exists $A \in \mathcal{A}$ such that $O \subset A$. Now $\mathcal{B}_A \in \mu_{LF}$ and so there exists a locally finite open cover $\mathcal{U}_O$ that refines $\mathcal{B}_A$.

Now $\mathcal{O} \wedge \mathcal{U}_O = \{O \cap U : \ O \in \mathcal{O}$ and $U \in \mathcal{U}_O\}$ is a locally finite open cover which refines $\{A \cap B : \ A \in \mathcal{A}$ and $B \in \mathcal{B}_A\}$. Hence (X, μ_{LF}) is a locally finite nearness space.

Bentley [5], in his interesting paper on paracompact nearness spaces defines a collection $\mathcal{C}$ of sets to be locally finite provided there exists a uniform cover $\mathcal{U}$ such that each member of $\mathcal{U}$ meets only finitely many members of $\mathcal{C}$. In order to distinguish this important concept from the usual topological concept of locally finite we will for the purposes of this paper only refer to his concept as "uniformly" locally finite. Then his definition of a paracompact nearness space can be stated as follows:

Definition 2.11. A nearness space (X, μ) is called paracompact if every uniform cover of (X, μ) is refined by some uniformly locally finite uniform cover.

Specifically, if $\mathcal{U} \in \mu$ then there exists $\mathcal{V} \in \mu$ and $\mathcal{W} \in \mu$ such that $\mathcal{V}$ refines $\mathcal{U}$ and each $W \in \mathcal{W}$ meets only finitely many members of $\mathcal{V}$.

Theorem 2.12. *Every paracompact nearness space is a locally finite nearness space.*

Theorem 2.13. (Bentley [5]) *A topological nearness space is paracompact (as a nearness space) iff it is paracompact as a topological space in the usual sense.*

In the following section, the class of topological spaces for which μ_{LF} is paracompact plays a crucial role.

It is well-know that the category of symmetric topological spaces is isomorphic to the subcategory of Near consisting of all topological nearness spaces and uniformly continuous maps. Since they are isomorphic, we identify them and call it TOP. The isomorphism maps (X, t) to (X, ξ_t).

It was shown in [6] that the categories of Pervin nearness space and nearness maps and Lindelöf nearness spaces and uniformly continuous maps are each isomorphic to TOP.

Let LFNEAR denote the full subcategory of NEAR consisting of all locally finite nearness spaces and uniformly continuous maps.

Define T : NEAR $\rightarrow$ LFNEAR by $T(X,\mu) = (X, T(\mu))$, where $T(\mu) = \{\mathcal{H} \in \mu$: there exists $\mathcal{L} \in \mu$ which refines $\mathcal{H}$ and is locally finite $\}$.

Theorem 2.14. *1.* LFNEAR *is a bireflective full subcategory of* NEAR.

2. The restriction of T to TOP *is an isomorphism.*

Proof: Let (X,μ) be a nearness space and $g : (X,\mu) \rightarrow (Y,\nu)$ be a uniformly continuous map where (Y,ν) is a locally finite nearness space. Define $h : (X, T(\mu)) \rightarrow (Y,\nu)$ by $h(x) = g(x)$ for all $x \in X$.

To see that h is uniformly continuous let $\mathcal{C} \in \nu$. Then there exists $\mathcal{P} \in \nu$ such that $\mathcal{P}$ is locally finite and refines $\mathcal{C}$. Now $\mathrm{int}(\mathcal{P})$ is in ν and is also locally finite. $g^{-1}(\mathrm{int}(\mathcal{P}))$ is an open locally finite cover of X. Hence $g^{-1}(\mathrm{int}(\mathcal{P}))$ belongs to $T(\mu)$ and therefore $h^{-1}(\mathcal{C}) \in T(\mu)$.

To see (2), note that $T(X,\mu_t) = (X,\mu_{LF})$. The only part of the proof that is not immediately evident is if $f : (X,t) \rightarrow (Y,s)$ is a continuous map then $f : (X,\mu_{LF}) \rightarrow (Y,\nu_{LF})$ is uniformly continuous. Let $\mathcal{C} \in \nu_{LF}$. Then there exists a locally finite open cover $\mathcal{P}$ of Y that refines $\mathcal{C}$. Since f is continuous, $f^{-1}(\mathcal{P})$ is a locally finite open cover of X. Hence $f^{-1}(\mathcal{C}) \in \mu_{LF}$ and f is uniformly continuous.

