Topology Proceedings

Web:	http://topology.auburn.edu/tp/
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E-mail:	topolog@auburn.edu
ISSN:	0146-4124

Topology Proceedings
Vol. 19, 1994

THE ONE-DIMENSIONAL ČECH COHOMOLOGY OF THE HIGSON COMPACTIFICATION AND ITS CORONA

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ABSTRACT. Let X be a locally compact metric space with proper metric d. In this paper it is shown that under very general conditions the 1-dimensional Čech cohomology of the Higson compactification contains a subgroup isomorphic to the additive real numbers. This is shown to be true for the Higson corona as well. These results are used to give a general class of counterexamples to a conjecture due to N. Higson concerning the Higson compactification.

Introduction. Let X be a metric space with metric d. We say that the metric d is proper if every closed bounded set is compact. Of course, only locally compact metric spaces can have proper metrics. If X is a metric space with proper metric d, then there is a compactification called the Higson compactification, denoted $\overline{X}^d$, which depends on the metric d. The corona of this compactification is the set $\overline{X}^d \backslash X$ with the subspace topology. We denote the corona of X by $\nu_d X$. A detailed description of this compactification and its elementary properties is given in §1. In this paper we show that in very general circumstances the 1-dimensional Čech cohomology of the Higson compactification contains a subgroup isomorphic to the additive real numbers. This is true of the Higson corona as well. The theorems we prove provide the basis for a counterexample to a conjecture due to N. Higson [12, 6.35, p. 83] concerning the Higson compactification.

Conjecture (N. Higson). *Suppose that X is a noncompact metric space with proper metric d. Suppose also that X is uniformly contractible with respect to d. Then the Čech cohomology of the Higson compactification of X is trivial.*

The main result of the paper is Theorem 1. Theorem 1 implies that whenever the metric space X is noncompact and connected with proper metric d, $\check{H}^1(\overline{X}^d)$ contains a subgroup isomorphic to the additive reals. If we let $X = R^n$ with the usual Euclidean metric, then X will be uniformly contractible and $\check{H}^1(\overline{X}^d)$ will be highly nontrivial. This gives a specific counterexample to the Higson conjecture. However, whenever X is noncompact and d is a proper metric on X, the requirement that X be uniformly contractible implies that X is also connected and this forces $\check{H}^1(\overline{X}^d)$ to contain a copy of the additive reals. So, Theorem 1 not only provides a counterexample to the Higson Conjecture, it actually shows that the requirement that X be uniformly contractible actually *forces* the conjecture to be false. It does not seem likely that the conjecture can easily be modified to be true.

Theorem 1. *Suppose that X is a noncompact connected metric space and suppose that d is a proper metric on X. Then the following exact sequence holds.*

$$0 \to C_d^*(X) \to C_d(X) \to \check{H}^1(\overline{X}^d)$$

If, in addition, for every $r > 0$, there is a compact set $K_r \subset X$ that for all $x \in X \backslash K_r$, the ball $B_r(x)$ is connected, then the following exact sequence holds.

$$0 \to C_d^*(X) \to C_d(X) \to \check{H}^1(\overline{X}^d) \to \check{H}_d^1(X) \to 0$$

It will be shown later that the group $C_d(X)/C_d^*(X)$ is isomorphic to the additive group of real numbers. Thus, either exact sequence of Theorem 1 implies that for all noncompact connected X with proper metric d, the Higson compactification of X has $\check{H}^1(\overline{X}^d) \cong R \oplus G$ for some abelian group G, where R

is the additive reals. In particular, this implies that $\check{H}^1(\overline{X}^d)$ is a very large nontrivial group. We will give the definition of $C_d^*(X)$ and $C_d(X)$ in §1. In §3 $\check{H}_d^1(X)$ will be defined.

The techniques proving Theorem 1 also prove the following theorem.

