# Topology Proceedings


| | |
|---|---|
| **Web:** | http://topology.auburn.edu/tp/ |
| **Mail:** | Topology Proceedings<br>Department of Mathematics & Statistics<br>Auburn University, Alabama 36849, USA |
| **E-mail:** | topolog@auburn.edu |
| **ISSN:** | 0146-4124 |

Topology Proceedings
Vol. 19, 1994

# ON A CONSTRUCTION OF HOMOGENEOUS, NON-BIHOMOGENEOUS CONTINUA OF P. MINC

KAZUHIRO KAWAMURA*

ABSTRACT. We show with the help of Menger manifold theory that the construction of P. Minc of infinite dimensional homogeneous, non-bihomogeneous continua can be modified to produce such examples in any dimension except 1.

## 1. INTRODUCTION.

A continuum (= a compact connected metric space) $X$ is said to be *homogeneous* if, for each pair of points $x, y$ of $X$ there exists a homeomorphism $h : X \to X$ such that $h(x) = y$. If in addition, the above homeomorphism can be chosen so that $h(y) = x$, that is, $h$ exchanges $x$ and $y$, then $X$ is said to be *bihomogeneous*. A homogeneous, non-bihomogeneous continuum was first constructed by K. Kuperberg [K]. It is a 7-dimensional locally connected continuum. P. Minc [M] provided another way to construct an infinite dimensional, non-locally connected example, and observed that his method can be modified to provide a 4-dimensional non-locally connected example (a personal communication). We show that the idea

* The author is supported by NSERC International Fellowship.

of Minc can be modified further to produce 2-dimensional non-locally connected example by applying results of Menger manifolds. However, we do not know whether there is an one-dimensional homogeneous non-bihomogeneous continuum.

The following is the main result of this paper.

**Main Theorem.** *For each integer $n \geq 2$, there is an $n$-dimensional homogeneous, non-bihomogeneous continuum.*

## 2. Preliminaries.

As in [M], we use properties of solenoids for the construction. To simplify the notation, we restrict ourselves to dyadic solenoid $\Sigma_2$, but the construction works for any solenoid. Recall that $\Sigma_2$ is a topological group and the unit element is denoted by $e$. Let $S$ be the unit circle in the complex plane $\mathbb{C}$. Then we can represent $\Sigma_2$ as an inverse limit, i.e. $\Sigma_2 = \varprojlim(S \xleftarrow{f} S \xleftarrow{f} \dots)$, where $f(z) = z^2$ for $z \in \mathbb{C}$. The projection to the $i$-th factor is denoted by $p_i : \Sigma_2 \to \mathbb{C}$.

In order to clarify the idea of the construction, first we recall the Minc's construction. For points $a, b \in \Sigma_2$, the homeomorphism $h_{a,b} : \Sigma_2 \to \Sigma_2$ defined by $h_{a,b}(x) = ax^{-1}b$, exchanges $a$ and $b$. Theorem 1 below states that for "most" pairs of points of $\Sigma_2$, any homeomorphism exchanging these points is homotopic to a homeomorphism of the above type. In order to eliminate the "orientation reversing" homeomorphism as above, each arc component is replaced by a countable union of the mapping cylinders of the degree 2 covering map $S \to S$. More precisely, the continuum $\Delta_2$ obtained in this way is represented as the limit of an inverse sequence, i.e. $\Delta_2 = \varprojlim(K_1 \leftarrow K_2 \leftarrow K_3 \leftarrow \dots)$, where each $K_i$ is a finite union of the mapping cylinders of the degree 2 covering map $f : S \to S$ (recall that $S$ is a simple closed curve), and each bonding map $K_{i+1} \to K_i$ is (in our setting of dyadic solenoid) the "degree 2" covering map. Then a map $q : \Delta_2 \to \Sigma_2$ is naturally defined so that the inverse image of any arc component of

$\Sigma_2$ is a countable union of the mapping cylinders of the map $f$. Then $\Delta_2$ is not bihomogeneous by this replacement trick, but is not homogeneous either. By multiplying the Hilbert cube $I^\infty$, the homogeneity is recovered, keeping the non-bihomogeneity.