Since NEAR is a properly fibred topological category, it is known, Herrlich [10], that an epireflective subcategory of NEAR is closed under the formation of subobjects and products. Thus, we have the following result.

Corollary 2.15. *The product of a nonempty family of locally finite nearness space is a locally finite nearness space. The* *subspaces of a locally finite nearness space is a locally finite nearness space.*

3. Extensions

All spaces in this section are assumed to be T_1. An extension Y of a space X is a space in which X is densely embedded. Unless otherwise noted, we will assume for notational convenience that $X \subset Y$. It is well known that for any extension Y of X there exists an equivalent extension Y' with $X \subset Y'$.

It Y is an extension of X then $\xi = \{\mathcal{A} \subset \mathcal{P}(X) : \cap \operatorname{cl}_Y \mathcal{A} \neq \emptyset\}$ is called the nearness structure on X induced by Y. Equivalently, $\mu = \{\mathcal{U} \subset \mathcal{P}(X) : \cup \operatorname{op}(\mathcal{U}) = Y\}$ where $\operatorname{op}(\mathcal{U}) = \{\operatorname{op}(U) : U \in \mathcal{U}\}$ and $\operatorname{op}(U) = Y - \operatorname{cl}_Y(X - U)$.

Let (Y, t) be a topological space and $\bar{X} = Y$. For each $y \in Y$, set $\mathcal{O}_y = \{O \cap X : y \in O \in t\}$. Then $\{\mathcal{O}_y : y \in Y\}$ is called the filter trace of Y on X.

The strict extension topology (See Banaschewski [1]) on Y is generated by the base $\{O^* : O \in t(X)\}$, where $O^* = \{y \in Y : O \in \mathcal{O}_y\}$. Let Y be a T_1 extension of X. Then Y is a strict extension of X if and only if $\{\operatorname{cl}_Y A : A \subset X\}$ is a base for the closed sets in Y.

Herrlich's completion of a nearness space [9] can be described as follows. Let (X, ξ) be a T_1 nearness space. Let X^* be the set of all ξ-clusters and for $A \subset X$ let $\operatorname{cl}(A) = \{\mathcal{A} \in X^* : A \in \mathcal{A}\}$. A nearness structure ξ^* is defined on X^* as follows:

$\mathcal{B} \in \xi^*$ provided $\{A \subset X :$ there exists $B \in \mathcal{B}$ with $B \subset \operatorname{cl}(A)\} \in \xi$. (X^*, ξ^*) is a complete nearness space and $\operatorname{cl}_\xi X = X^*$. Also, for $A \subset X$, $\operatorname{cl}_{\xi^*}(A) = \operatorname{cl}(A)$.

Herrlich and Bentley [3], also describe the completion in the following equivalent manner. The function $e : X \to X^*$ maps every $x \in X$ onto the cluster $\mathcal{A}_x$, consisting of those subsets of X which have x as an adherent point. For any subset B of X, the set B^* denotes the set of all $p \in X^*$ such that B meets every member of the cluster p. A cover $\mathcal{C}$ of X^* belongs to μ^* provided there exists $\mathcal{P} \in \mu$ such that $\{P^* : P \in \mathcal{P}\}$ refines $\mathcal{C}$.

The following two important theorems are due to Bentley and Herrlich [2].

Theorem A. *For any T_1 nearness space (X, ξ) the following conditions are equivalent.*

(1) *ξ is a nearness structure induced on X by a strict extension.*
(2) *The completion X^* of X is topological.*
(3) *ξ is concrete.*

Theorem B. *Strict extensions are equivalent if and only if they induce the same nearness structure.*

Theorem C. (Carlson [6]) *Let (X, t) be a T_1 topological space. Then:*

(1) *ξ_P is a compatible concrete contigual nearness structure on X.*
(2) *The ξ_P-clusters are of the form $\mathcal{A}(\mathcal{F}) = \{A \subset X : \bar{A} \in \mathcal{F}\}$, where $\mathcal{F}$ is a closed ultrafilter on X.*
(3) *$\xi_P = \{\mathcal{A} \subset \mathcal{P}(X) : \mathcal{A} \subset \mathcal{A}(\mathcal{F})$ for some closed ultrafilter $\mathcal{F}\}$.*
(4) *(X^*, ξ_p^*) is the Wallman compactification of (X, t).*
(5) *If (X, t) is normal then (X^*, ξ_P^*) is the Stone-Čech compactification of (X, t).*

