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T-PROTOPOLOGICAL GROUPS

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ABSTRACT. A protopological group is a group G with a topology t with the property that there exists a collection $\mathcal{N}$ of normal subgroups such that (1) for every neighborhood U of the identity there exists $N \in \mathcal{N}$ such that $N \subset U$ and (2) G/N with the quotient topology is a topological group for every $N \in \mathcal{N}$. We say that $\mathcal{N}$ converges to the identity and call $\mathcal{N}$ a normal system. A t-protopological group is a protopological group with normal system $\mathcal{N}$ such that for all $N \in \mathcal{N}$ we have that UN is open. In this paper we examine basic properties of t-protopological groups. We introduce a method of describing the associated Graev topology for a protopological group. Finally, we prove that a compact Hausdorff t-protopological group is a topological group.

1. INTRODUCTION

Montgomery and Zippin [6] in 1955 gave the following definition: a group G is approximated by Lie groups if every neighborhood of the identity contains an invariant subgroup H such that G/H is topologically isomorphic to a Lie group. Bagley, Wu and Yang [1] in 1985 gave a definition of pro-Lie which is similar to Montgomery and Zippin's definition of approximated by Lie groups. In this paper we will make a definition similar to Montgomery and Zippin's definition of approximated by Lie groups which deals with the more general case of topological groups.

Definition. A protopological group is a group G with a topology t with the property that there exists a collection $\mathcal{N}$ of normal subgroups such that (1) for every neighborhood U of the identity there exists $N \in \mathcal{N}$ such that $N \subset U$ and (2) G/N with the quotient topology is a topological group for every $N \in \mathcal{N}$. We say that $\mathcal{N}$ converges to the identity and call $\mathcal{N}$ a normal system. If each $N \in \mathcal{N}$ is closed, we say that $\mathcal{N}$ is a closed normal system.

Ellis has proven that if (G, t) is a locally compact Hausdorff group with continuous multiplication, then (G, t) is a topological group [4]. There are other variations on this result; however, all have placed some type of restriction on the multiplication map. In [3] we see that a protopological group with continuous multiplication is a topological group. In this paper, we are able to get a more general result without placing any conditions on the multiplication map.

We introduce some notation which will be used throughout this paper. If G is a group and N a normal subgroup, we use xN to denote the equivalence class of x in G/N. We shall denote the quotient map by π where $\pi : G \to G/N$ is defined by $\pi(x) = xN$. If we need to distinguish between quotient maps on a group G, we will subscript π with the normal subgroup being used, for example π_N. If G has a topology t and we place the quotient topology on G/N we denote the quotient topology by t_π or if needed by t_{π_N}.

Let (G, t) be a topological group and N a normal subgroup. The quotient map $\pi : (G, t) \to (G/N, t_\pi)$ is an open map, since if U is open in t, $\pi^{-1}(\pi(U)) = NU$ is open in t and hence, $\pi(U)$ is open in t_π. However, the quotient map is not necessarily open if (G, t) is a protopological group. To see this define a topology $t = \{\emptyset, \mathbb{R}, \mathbb{Z} \cup \{\frac{1}{2}\}\}$ on the real numbers. Clearly, $(\mathbb{R}, t)$ is a protopological group with normal system consisting only of $\mathbb{Z}$; however, $\pi : \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ is not open. If (G, t) is a group with a topology and N is a normal subgroup, then $\pi : G \to G/N$ is an open map if and only if UN is open in G for all U open in t.

2. T-PROTOPOLOGICAL GROUPS

The above observation motivates the following definition:

Definition. A t-protopological group is a protopological group (G, t) with normal system $\mathcal{N}$ such that for all open sets U and for all $N \in \mathcal{N}$ we have that UN is open. We say that $\mathcal{N}$ is a t-normal system.

In a t-protopological group, $\pi_N : G \to G/N$ is an open map for all $N \in \mathcal{N}$ since UN is open by definition of t-protopological group. An example is given in [3] of a protopological group for which the required quotient maps are not open, hence the group is not a t-protopological group. Thus, it is possible to have a normal system that is not a t-normal system.

The category of topological groups has products, quotients, subgroups, and joins. In a topological group, the closure of a normal subgroup is a normal subgroup. This is not necessarily the case, even in a group with continuous multiplication. To see this, we place the open right ray topology on the real numbers and note that $\overline{0}$ is not a subgroup. We will investigate which of the above constructions can be carried out in the category of t-protopological groups.