In order to state Theorem 1, we need some terminology. Let $r$ be a rational number. A map $g : \Sigma_2 \to \Sigma_2$ is called a *power r-map* if for each $i \geq 1$, there is a $j > i$ such that $p_i(g(x)) = (p_j(x))^{r \cdot 2^{j-i}}$ for each $x \in \Sigma_2$. Every map $g : \Sigma_2 \to \Sigma_2$ is homotopic to a map $h : \Sigma_2 \to \Sigma_2$ of the form $h(x) = g(e) \cdot \rho(x)$ ($\cdot$ denotes the group multiplication) ($x \in \Sigma_2$), where $\rho$ is a power $r$-map, and the $r$ is uniquely determined by $g$ ([M], Proposition 4), called the *order* of $g$.

Let $E$ be the arc component of $\Sigma_2$ containing $e$.

**Theorem 1** ([M], Theorem 1). *There is a countable collection $\mathcal{K}$ of arc components of $\Sigma_2$ such that each $g : \Sigma_2 \to \Sigma_2$ with $g(e) \notin \cup\mathcal{K}$ and $g(g(e)) \in E$ has order -1.*

**Proposition 2** ([M], Proposition 7). *Suppose that $h : \Delta_2 \to \Delta_2$ is a continuous map such that $h(h(e)) = e$ and $q \circ h(e)$ does not belong to any of the components in the collection $\mathcal{K}$ in Theorem 1. Then $h|q^{-1}(e) \simeq 0$.*

It is important to notice that the proof of the non-bihomogeneity of $\Delta_2$, based on Proposition 2 above, is obtained by the homotopy condition which is preserved under the multiplication of the Hilbert cube $I^\infty$. If, instead of multiplying $I^\infty$, we can find a low dimensional *homogeneous* continuum which keeps the above information, then it would be possible to construct a low dimensional example. This is the situation to which Menger manifold theory is applied.

Now we review briefly Menger manifold theory. Let $I^{2n+1}$ be the $(2n+1)$-cell with a standard triangulation $L$. Take the second barycentric subdivision $\beta^2 L$ of $L$ and let $M_1 = st(|L^{(n)}|, \beta^2 L)$ and $L_1 = \beta^2 L \mid M_1$. Take $\beta^2 L_1$ and let $M_2 = st(|L_1^{(n)}|, \beta^2 L_1)$, and so on. Then we have a decreasing sequence of compacta $\{M_i | i \in \mathbb{N}\}$ and $\mu^n = \cap_{i=1}^\infty M_i$ is called the *n-dimensional universal Menger compactum.* A *$\mu^n$-manifold* is a locally compact separable metric space each point of which has a neighbourhood homeomorphic to $\mu^n$. The fundamental theorem of M. Bestvina [B] states that a locally compact separable metric space is a $\mu^n$-manifold if and only if it is an $n$-dimensional $LC^{n-1}$ (=locally (n-1)-connected) space with the $DD^nP$, where the $DD^nP$ of a metric space $X$ means the following property;

> ($DD^nP$) For each pair of maps $\alpha, \beta : I^n \to X$ and for each $\epsilon > 0$, there exist maps $\alpha'$ and $\beta'$ : $I^n \to X$ which are $\epsilon$-close to $\alpha$ and $\beta$ respectively such that im $\alpha' \cap$ im $\beta' = \emptyset$.

In connection with our construction, the following result is important.

**Theorem 3** ([B], Theorem 3.2.2). *Any $\mu^n$-manifold $X$ is strongly locally homogeneous, that is, for each point $x$ of $X$ and each neighbourhood $U$ of $x$, there exists a neighbourhood $V \subset U$ of $x$ such that, for any $y \in V$, there exists a homeomorphism $f : X \to X$ such that $f(x) = y$ and $f|X - U =$id.*

A compactum $X$ is said to be $UV^k$ if, for any embedding $e : X \to M$ into an ANR $M$, the following condition holds.

> ($UV^k$) for each neighbourhood $U$ of $e(X)$, there exists a neighbourhood $V \subset U$ of $e(X)$ such that $\pi_i(V) \to \pi_i(U)$ is trivial for each $i = 0, 1, \ldots, k$.