Theorem D. (Carlson [6]) *Let (X, ξ) be a T_1 nearness space and (X^*, ξ^*) its completion. Then the trace filters on X are given by*

(1) *$\mathcal{O}_x = \{O \in t(\xi) : x \in O\}$ for $x \in X$, and*
(2) *$\mathcal{O}_{\mathcal{A}} = \{O \in t(\xi) : X - O \notin \mathcal{A}\}$ for $\mathcal{A} \in X^* - X$.*

Thus, if X is a T_1 topological space and ξ_P the Pervin nearness structure on X then the trace filters of (X^*, ξ_P^*) are of the form $\mathcal{O}_{\mathcal{A}(\mathcal{F})} = \mathcal{O}(\mathcal{F})$ where $\mathcal{F}$ is a closed ultrafilter on X.

Let (X, ξ) be a T_1 nearness space and (X^*, ξ^*) its completion. Recall that $\operatorname{op}(A) = X^* - \operatorname{cl}_{\xi^*}(X - A)$ for $A \subset X$. Moreover, the notation U^*, for $U \subset X$, has been used to indicate the collection of ξ-clusters $\mathcal{A}$ such that $U \cap A \neq \emptyset$ for each $A \in \mathcal{A}$

and also, if U is open in X, to denote the family of ξ-clusters $\mathcal{A}$ such that $\mathcal{O}_{\mathcal{A}}$, the trace filter of $\mathcal{A}$, contains U. The following theorem notes that this notation is consistent.

Theorem 3.1 *Let (X,ξ) be a T_1 nearness space and O an open set in X. Then:*

$$\begin{aligned} op(O) &= \{\mathcal{A} \in X^* : O \cap A \neq \emptyset \text{ for each } A \in \mathcal{A}\} \\ &= \{\mathcal{A} \in X^* : O \in \mathcal{O}_{\mathcal{A}}\} = O^*. \end{aligned}$$

Theorem 3.2. *Let (X,μ) be a T_1 nearness space. Then:*

(1) *$\mathcal{U} \in \mu$ iff $\mathcal{U}^* \in \mu^*$*

(2) *Let $\mathcal{O}$ be an open cover of X. Then $\mathcal{O} \in \mu$ iff $\mathcal{O}^* \in \mu^*$.*

It follows that if (X,μ_{LF}) is concrete then $u \in \mu_{LF}{}^*$ if and only if there exists a locally finite open cover $\mathcal{P}$ of X such that $\mathcal{P}^*$ refines $\mathcal{U}$.

Recall that the Wallman compactification for a T_1 topological space can be described as the collection wX of closed ultrafilters on X with the base for the closed sets given by $\{F^* : F \text{ closed in } X\}$ where $F^* = \{\mathcal{F} \in wX : F \in \mathcal{F}\}$. Now $wX \cong (X^*,\mu_p^*)$ and the elements of X^* are of the form $\mathcal{A}(\mathcal{F})$ for $\mathcal{F}$ a closed ultrafilter on X.

Definition 3.3. Let (X,t) be a T_1 topological space. Let $S = \{\mathcal{F} \in wX : X\backslash\mathcal{F} \text{ does not have a locally finite subcover}\}$ $S' = \{\mathcal{A}(\mathcal{F}) : \mathcal{F} \in S\}$.
Now $S \subset wX$ and S' is a subset of (X^*,μ_p^*). Since S' is the image of S under the homeomorphism from wX to (X^*,μ_p^*) we will identify them and by abuse of the notation not distinguish between them when no confusion can occur.

Definition 3.4. Let (X,t) be a T_1 topological space and μ_p the Pervin nearness structure on X. Let $\mathcal{O}$ be an open cover of X and set

$$Y(\mathcal{O}) = \cup\{O^* : O \in \mathcal{O}\}.$$

Theorem 3.5. *Let (X,t) be a T_1 topological space. Then:*

(1) *If $\mathcal{O}$ is an open cover and $\mathcal{O} \in \mu_{LF}$ then $S \subset Y(\mathcal{O})$.*

(2) $S = \cap\{Y(\mathcal{O}) :$ *$\mathcal{O}$ an open cover of X and $\mathcal{O} \in \mu_{LF}$*$\}$.
(3) $S = \cap\{Y(\mathcal{O}) :$ *$\mathcal{O}$ is a locally finite open cover of X*$\}$.