Theorem 2. *Suppose that X is a noncompact metric space with proper metric d which has the property that for every compact subset K of X, there is a bounded open set U of X which contains K such that $X \backslash U$ is connected. Then there is a subgroup of $\check{H}^1(\nu_d X)$ which is isomorphic to the additive reals.*

The above theorems imply some interesting topological properties for the Higson compactification of X based on results from [9]. Under the hypotheses of the above theorems, neither $\overline{X}^d$ nor $\nu_d X$ can be locally connected because there is always a map of each of these spaces onto the rational solenoid Σ_ω. The existence of such a map onto a solenoid also implies that neither $\overline{X}^d$ nor $\nu_d X$ can be arcwise connected, nor can every pair of points be connected by some locally connected subcontinuum of the space.

The techniques of the paper make use of methods developed by the author and R. B. Sher in [11]. The methods were developed further in subsequent papers ([5], [6], [7], and [10]). The techniques in these papers will likely lead to the determination of further properties of the Higson compactification and its corona. The techniques of A. Calder and J. Siegel ([1] and [2]) are relevant to the study of the higher-dimensional Čech cohomology of $\overline{X}^d$ and $\nu_d X$.

The author wishes to acknowledge fruitful discussions with A. Dranishnikov who gave a stimulating account of the Higson compactification and its applications. The author has also had very useful conversations on the subject with his colleagues C. Stark and J. Martinez. I am especially indebted to Yutaka Iwamoto of the University of Tsukuba who pointed out a subtle

error in a former version of Theorem 1 and suggested a useful correction.

0. Preliminaries and Notation

We assume that the reader is familiar with the basic theory of compactifications. Gillman and Jerison [3], Isbell [4], and Walker [14] are good references on this subject. It would be helpful if the reader were familiar with Čech cohomology. We use the notation $\check{H}^n(X)$ to denote the n-dimensional Čech cohomology of X with integer coefficients using the (numerable) open covers of X. Let Z denote the group of integers. Let $K(Z,n)$ be any connected CW-complex having the property that

$$\pi_k(K(Z,n)) = \begin{cases} Z & k = n \\ 0 & k \neq n \end{cases}$$

Then it is well-known that $\check{H}^n(X) \cong [X, K(Z,n))]$ where $[X, K(Z,n))]$ is the collection of all homotopy classes of functions from X into the space $K(Z,n)$. Describing the group structure of this collection of homotopy classes would take us on a tangent, but we will be using a special case, that of $n = 1$. In this case it can be described easily. For $n = 1$ it is convenient to use the circle S^1 as our $K(Z,1)$. There is a topological group structure on S^1 and the group structure on $[X, S^1]$ can be derived from pointwise multiplication of the functions from X to S^1. We also assume that the reader is familiar with the basic properties of covering-spaces.

1. Basic Properties of the Higson Compactification

In this section we will give the basic definitions and outline the basic results concerning the Higson compactification and its corona. Our approach is slightly different than that taken in [12], but equivalent. We will emphasize certain details that will be used later.

The Higson compactification is a compactification which is defined for all locally compact metric spaces endowed with certain metrics. We say that a metric d on X is *proper* provided that every bounded set in X has compact closure. For X to have a proper metric, obviously X must be locally compact. So, we will assume that X is locally compact metric throughout. Suppose that X is noncompact with d a proper metric. Let $f : X \to Y$ be a continuous function into a metric space Y with specific metric. We say that the function f satisfies $(*)$ provided that

$$(*) \qquad \lim_{x \to \infty} diam(f(B_r(x))) = 0 \quad \forall r > 0.$$

Property $(*)$ means that for each $r > 0$ and each $\epsilon > 0$, there is a compact set $K = K_{r,\epsilon}$ in X such that for all $x \notin K$, $diam(f(B_r(x))) < \epsilon$. We now define $C_d^*(X)$ and $C_d(X)$. Recall the standard notation of [3] that $C(X)(C^*(X))$ denotes the set of all (bounded) real-valued continuous functions on X. These are rings under pointwise addition and multiplication with $C^*(X)$ a subring of $C(X)$. By analogy with these definitions we define $C_d(X)$ and $C_d^*(X)$ as follows.