Theorem 1. *If (G, t) is a t-protopological group with t-normal system $\mathcal{N}$ and if $f : (G, t) \to (G', t')$ is a continuous, open, onto homomorphism, then (G', t') is a t-protopological group with t-normal system $\mathcal{N}' = \{f(N) \mid N \in \mathcal{N}\}$.*

Proof: Clearly, $\mathcal{N}'$ converges to the identity in G'. We wish to show that the quotient map $p : (G', t') \to (G'/f(N), t'_p)$ is an open map. If V is open in t', then $f^{-1}(V)$ is open in t. Hence, for all $N \in \mathcal{N}$, $Nf^{-1}(V)$ is open in t. Thus, $f(Nf^{-1}(V)) = f(N)V$ is open for all $f(N) \in \mathcal{N}'$ and p is open. To see that $(G'/f(N), t'_p)$ is a topological group, we use the following commutative diagram:

$$\begin{array}{ccc} (G,t) & \xrightarrow{\ f\ } & (G',t') \\ \pi \downarrow & & \downarrow p \\ (G/N,\ t_\pi) & \xrightarrow{\ f'\ } & (G'/f(N),\ t'_p) \end{array}$$

Since we are factoring out the image of N we get an induced continuous, onto homomorphism f' such that $f' \circ \pi(g) = p \circ f(g)$ for all $g \in G$. Let U be open in t_π. Since $f'(U) = p \circ f\left(\pi^{-1}(U)\right)$ we have that $f'(U)$ is open in t'_p. Thus, f' is a continuous, open, onto homomorphism and since $(G/N,\ t_\pi)$ is a topological group, $(G'/f(N),\ t'_p)$ is also. Therefore, $(G',\ t')$ is a t-protopological group. □

The following theorems show how to form new t-protopological groups from given t-protopological groups. If $\{(G_\alpha, t_\alpha)\}_{\alpha\in\Gamma}$ is a family of topological groups, then $\prod\limits_{\alpha\in\Gamma} G_\alpha$ is a topological group with the product topology. We get a similar result if we have a family of t-protopological groups.

Theorem 2. *If $\{(G_\alpha,\ t_\alpha)\}_{\alpha\in\Gamma}$ is a collection of groups with topologies and τ denotes the product topology on $\prod\limits_{\alpha\in\Gamma} G_\alpha$, then $(\prod\limits_{\alpha\in\Gamma} G_\alpha,\ \tau)$ is a t-protopological group if and only if (G_α, t_α) is a t-protopological group for all $\alpha \in \Gamma$.*

Proof: If $(\prod\limits_{\alpha\in\Gamma} G_\alpha, \tau)$ is a t-protopological group, then since $p_\beta : \prod\limits_{\alpha\in\Gamma} G_\alpha \to G_\beta$ is a continuous, open, onto homomorphism we have that (G_β, t_β), for all $\beta\in \Gamma$, is a t-protopological group by Theorem 1.

Let $\{(G_\alpha, t_\alpha)\}_{\alpha\in\Gamma}$ be a family of t-protopological groups where $\mathcal{N}_\alpha$ denotes the t-normal system for G_α. We form $\mathcal{N} = \{\prod_{\alpha\in\Gamma} N_\alpha \mid N_\alpha \in \mathcal{N}_\alpha\}$ and note that $\mathcal{N}$ converges to the identity. Let $\prod_{\alpha\in\Gamma} W_\alpha$ be a basic open set in $\prod_{\alpha\in\Gamma} G_\alpha$. Then $\left(\prod_{\alpha\in\Gamma} N_\alpha\right)\left(\prod_{\alpha\in\Gamma} W_\alpha\right) = \prod_{\alpha\in\Gamma} N_\alpha W_\alpha$ which is open in $\prod_{\alpha\in\Gamma} G_\alpha$. To see that $\prod_{\alpha\in\Gamma} G_\alpha / \prod_{\alpha\in\Gamma} N_\alpha$ is a topological group, we define $f : \prod_{\alpha\in\Gamma} G_\alpha/N_\alpha \to \prod_{\alpha\in\Gamma} G_\alpha / \prod_{\alpha\in\Gamma} N_\alpha$ by $f(\langle x_\alpha N_\alpha\rangle) = \langle x_\alpha\rangle \prod_{\alpha\in\Gamma} N_\alpha$ and define $g : \prod_{\alpha\in\Gamma} G_\alpha / \prod_{\alpha\in\Gamma} N_\alpha \to \prod_{\alpha\in\Gamma} G_\alpha/N_\alpha$ by $g(\langle x_\alpha\rangle \prod_{\alpha\in\Gamma} N_\alpha) = \langle x_\alpha N_\alpha\rangle$. Then f and g are inverse isomorphisms. We have the following commutative diagram where $p = \prod_{\alpha\in\Gamma} \pi_{N_\alpha}$:

$$
\begin{array}{ccc}
 & (\prod_{\alpha\in\Gamma} G_\alpha, \tau) & \\
 {}^{p}\swarrow & & \searrow^{\pi} \\
(\prod_{\alpha\in\Gamma} G_\alpha/N_\alpha, \prod_{\alpha\in\Gamma} t_{\alpha_\pi}) & \underset{g}{\overset{f}{\rightleftarrows}} & (\prod_{\alpha\in\Gamma} G_\alpha / \prod_{\alpha\in\Gamma} N_\alpha, \tau_\pi)
\end{array}
$$