A proper map $f : X \to Y$ is called a *$UV^k$-map* if each $f^{-1}(y)$ is a $UV^k$-compactum. A $UV^k$-map between $LC^k$ compacta induces an isomorphism between $j$-th homotopy groups for $j = 0, \ldots, k$. A compact set $Z$ of a $\mu^n$-manifold $M$ is called a *$Z$-set* if for each $\epsilon > 0$, there exists a map $f : M \to M - Z$ which is $\epsilon$-close to $\mathrm{id}_M$. We have the following "sum theorem" for $\mu^n$-manifolds.

**Theorem 4** ([$C_3$, Proposition 2.4]). *Let $M = M_1 \cup M_2$, where $M_1, M_2$ and $M_0 = M_1 \cap M_2$ are $\mu^n$-manifolds. If $M_0$ is a $Z$-set in $M_i (i = 1, 2)$, then $M$ itself is a $\mu^n$-manifold.*

The following is an easy consequence of [$C_1$, Theorem 1.3] and [$C_2$, Theorem 1.5].

**Theorem 5.** *For each compact polyhedron $K$, there exists a $UV^{n-1}$ map $\varphi_K : M_K \to K$ of a $\mu^n$-manifold $M_K$ such that*

*(1) for each compactum $Z$ with $\dim Z \leq n$, there is a $Z$-embedding $s : Z \to M_K$ such that $\varphi_K \circ s = id$, and*

*(2) for each $Z$-set $A$ in $K$, $\varphi_K^{-1}(A)$ is a $Z$-set in $M_K$.*

Finally the homogeneity of our example is proved by applying the following theorem.

**Theorem 6** ([R], Theorem 2). *Let $X = \varprojlim\{X_i, f_i : X_{i+1} \to X_i\}$ be the limit of an inverse sequence of continua such that each $f_i$ is a regular covering map. The projection onto the $i$-th factor is denoted by $f_{i\infty} : X \to X_i$. Suppose that $X_1$ is strongly locally homogeneous and $f_{1\infty}$ is a locally trivial bundle with Cantor set fibers, then $X$ is* homogeneous.

## 3. Construction.

Let $n \geq 2$ be a given natural number. Recall that $f : S \to S$ is the degree 2 covering map of the simple closed curve $S$, and take the mapping cylinder $C = S \times [0,1] \oplus S/(x,1) \sim f(x)$ and take a $\mu^n$-manifold $M'$ and a $UV^{n-1}$map $\varphi' : M' \to C$ satisfying the conditions of Theorem 5. Shrink each fibre $\varphi'^{-1}(x), x \in S \times 0$ to a point and let $M$ be the resulting continuum which is also a $\mu^n$-manifold containing a homeomorphic copy $M_0$ of $S \times 0$. Observe that $M_0$ is a $Z$-set, and also that $\varphi'$ induces a $UV^{n-1}$map $\varphi : M \to C$. By (1) of Theorem 5, we have that there is a $Z$-embedding $s : S \times 1 \to M$ such that $\varphi \circ s = \mathrm{id}$. The set $M_1 = s(M \times 1)$ is homeomorphic to $S$. Let $c : C \to S$ be the standard CE-retraction of the mapping cylinder defined by

$$c([x,t]) = f(x) \text{ for } (x,t) \in S \times [0,1] \text{ and}$$

$$c(y) = y \text{ for } y \in S.$$

Define a $UV^{n-1}$ retraction $r : M \to M_1$ by $r = s \circ c \circ \varphi$. Since $\varphi$ is $UV^1$ (recall that $n \geq 2$), it is easy to see that

(a) $r|M_0 \simeq (M_0 \hookrightarrow M)$.

For each $n \geq 1$, we define $K_n$ and $L_n$ as follows. Fix homeomorphisms $\alpha : M_1 \to M_0$ and $\beta : S \times 1 \to S \times 0$ and let

$L_n = M \times \{0, 1, \ldots, 2^n - 1\}/(x, i) \sim (\alpha(x), i+1) \bmod 2^n, x \in M_1$

$K_n = C \times \{0, 1, \ldots, 2^n - 1\}/(x, i) \sim (\beta(x), i+1) \bmod 2^n, x \in S \times 1$.