$$C_d(X) = \{f \in C(X) \mid f \text{ satisfies } (*)\}$$

$$C_d^*(X) = \{f \in C^*(X) \mid f \text{ satisfies } (*)\}$$

With the supremum norm on $C^*(X), C_d^*(X)$ is a closed subring of $C^*(X)$ containing all the constant functions. Because the metric d on X is proper, $C_d^*(X)$ generates the topology of X. It is well-known that the compactifications of X are in one-to-one correspondence with the closed subrings F of $C^*(X)$ which contain the constants and generate the topology of X. For a given such subring F the compactification associated with F can be produced several equivalent ways. One way is to let the points of the compactification be the maximal ideals of the ring F with the hull-kernel topology with the point x of X being identified with the fixed maximal ideal $M_x = \{f \in F \mid f(x) = 0\}$.

Another way to construct the compactification associated with F is to embed the space X in the product space $\prod_{f\in F} I_f$, where I_f is the smallest closed interval containing the image of X under f. The embedding of X is $e : X \to \prod_{f\in F} I_f$ given by the formula $e(x)_f = f(x)$. The compactification associated with F is then the closure of $e(X)$ in $\prod_{f\in F} I_f$. Both of these methods are well-known and widely used as well as other methods. The compactification that is produced is characterized by the property that if f is a real-valued continuous function on X, then f extends continuously to the compactification associated with F if and only if $f \in F$.

We are now in a position to define the Higson compactification and its corona. The Higson compactification is the compactification associated with the closed subring $F = C_d^*(X) \subset C^*(X)$. We denote the Higson compactification by $\overline{X}^d$. It is characterized as the compactification $\overline{X}^d$ such that the real-valued continuous functions on X that extend to $\overline{X}_d$ are precisely the ones in $C_d^*(X)$.

We now want to give another characterization of $\overline{X}^d$. We first observe that if d is any proper metric on X and $f : X \to Y$ is any continuous map into a metric space Y endowed with a specific metric, then the property $(*)$ makes perfectly good sense. We also want to observe that if Y happens also to be compact, then property $(*)$ holds for the map f for one metric on Y if and only if it holds for any other metric on Y.

Proposition 1. *Suppose that X is noncompact and that d is a proper metric on X. The Higson compactification $\overline{X}^d$ is the unique compactification of X such that if Y is any compact metric space and $f : X \to Y$ is continuous, then f has a continuous extension to $\overline{X}^d$ if and only if f has property $(*)$.*

Proof: Suppose that Y is compact metric and that $\overline{X}^d$ is the Higson compactification of X. Suppose that Y is compact metric and let $f : X \to Y$ be a continuous mapping which satisfies $(*)$. As observed above, since all metrics on Y are equivalent we may choose any convenient metric on Y and $(*)$ will be satisfied. So, let us imagine that Y is embedded in the Hilbert cube $\prod_{i=1}^{\infty}[0,1]_i$ and assume that the metric d on the Hilbert cube is given by $d((x_i),(y_i)) = \max_{1 \le i < \infty} \frac{|x_i - y_i|}{2^i}$ and let Y inherit this metric as a subspace. Let $\pi_j : \prod_{i=1}^{\infty}[0,1]_i \to [0,1]_j$ be the projection onto the jth coördinate of the Hilbert cube. It is easy to see that $\pi_j \circ f : X \to [0,1]$ satisfies $(*)$. Thus, this map has an extension to X^d since it is a bounded map into the reals satisfying $(*)$. Call this map $\overline{\pi_j \circ f} : \overline{X}^d \to [0,1]_j$. The existence of this collection of maps implies that there is a map $\overline{f} : \overline{X}^d \to \prod_{i=1}^{\infty}[0,1]_i$ which clearly extends f. Now Y is compact and since $f(X) \subset Y$, we must also have that $\overline{f}(\overline{X}^d) \subset Y$. We have proved that the map $f : X \to Y$ to a compact metric space Y satisfying $(*)$ has an extension $\overline{f} : \overline{X}^d \to Y$. On the other hand suppose that Y is compact metric and that $f : X \to Y$ is a continuous mapping which does not satisfy $(*)$. Then embed Y in the Hilbert cube as above with the same metric. Now think of the mapping f as going into the Hilbert cube via the embedding of Y, $f : X \to Y \subset \prod_{i=1}^{\infty}[0,1]_i$. Then this map into the Hilbert cube cannot satisfy $(*)$. It therefore follows that for some $j, \pi_j \circ f : X \to [0,1]_j$ also cannot satisfy $(*)$. Thus, this function does *not* have an extension to $\overline{X}^d$. This implies that f cannot have an extension to $\overline{X}^d$ or all of the functions $\pi_i \circ f : X \to [0,1]_i$ would have extensions as well and we have just argued that one of them does not.