Since π is a quotient map and p is continuous, g is continuous. Since each π_{N_α} is open, p is a quotient map and f is continuous. Since $(\prod_{\alpha\in\Gamma} G_\alpha/N_\alpha, \prod_{\alpha\in\Gamma} t_{\alpha_\alpha})$ is a topological group and f is a topological isomorphism, $(\prod_{\alpha\in\Gamma} G_\alpha / \prod_{\alpha\in\Gamma} N_\alpha, \tau_\pi)$ is a topological group. Hence, $(\prod_{\alpha\in\Gamma} G_\alpha, \tau)$ is a t-protopological group. $\square$

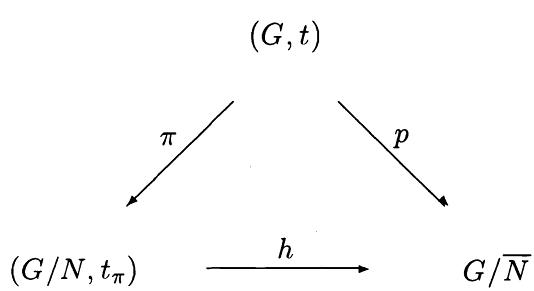
In a topological group, any subgroup S with the inherited topology is a topological group. In a protopological group, this is not necessarily the case. However, an open subgroup of a protopological group is a protopological group [3]. Similar results are true for t-protopological groups.

As mentioned before, in a topological group the closure of a normal subgroup is a normal subgroup. In a t-protopological group, if N is an element of a t-normal system, then $\overline{N}$ is a normal subgroup.

Theorem 3. *If (G,t) is a t-protopological group with t-normal system $\mathcal{N}$, then for every $N \in \mathcal{N}$, $\overline{N}$ is a normal subgroup and $G/\overline{N}$ is a topological group with the quotient topology. Furthermore, $p : (G,t) \to (G/\overline{N}, t_p)$ is an open map.*

Proof: For any $N \in \mathcal{N}$, we denote the identity in G/N by e and note that $\overline{e}$ is a normal subgroup in G/N. A direct argument shows that $\pi_N^{-1}(\overline{e}) = \overline{N}$ and hence, $\overline{N}$ is a normal subgroup.

Next, we will show that $G/\overline{N}$ is a topological group with the quotient topology from the quotient map $p : G \to G/\overline{N}$. By the first isomorphism theorem for groups [5], since $\overline{N}/N$ is a normal subgroup of G/N, $(G/N)/(\overline{N}/N)$ is algebraically isomorphic to $G/\overline{N}$. The quotient map $h : (G/N, t_\pi) \to \big((G/N)/(\overline{N}/N), t_{\pi_h}\big)$ is an open map and $\big((G/N)/(\overline{N}/N), t_{\pi_h}\big)$ is a topological group. We can think of h as taking G/N onto $G/\overline{N}$ since $G/\overline{N}$ and $(G/N)/(\overline{N}/N)$ are algebraically isomorphic. Thus, we have the following commutative diagram:



We can place two topologies on $G/\overline{N}$, either t_p or t_{π_h}. We have that U is open in t_{π_h} if and only if $\pi^{-1}(h^{-1}(U))$ is open in t. But, $\pi^{-1}(h^{-1}(U))$ is open in t if and only if $p^{-1}(U)$ is open in t. Equivalently, $p^{-1}(U)$ is open in t if and only if U is open in t_p. Thus, $t_p = t_{\pi_h}$. Therefore, $(G/\overline{N}, t_p)$ is a topological group since $((G/N)/(\overline{N}/N), t_{\pi_h})$ is a topological group. Since the diagram commutes and π and h are open maps, p is an open map. □

Corollary 4. *If (G,t) is a regular t-protopological group with t-normal system $\mathcal{N}$, then $\mathcal{N}' = \{\overline{N} \mid N \in \mathcal{N}\}$ is a closed t-normal system for (G,t).*

3. The Associated Graev Topology

Let (G,t) be a group with a topology., then the associated Graev topology is the finest group topology τ such that $\tau \subset t$ [2]. It is clear that the associated Graev topology exists, but in general it is difficult to describe this topology. We shall introduce a method of describing the associated Graev topology for a protopological group.