For each $n$, there are natural degree 2 covering maps $g_n : L_{n+1} \to L_n$ and $h_n : K_{n+1} \to K_n$, and a $UV^{n-1}$map $\varphi_n : L_n \to K_n$ induced by $\varphi$ such that

$$h_n \circ \varphi_{n+1} = \varphi_n \circ g_n.$$

Let $L_\infty = \varprojlim(L_1 \xleftarrow{g_1} L_2 \xleftarrow{g_2} L_3 \leftarrow \ldots)$

and $\Delta = \varprojlim(K_1 \xleftarrow{h_1} K_2 \xleftarrow{h_2} K_3 \leftarrow \ldots)$.

As in [M], the space $\Delta$ admits a surjection $q : \Delta \to \Sigma_2$ such that $q^{-1}(x)$ is a simple closed curve for each $x \in \Sigma_2$. Let $\varphi_\infty = \varprojlim \varphi_n$ and $p = q \circ \varphi_\infty$. Then the maps $\varphi_\infty$ and $p$ have the following properties;

(b) each $L_n$ is a connected $\mu^n$-manifold, by Theorem 4. Theorem 3 implies that $L_n$ is strongly locally homogeneous.

(c) The set $p^{-1}(e)$ is homeomorphic to $\varphi^{-1}(S \times 1)$ which contains a homeomorphic copy $S_e$ of $M_0$.

We apply Theorem 6 to the space $L_\infty$. Each $g_n$ is a regular covering map and it is easy to see that the local product condition of Theorem 6 is satisfied, thus from Theorem 6 with the help of the part (b), we have that;

(d) the space $L_\infty$ is homogeneous.

Recalling that the proof of Proposition 2 is, in essence, obtained by considering fundamental groups, the same proof with the help of the part (a) gives that;

(e) if $h : L_\infty \to L_\infty$ is a continuous map such that $h(h(e)) = e$ and $p(h(e)) \notin \cup\mathcal{K}$, then $h|S_e \simeq 0$.

This enables us to proceed in exactly the same way as in Proposition 2 to prove that $L_\infty$ is not bihomogeneous (see [M], Proposition 7 and Theorem 2). Clearly dim $L_\infty = n$ and this is the required example.

**Acknowledgement.** The author would like to express his sincere thanks to Professor P. Minc for the discussion of his example. Also the author would like to express his thanks to the referee whose suggestion was very helpful to make the present paper readable.

## References

[B] M. Bestvina, *Characterizing k-dimensional universal Menger compacta*, Mem. Amer. Math. Soc., 380 **71** (1988).

[C1] A. Chigogidze, *$UV^n$-equivalence and n-equivalence*, Top. Appl. **45** (1992), 283–291.

[C2] ——, *Classification theorem for Menger manifolds*, Proc. Amer. Math. Soc., **116** (1992), 825–832.

[C3] ——, *Finding a boundary for Menger manifolds*, ibidem, **121** (1994), 631-640.

[CKT] ——, K. Kawamura and E. D. Tymchatyn, *Menger manifolds*, Continua with Houston Problem book, ed. by H. Cook, W. T. Ingram, K.T. Kuperberg, A. Lelek and P. Minc, Marcel Dekker 1995, 37–88.

[K] K. Kuperberg, *On the bihomogeneity problem of Knaster*, Trans. Amer. Math. Soc., **321** (1990), 129-143.

[M] P. Minc, *Solenoids and bihomogeneity*, in Continua with Houston Problem book, ed. by H. Cook, W. T. Ingram, K. T. Kuperberg, A. Lelek, and P. Minc, Marcel Dekker 1995, 297–304.

[R] J. T. Rogers, *An aposyndetic homogeneous curve that is not locally connected*, Houston J. Math., **9** (1983), 433-440.

University of Saskatchewan
Saskatoon, Saskatchewan
S7N 0W0 Canada

*current address*:
University of Tsukuba
Tsukuba-shi, Ibaraki 305, JAPAN
*e-mail*: kawamura@math.tsukua.ac.jp