Now suppose that we have any compactification C of X having the property that if Y is any compact metric space and $f : X \to Y$ is continuous, then f has a continuous extension to

$\overline{X}^d$ if and only if f has property $(*)$. It follows that the set of real-valued continuous functions that extend to C is precisely the closed subring $C_d^*(X)$ of $C^*(X)$. Thus, the compactification C is $\overline{X}^d$ □.

The Higson corona is simply $\overline{X}^d$ less the embedded copy of X. We denote the Higson corona by $\nu_d X = \overline{X}^d \backslash X$. The letter d used in the designation of the Higson compactification and Higson corona emphasizes the dependence on the proper metric d. For different metrics one may get different Higson compactifications and coronas.

One motivation for the Higson compactification is to study limit properties of balls at infinity for a Riemannian manifold. However, the definition only requires that X be locally compact and not necessarily a manifold. It is good to bear the motivation in mind and to observe how the theorems in this paper apply when X is a manifold.

Throughout the rest of the paper whenever there is discussion of the Higson compactification of a space X it will be assumed that X is noncompact metric space having a proper metric d and therefore locally compact.

2. $\beta N \subset \overline{X}^d$.

Let βN denote the Stone-Čech compactification of the positive integers. The Higson compactification of X is almost never equivalent to the Stone-Čech compactification of X. Nevertheless, it always contains a copy of βN. This implies that the Higson compactification is at least as complicated as βN. Since βN is not metrizable, $\overline{X}^d$ cannot be metrizable either. Since βN has 2^c points (where $c = 2^{\aleph_0}$ denotes the cardinality of the reals), it follows that $\overline{X}^d$ will also have 2^c points. We now include a simple proof that $\beta N \subset \overline{X}^d$.

Suppose that A is a subset of a compact space Z. Then the closure of A in Z is a compactification of A. Let $F = \{f|_A | f \in C(Z)\}$. Now F is a closed subring of $C^*(A)$. Thus, the closure of A in Z is the compactification associated with this closed subring. So, if $F = C^*(A)$, then the closure of A in Z is equivalent to the Stone-Čech compactification of A. We now proceed to the proof.

Theorem 3. *If X is any noncompact locally compact metric space with proper metric d, then $\overline{X}^d$ contains a copy of βN.*

Proof: Since X is not compact and metric, there is a countable closed discrete subset. Let us denote this set by $N = \{x_i | i = 1, 2, ...\}$. We now construct a subset $P \subset N$ as a subsequence. Let $p_1 = x_1$. Since the metric d on X is proper, there must be an $x_i \notin B_2(p_1)$. Choose such an x_i and let $p_2 = x_i$. Similarly, there must be an $x_i \notin B_3(p_1) \cup B_3(p_2)$. Choose such an x_i and let $p_3 = x_i$. Continuing in this fashion we obtain a subset P of N such that for any $r > 0$, there is an M such that for all, $i > M$, $B_r(p_i) \cap P = \{p_i\}$. Now let P inherit the metric of X. Then note that $C_d^*(P) = C^*(P)$. Thus, the Higson compactification of P with the metric it inherits from X is the Stone-Čech compactification of P. Because of the remarks made just preceding the statement of Theorem 3, we only need to show that every bounded map $f : P \to R$ extends to a bounded function $\overline{f} : X \to R$ such that $\overline{f} \in C_d^*(X)$. Then the closure of P in $\overline{X}^d$ will be homeomorphic to the Stone-Čech compactification of P. Let $f : P \to R$ be any bounded function defined on P. For each i, let $r_i = \frac{i}{2}$. Then the collection of open balls $\{B_{r_i}(p_i)\}_{i=1}^{\infty}$ has the property that if x is any point in X, then the ball centered at x of radius $\frac{1}{2}$, $B_{\frac{1}{2}}(x)$, can intersect at most one of the balls in the collection. Suppose this is not the case. Then suppose that we have $B_{\frac{1}{2}}(x) \cap B_{r_i}(p_i) \neq \emptyset$ and $B_{\frac{1}{2}}(x) \cap B_{r_j}(p_j) \neq \emptyset$ with $i > j$. Let $z_i \in B_{\frac{1}{2}}(x) \cap B_{r_i}(p_i)$ and $z_j \in B_{\frac{1}{2}}(x) \cap B_{r_j}(p_j)$. Then