If μ_{LF} is concrete and each ξ_{LF}-cluster is a ξ_P-cluster then $(X^*, \mu_{LF}*)$ is a subspace of (X^*, μ_p*), the Wallman compactification of X.

Condition A. For each open cover $\mathcal{O}$ that has no locally finite open refinement there exists $\mathcal{F} \in S$ such that $\mathcal{O} \subset X\backslash\mathcal{F}$.

Condition B. For every locally finite open cover $\mathcal{U}$ there exists a locally finite open cover $\mathcal{V}$ that refines $\mathcal{U}$ and a locally finite open cover $\mathcal{W}$ such that each member of $\mathcal{W}$ meets only finitely many members of $\mathcal{V}$.

Condition C. For each locally finite open cover $\mathcal{U}$ there exists a locally finite open cover $\mathcal{V}$ that refines $\mathcal{U}$ such that for each $\mathcal{F} \in S$ there exists $O \in \mathcal{O}(\mathcal{F})$ such that O meets only finitely many members of $\mathcal{V}$.

Theorem 3.6. *Let (X, t) be a symmetric topological space. Then:*

(1) *(X, t) satisfies Condition A iff $\mu_{LF} = \mu_S$.*
(2) *(X, t) satisfies Condition A iff μ_{LF} is concrete and every ξ_{LF}-cluster is a ξ_P-cluster.*
(3) *(X, t) satisfies condition B iff μ_{LF} is paracompact.*

Theorem 3.7. *Let (X, t) be a topological space. Then:*

(1) *Condition B implies Condition C.*
(2) *Condition A plus Condition C implies Condition B.*

Proof: (1) Let $\mathcal{U}$ be a locally finite open cover of X. By Condition B there exists a locally finite open cover $\mathcal{V}$ that refines $\mathcal{U}$ and a locally finite open cover $\mathcal{W}$ such that each $W \in \mathcal{W}$ meets only finitely many members of $\mathcal{V}$. Let $\mathcal{F} \in S$. Then $\mathcal{W} \not\subset X\backslash\mathcal{F}$ and thus there exists a $W \in \mathcal{W}$ such that $W \in \mathcal{O}(\mathcal{F})$.

(2) Let $\mathcal{U}$ be a locally finite open cover of X. By condition C there exists a locally finite open cover $\mathcal{V}$ that refines $\mathcal{U}$ such

that for each $\mathcal{F} \in S$ there exist $O \in \mathcal{O}(\mathcal{F})$ such that O meets only finitely many members of $\mathcal{V}$.

Set $\mathcal{O} = \{O : O$ is open and O meets only finitely many members of $\mathcal{V}\}$.

$\mathcal{O}$ is an open cover of X since $\mathcal{V}$ is locally finite. By the definition of $\mathcal{O}$ any open refinement of $\mathcal{O}$ must be a subcover of $\mathcal{O}$. If $\mathcal{O}$ has a locally finite open refinement we are through. Suppose it does not. Then, by Condition A, there exists $\mathcal{F} \in S$ such that $\mathcal{O} \subset X \backslash \mathcal{F}$. but, by Condition C, there exists $O \in \mathcal{O}(\mathcal{F})$ that meets only finitely many members of $\mathcal{V}$. But this implies that $O \in \mathcal{O}$ and we have a contradiction. Hence, there exists a locally finite open cover $\mathcal{W}$ that refines $\mathcal{O}$. Therefore, $\mathcal{W} \subset \mathcal{O}$ and thus each member of $\mathcal{W}$ meets only finitely many members of $\mathcal{V}$. Hence Condition C holds.

Theorem 3.8. *Let (X,t) be a T_1 topological space satisfying Condition A and Condition C. Then S is paracompact.*

Proof: Every basic open cover of S is of the form $\mathcal{U}^* = \{U^* : U \in \mathcal{U}\}$ where $\mathcal{U}$ is an open cover of X and $S \subset \cup\mathcal{U}^*$. Let $\mathcal{U}^*$ be a basic open cover of S. Suppose $\mathcal{U}$ is not locally finite. Then, since Condition A holds, there exists $\mathcal{F} \in S$ such that $\mathcal{U} \subset X \backslash \mathcal{F}$. But then $\mathcal{F} \notin \cup \mathcal{U}^*$ and we have a contradiction. Thus, $\mathcal{U}$ must be locally finite and by Condition C there exists a locally finite open cover $\mathcal{V}$ that refines $\mathcal{U}$ such that for each $\mathcal{F} \in S$ there exists an $O \in \mathcal{O}(\mathcal{F})$ that meets only finitely many members of $\mathcal{V}$. Then $\mathcal{V}^*$ refines $\mathcal{U}^*$ and $\mathcal{F} \in O^*$ where O^* meets only finitely many members of $\mathcal{V}^*$. Hence S is paracompact.