Let (G,t) be a protopological group with normal system $\mathcal{N}$. Then for each $N \in \mathcal{N}$, $(G/N,\ t_\pi)$ is a topological group with $\pi_N : (G,t) \to (G/N,\ t_\pi)$. We call η the natural map where $\eta : G \to \prod_{N\in\mathcal{N}} G/N$ is defined by $\eta(x) = \langle \pi_N(x) \rangle$. Since each function π_N is a continuous homomorphism, η is a continuous homomorphism when we place the product topology on $\prod_{N\in\mathcal{N}} G/N$. Thus, $(\prod_{N\in\mathcal{N}} G/N,\ \tau)$, where τ denotes the product topology, is a topological group. We can use η to place a topology P on G where P is the pullback of the product topology on $\prod_{N\in\mathcal{N}} G/N$. We can also describe P as the initial or weak topology for the family of maps $\{\pi_N\}_{N\in\mathcal{N}}$. Thus, (G,P) is a topological group since the pullback of a group topology is a group topology.

If (G,t) is a protopological group with normal system $\mathcal{N}$ such that $\bigcap_{N\in\mathcal{N}} N = e$, then η is one-to-one. Therefore, if (G,t) is a T_1 protopological group, then η is one-to-one.

Theorem 5. *If (G,t) is a protopological group, then P is the associated Graev topology.*

Proof: Since P is the coarsest topology on G such that the family of maps $\{\pi_N\}_{N\in\mathcal{N}}$ is continuous, we have $P \subset t$.

Next, let τ be any group topology on G such that $\tau \subset t$. Let U be a neighborhood of e in τ. Then there exists W, a neighborhood of e in τ, such that $W^2 \subset U$. Then, since $\tau \subset t$, W is also a neighborhood of e in t. Hence, there exists $N \in \mathcal{N}$ with $N \subset W$. Since (G,τ) is a topological group, we have that $\pi_N(W)$ is open in $(G/N, \tau_\pi)$. Since $\tau \subset t$, $\tau_\pi \subset t_\pi$ and hence $\pi_N(W)$ is open in t_π. Then $\pi_N^{-1}(\pi_N(W))$ is open in P since π_N is continuous when G is endowed with P. Hence, $\pi_N^{-1}(\pi_N(W)) = NW$ is open in P and $NW \subset W^2 \subset U$. Thus, $e \in NW \subset U$ and hence U is an element of P. Therefore, $\tau \subset P$ for every group topology $\tau \subset t$, and hence P is the associated Graev topology for (G,t). □

We are often interested in what topological properties the associated Graev topology will posses. Using the fact that P is the associated Graev topology for a protopological group, we can obtain some insight.

Corollary 6. *If (G,t) is a regular Hausdorff t-protopological group, then the associated Graev topology is Hausdorff.*

Corollary 7. *If (G,t) is a T_1 protopological group with a closed normal system, then the associated Graev topology is Hausdorff.*

Corollary 8. *If (G,t) is a T_1 protopological group with a closed normal system, then (G,t) is Hausdorff.*

Theorem 9. *If (G, t) is a regular Hausdorff first countable t-protopological group, then the associated Graev topology is metric.*

Proof: Since (G, t) is first countable, there is a countable closed normal system $\mathcal{N}$ for (G, t). We can think of (G, P) as being embedded in $\prod_{N \in \mathcal{N}} G/N$, a countable product of metric spaces. Therefore, (G, P) is a metric space.

Ellis's Theorem [4] states that if (G, t) is a compact Hausdorff group with a topology t such that multiplication is continuous, then (G, t) is a topological group. We have a similar result if we replace the continuity condition with the protopological group condition.

Theorem 10. *If (G, t) is a compact Hausdorff t-protopological group, then (G, t) is a topological group.*

Proof: We know that (G, t) has a closed normal system $\mathcal{N}$ by Corollary 4. Then for each $N \in \mathcal{N}$, $(G/N,\ t_{\pi_N})$ is compact Hausdorff. Hence, $\prod_{N \in \mathcal{N}} G/N$ is a compact Hausdorff topological group. We have $\eta : (G, t) \to \prod_{N \in \mathcal{N}} G/N$ where η is a continuous, closed, one-to-one homomorphism. Hence, η is an embedding. Since $\eta(G)$ is a subgroup of the topological group $\prod_{N \in \mathcal{N}} G/N$, then $\eta(G)$ is a topological group. Therefore, (G, t) is a topological group. □

Corollary 11. *If (G, t) is a compact Hausdorff protopological group with a closed normal system, then (G, t) is a topological group.*

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