$$d(p_j, p_i) \leq d(p_j, z_j) + d(z_j, x) + d(x, z_i) + d(z_i, p_i) < \frac{j}{2} + \frac{1}{2} + \frac{1}{2} + \frac{i}{2} < i + 1.$$

However, this contradicts the choice of p_i, $d(p_i, p_j) > i + 1$.

We now define an extension $\overline{f} \in C_d^*(X)$ for f. The function is defined as follows.

$$\overline{f}(x) = \begin{cases} 0 & x \notin \bigcup_i^\infty B_{r_i}(p_i) \\ f(p_i) \cdot \frac{r_i - d(x,p_i)}{r_i} & x \in B_{r_i}(p_i) \end{cases}$$

This function will agree with f at the points of P. If we assume that M is a bound for the function $|f|$, then M is also a bound for the function $|\overline{f}|$. Clearly $\overline{f}$ is continuous. We only need to verify that $\overline{f}$ satisfies $(*)$. To that end let $r > 0$ be fixed and let $\epsilon > 0$. There is a compact set $K = K_{r,\epsilon}$ contained in X such that if $x \notin K$ and $B_r(x) \cap B_{r_i}(p_i) \neq \emptyset$, then $r_i > \frac{4rM}{\epsilon}$ where M is the bound for $|f|$. We will now show that $diam(\overline{f}(B_r(x))) < \epsilon$. We will show this by showing that $d(\overline{f}(x), \overline{f}(z)) < \frac{\epsilon}{2}$ for every $z \in B_r(x)$. So, let $z \in B_r(x)$. Suppose that z is in $B_{r_i}(p_i)$. If x is also in $B_{r_i}(p_i)$, then $\overline{f}(x) = f(p_i) \cdot \frac{r_i - d(x,p_i)}{r_i}$ and $\overline{f}(z) = f(p_i) \cdot \frac{r_i - d(zp_i)}{r_i}$. Thus, $|\overline{f}(x) - \overline{f}(z)| = |f(p_i)| \cdot \frac{|d(x,p_i) - d(z,p_i)|}{r_i} < \frac{rM}{r_i} < \frac{\epsilon}{4}$. If x is not in any other $B_{r_j}(p_j)$, then $\overline{f}(x) = 0$ and

$$|\overline{f}(x) - \overline{f}(z)| = |f(p_i)| \cdot \frac{|(r_i - r_i) - (r_i - d(z,p_i))|}{r_i}$$
$$= |f(p_i)| \cdot \frac{|r_i - d(z,p_i)|}{r_i} \leq |f(p_i)| \cdot \frac{|d(x,p_i) - d(z,p_i)|}{r_i} < \frac{rM}{r_i} < \frac{\epsilon}{4}$$