Theorem 3.9. *Let (X,t) be a T_1 topological space. Then:*

(1) $\mu_P \subset \mu_{LF} \subset \mu_S \subset \mu_t$
(2) $\xi_t \subset \xi_S \subset \xi_{LF} \subset \xi_P$

Theorem 3.10. *Let Y be a strict T_1 extension of X. If Y is paracompact then $\mu_Y \subset \mu_{LF}$.*

Theorem 3.11. (Bentley [5]) *A nearness space is concrete and paracompact iff its completion is topological and paracompact.*

Corollary 3.12. *Let (X, t) be a T_1 topological space. The following statements are equivalent.*

(1) (X, μ_{LF}) *is concrete and paracompact.*
(2) $(X^*, \mu_{LF}*)$ *is topological and paracompact.*

Theorem 3.13. *Let (X, t) be a T_1 topological space such that μ_{LF} is concrete. Then $S' = (X^*, \mu_{LF}*)$ iff each ξ_{LF}-cluster is a ξ_p-cluster.*

Definition 3.14 A closed filter $\mathcal{F}$ is called a p-filter if $x \backslash \mathcal{F}$ is not refined by a locally finite open cover. Let Y be a topological space and $X \subset Y$. X is said to be relatively paracompact in Y if for each open cover $\mathcal{O}$ of Y there exists a locally finite open cover of X that refines $\{O \cap X : O \in \mathcal{O}\}$.

It is evident that a space is relatively paracompact in itself if and only if it is paracompact. It is also clear that every subspace of a paracompact space is relatively paracompact in that space.

Theorem 3.15. *Let Y be a topological space and $X \subset Y$. Then X is relatively paracompact in Y iff each p-filter on X clusters in Y.*

Proof: Suppose X is relatively paracompact in Y and let $\mathcal{F}$ be a p-filter in X. If $\mathcal{F}$ does not cluster in Y then for each $y \in Y$ there exists an open set O_y in Y containing y and an $F_y \in \mathcal{F}$ such that $O_y \cap F_y = \emptyset$. Then $\mathcal{O} = \{O_y : y \in Y\}$ is an open cover of Y and hence $\mathcal{O} \wedge \{X\}$ is an open cover of X. Since X is relatively paracompact in Y there exists a locally finite open cover $\mathcal{S}$ of X that refines $\mathcal{O} \wedge \{X\}$. Hence $\mathcal{S}$ refines $X \backslash \mathcal{F}$ which is impossible since $\mathcal{F}$ is a p-filter. Thus, each p-filter on X must cluster in Y.

On the other hand, suppose that each p-filter on X does cluster in Y and suppose that $\mathcal{O}$ is an open cover of Y such that $\mathcal{O} \wedge \{X\}$ is not refined by a locally finite open cover of X. Let $\mathcal{F}$ be the closed filter generated by the closed filter subbase

$\{X - (O \cap X) : O \in \mathcal{O}\}$. Then $\mathcal{F}$ is a p-filter on X that does not cluster in Y and we have a contradiction. Therefore, X is relatively paracompact in Y.

Theorem 3.16. *Let (X, t) be a T_1 topological space. The following statements are equivalent.*

(1) *S is paracompact.*
(2) *μ_{LF} is concrete and paracompact and each ξ_{LF}-cluster is a ξ_p-cluster.*
(3) *(X^*, μ_{LF^*}) is a paracompact topological space and each ξ_{LF}-cluster is a ξ_p-cluster.*
(4) *$\mu_{LF} = \mu_S$ and μ_{LF} is paracompact.*
(5) *X is relatively paracompact in S and μ_{LF} is paracompact.*

Proof: (1) implies (2). Suppose ξ_{LF} is not concrete. Then there exists a p-filter $\mathcal{F}$ such that $\mathcal{F}$ is not contained in any ξ_{LF}-cluster. Thus $\mathcal{F} \not\subset \mathcal{M}$ for each $\mathcal{M} \in S$. Set $\mathcal{O} = X \backslash \mathcal{F}$. Then $S \subset \cup\, \mathcal{O}^*$ and since $\mathcal{O}$ has no locally finite open refinement it follows that S is not paracompact. But this is impossible and hence ξ_{LF} must be concrete and each ξ_{LF}-cluster must be a ξ_p-cluster.