Now if it is the case that neither x nor z are in any $B_{r_i}(p_i)$, then clearly $|\overline{f}(x) - \overline{f}(z)| = 0$. The only other case to consider is that $x \in B_{r_i}(p_i)$ for some i and $z \in B_{r_j}(p_j)$ for some $j \neq i$. In that case it can be shown that $|\overline{f}(x)| < \frac{\epsilon}{4}$ and $|\overline{f}(z)| < \frac{\epsilon}{4}$. Thus, $|\overline{f}(x) - \overline{f}(z)| < \frac{\epsilon}{2}$ in this case as well. This completes the proof that $\lim_{n\to\infty} diam(\overline{f}(B_r(x))) = 0$. Thus f has a bounded continuous extension to X which satisfies $(*)$ and thus has an extension to $\overline{X}^d$. Thus, the closure of P in $\overline{X}^d$ is equivalent to βP and this is just the copy of βN claimed in the theorem. □

It is worth mentioning at this point, that the Higson compactification of the positive integers N endowed with the usual metric is *not* equivalent to the Stone-Čech compactification of N. For instance, let $f : N \to [0,1]$ be the function defined by the formula

$$f(n) = \begin{cases} 0 & n \text{ even} \\ 1 & n \text{ odd} \end{cases}$$

Then $f \in C^*(N)$, but $f \notin C^*_d(N)$. The reason that $f \notin C^*_d(N)$ is that $\lim_{n\to\infty} diam(f(B_2(n))) \neq 0$ and thus f does not satisfy $(*)$. However, there is a subset $P \subset N$ identified in the preceeding proof which has the property that $\beta P \subset \overline{N}^d$.

3. Proof of the Main Theorems.

In this section we prove Theorems 1 and 2. A few preliminary remarks are in order. In §0 we discussed the natural isomorphism $\check{H}^1(X) \cong [X, S^1]$. We now want to define the notation $\check{H}^1_d(X)$ in Theorem 1. In Theorem 1, X is a locally compact metric space with proper metric d. Identify $\check{H}^1(X)$ with $[X, S^1]$ and then let $\check{H}^1_d(X)$ be the subgroup of $[X, S^1]$ of homotopy classes $[f]$ of functions $f : X \to S^1$ satisfying $(*)$. Note that we are only requiring *one* representative in the homotopy class to satisfy $(*)$. Also, we are not requiring that the homotopies between representatives satisfy $(*)$. We now prove Theorem 1.

Proof of Theorem 1. We first express the second exact sequence of Theorem 1 in a more convenient form. In the theorem the second exact sequence is written

$$0 \to C^*_d(X) \to C_d(X) \to \check{H}^1(\overline{X}^d) \to \check{H}^1_d(X) \to 0.$$

We let $\check{H}^1_d(X) := [X, S^1]_d := \Big\{ [f] \mid f : X \to S^1 \text{ satisfies } (*) \Big\}$. From the above remarks above the sequence has the following equivalent form.

$$0 \to C^*_d(X) \to C_d(X) \xrightarrow{\ a\ } [\overline{X}^d, S^1] \xrightarrow{\ b\ } [X, S^1]_d \to 0$$

We have defined the individual groups in the sequence, now we need to define the homomorphisms. The homomorphism

$C_d^*(X) \to C_d(X)$ is just inclusion. We now define $a : C_d(X) \to [\overline{X}^d, S^1]$. Suppose that $f : X \to R$ an element of $C_d(X)$. Then f satisfies $(*)$ by the definition of $C_d(X)$. Let $e : R \to S^1$ be the covering map given by $e(x) = e^{2\pi ix}$. Then $e \circ f$ also satisfies $(*)$. Thus, by Proposition 1, $e \circ f$ has an extension to $\overline{X}^d$, call the extension $\overline{e \circ f} : \overline{X}^d \to S^1$. We then define $a(f)$ to be the homotopy class of this extension, $a(f) = [\overline{e \circ f}] \in [\overline{X}^d, S^1]$. We now show that the kernel of a equals $C_d^*(X)$ in $C_d(X)$. First we show that $C_d^*(X) \subset \ker(a)$. Now suppose that g is in $C_d^*(X)$. Then g is a bounded real-valued continuous function satisfying $(*)$. Thus, g has an extension $\overline{g}$ to $\overline{X}^d$ mapping to R. Now $\overline{g}$ is a lift for the map $\overline{e \circ g} : \overline{X}^d \to S^1$. Thus the map $\overline{e \circ g}$ is null-homotopic. This proves that $a(C_d^*(X)) = 0$ and that $C_d^*(X) \subset \ker(a)$. We now show that $\ker(a) \subset C_d^*(X)$. Suppose that g is in $C_d(X)$ and that $a(g) = [\overline{e \circ g}] = 0$. Then $\overline{e \circ g}$ is null-homotopic and thus has a lift. Call the lift $q : \overline{X}^d \to R$. The image of q must be bounded since $\overline{X}^d$ is compact and thus $q|_X$ must also be bounded in R. However, $q|_X$ and g differ by a covering transformation since $q|_X$ and g are both lifts for the map $\overline{e \circ g}|_X$. This implies that g must also be bounded and thus that $g \in C_d^*(X)$. Thus, $\ker(a) \subset C_d^*(X)$ and thus $\ker(a) = C_d^*(X)$. The preceeding shows that the cosets in $C_d(X)/C_d^*(X)$ are in one-to-one correspondence with a subgroup of $[\overline{X}^d, S^1]$. We have now shown that the first sequence in Theorem 1 is exact under the assumption that X is a noncompact connected metric space having a proper metric d.

Now assume that X has the additional property that for every $r > 0$ there is a compact set $K_r \subset X$ such that for every $x \in X \backslash K_r$, $B_r(x)$, is connected. Now we need to show that the image of a is the kernel of b under this assumption. We first define b. Suppose that $[f] \in [\overline{X}^d, S^1]$. Then let $b([f]) = [f|_X]$. Clearly this definition will not depend on the representative f chosen in the homotopy class. Now we show that $\mathrm{Im}(a) \subset$

ker(b). Suppose that $f \in C_d(X)$. Then $a(f) = [\overline{e \circ f}]$ and $b([\overline{e \circ f}]) = [\overline{e \circ f}|_X] = [e \circ f]$. Now $e \circ f : X \to S^1$ has a lift, namely $f : X \to R$, and is thus null-homotopic. Thus, $a \circ b(f) = 0 \in [X, S^1]_d$ and we have proved that Im(a) $\subset$ ker(b). Now we will prove that Im(a) $\supset$ ker(b). Suppose that $[f] \in [\overline{X}^d, S^1]$ and that $b([f]) = [f|_X] = 0$. Then $f|_X : X \to S^1$ is null-homotopic. This implies that $f|_X$ has a lift to R, call the lift $g : X \to R$. Since $f|_X$ has an extension to $\overline{X}^d$, it must satisfy condition $(*)$. We now show that the lift g must also satisfy condition $(*)$. Let $r > 0$ and let K_r be a compact subset of X such that for all $x \in X \backslash K_r$, $B_r(x)$ is connected. Let $x_i \to \infty$ in X. We may assume without loss of generality that $x_i \notin K_r$ for all i. Since f satisfies $(*)$, we have that $diam(f(B_r(x_i))) \to 0$ as $i \to \infty$. Since $B_r(x_i)$ is connected, the lift of $B_r(x_i)$ is unique up to a covering transformation. Thus, it is also true that $\lim_{i\to\infty} diam(g(B_r(x_i))) = 0$. This implies that g satisfies $(*)$ and thus, $g \in C_d(X)$. Now, $a(g) = [\overline{e \circ g}] = [f]$ since g was a lift of f. This implies that Im(a) $\supset$ ker(b) and completes the proof that Im(a) =ker(b) . All that is left in the proof of Theorem 1 is to show that b is onto.

Suppose that $[f] \in [X, S^1]_d$ where f is the representative in the homotopy class which satisfies $(*)$. Then f has an extension to $\overline{X}^d$, call the extension $\overline{f} : \overline{X}^d \to S^1$. Clearly $[\overline{f}] \in [\overline{X}^d, S^1]$ and $b([\overline{f}]) = [\overline{f}|_X] = [f]$ and b is onto. This completes the proof of Theorem 1.