Finally $\mu_{LF} \subset \mu_S$ and if S is paracompact then, by theorem 3.10, $\mu_S \subset \mu_{LF}$. Hence, $\xi_{LF} = \xi_S$ and thus each ξ_{LF}-cluster is a ξ-cluster.

To see that μ_{LF} is paracompact let $\mathcal{U} \in \mu_{LF}$. Then $S \subset \cup\, \mathcal{U}^*$ and since S is paracompact there exists a locally finite open cover $\mathcal{O}$ of S that refines $\mathcal{U}^*$. Let $\mathcal{N} = \{N_p : p \in S\}$ be a collection of open neighborhoods of the points of S such that each N_p meets only finitely many members of $\mathcal{O}$.

Now $\mathcal{N}$ is also an open cover of S and since S is paracompact there exists a locally finite open cover $\mathcal{W}$ of S that refines $\mathcal{N}$. Let $\mathcal{V} = \{O \cap X : O \in \mathcal{O}\}$ and $\mathcal{H} = \{W \cap X : W \in \mathcal{W}\}$. Now $\mathcal{V} \in \mu_{LF}$ and $\mathcal{V}$ refines $\mathcal{U}$. Now $\mathcal{H} \in \mu_{LF}$ and for each $W \cap X \in \mathcal{H}$ there exists an $N_p \in \mathcal{N}$ with $W \subset N_p$. Since N_p meets only finitely many members of $\mathcal{O}$ it follows that $W \cap X$ meets only finitely many members of $\mathcal{V}$. Hence μ_{LF} is paracompact.

(2) implies (1). This holds by theorem 3.6 and theorem 3.8.

(2) iff (3). This holds by theorem 3.11.

(2) iff (4). Assume that (2) holds. By theorem 3.9, $\mu_{LF} \subset \mu_p$. Since (2) implies (1), S is paracompact and $\mu_S \subset \mu_{LF}$ by theorem 3.10. Hence (4) holds. That (4) implies (2) is clear.

(1) implies (5). If S is paracompact then, since $X \subset S$, it follows that X is relatively paracompact in S. Since (1) implies (2) it follows that μ_{LF} is paracompact.

(5) implies (2). A p-filter $\mathcal{F}$ on X clusters in S iff there exists $\mathcal{M} \in S$ with $\mathcal{F} \subset \mathcal{M}$. This is equivalent to μ_{LF} being concrete and each ξ_{LF}-cluster being a ξ_p-cluster. Thus (2) holds.

References

1. B. Banaschewski, *Extensions of topological spaces*, Can. Math. Bull., **7**, (1964) 1-22.
2. H.L. Bentley and H. Herrlich, *Extensions of topological spaces*, Topological Proc., Memphis State Univ. Conference, Marcel-Dekker, New York (1976) 120-184.
3. H. L. Bentley and H. Herrlich, *Completion as reflection*, Commentationes Math. Univ. Carolinae, **19, 3**, (1978) 541-568.
4. H. L. Bentley, M. S. Hastings, R. G. Ori, *Rings of uniformly continuous functions*, Categorical Topology Proc. Conference Toledo, Ohio, Helderman Verlag Berlin (1983) 46-70.
5. H. L. Bentley, *Paracompact spaces*, Topology and its Appl. **39**, (1991) 283-297.
6. J. W. Carlson, *Lindelöf Nearness Structures*, Categorical Topology Proc. Conference Toledo, Ohio, Helderman Verlag Berlin (1983) 185-196.
7. J. W. Carlson, *Pervin nearness spaces*, Top. Proc. **9**, (1984) 7-30.
8. J. W. Carlson, *Metacompact nearness spaces*, Top. Proc., **16** (1991) 17-28.
9. H. Herrlich, *A Concept of nearness*, Gen. Top Appl. **5**, (1974) 191-212.
10. H. Herrlich, *Topological structures*, Mathematical Centre Tracts **52**, Amsterdam, 1974.
11. S. Willard, General Topology, Addison-Wesley, Reading Massachusetts, 1970.

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