Corollary 1. *Suppose that X is a noncompact connected metric space and suppose that d is a proper metric on X. Then there is a subgroup of $\check{H}^1(\overline{X}^d)$ which is isomorphic to the additive real numbers.*

Proof: We only need to show that $C_d(X)/C_d^*(X)$ is isomorphic to the additive reals since this is a subgroup of $\check{H}^1(\overline{X}^d)$ by Theorem 1. Clearly the group $C_d(X)/C_d^*(X)$ is abelian, divisible and torsion free. So, the only thing that needs to be shown is that the group has rank $2^{\aleph_0}$. Let x_0 be any point in X. For each $r > 0$, define a function $f_r(x) = r \cdot \sqrt{d(x, x_0)}$. Then each of the functions $f_r : X \to R$ will be elements of $C_d(X)$ and for $r \neq s, f_r - f_s \notin C_d^*(X)$. This shows that the group $C_d(X)/C_d^*(X)$ has rank at least $2^{\aleph_0}$. On the other hand, X is separable metric. Thus, $C_d(X)$ itself cannot have rank greater than $2^{\aleph_0}$. Thus, $C_d(X)/C_d^*(X)$ has rank equal $2^{\aleph_0}$. □

Proof of Theorem 2. By assumption X is a noncompact metric space with d a proper metric on X. These assumptions imply that X must be separable metric. By the assumption in the theorem, for every compact subset K of X, there is a bounded open set U of X which contains K such that $X \backslash U$ is connected. Since X is separable metric, this assumption implies that we can find a sequence $\{U_i\}_{i=1}^{\infty}$ of bounded open subsets of X such that (1) each U_i has compact closure in X, (2) $\overline{U}_i \subset U_{i+1}$ for all i, (3) $\bigcup_{i=1}^{\infty} U_i = X$, and (4) $X \backslash U_i$ is connected for all i. Now for each i, let $X_i = X \backslash U_i$. For each i, d_i be the restriction of the metric d to X_i. Let $X_i \xleftarrow{e_i} X_{i+1}$ be the inclusion map. We now need the following claim.

Claim: For each i, e_i has an extension $\overline{e}_i : \overline{X}_{i+1}^d \to \overline{X}_i^d$.

Proof of Claim: The mapping $e_i : X_{i+1} \to X_i$ induces a homomorphism $e_i^* : C^*(X_i) \to C^*(X_{i+1})$ defined by $e_i^*(f) = f \circ e_i$. As is well-known, the map e_i has the extension $\overline{e}_i : \overline{X}_{i+1}^d \to \overline{X}_i^d$ if and only if $e_i^*(C_d^*(X_i)) \subset C_d^*(X_{i+1})$. However, if $f \in C_d^*(X_i)$ satisfies $(*)$, then $e_i^*(f) = f \circ e_i$ also clearly satisfies $(*)$ as well and thus $e_i^*(C_d^*(X_i)) \subset C_d^*(X_{i+1})$ as required. This proves the claim.

Now resuming the proof of Theorem 2, we have the inverse sequence of spaces

$$X_1 \xleftarrow{e_1} X_2 \xleftarrow{e_2} X_3 \ldots$$

which has given rise to the following inverse sequence of compact spaces.

$$\overline{X}_1^d \xleftarrow{\bar{e}_1} \overline{X}_2^d \xleftarrow{\bar{e}_2} \overline{X}_3^d \ldots$$

The mappings e_i are all isometric embeddings and consequently each of the $\bar{e}_i$ are also embeddings. Thus, the inverse limit of the last inverse sequence is just the Higson corona of X, $\nu_d X = \varprojlim\{\overline{X}_i^d, \bar{e}_i\}$. Now consider the functions $\{f_r : X \to R\}_{r>0}$ defined in the proof of the previous corollary such that $f_r - f_s$ is unbounded whenever $r \neq s$. For each i, let $f_{r,i} : X_i \to R$ be the restriction of each of the above functions to the space X_i. Then it is also true for this collection of functions that $f_{r,i} - f_{s,i}$ is also unbounded whenever $r \neq s$. Now for each i the $\overline{e \circ f_{r,i}} : \overline{X}_i^d \to S^1$ are essential maps no two of which are homotopic. We also observe that for each $\overline{e \circ f_{r,i}} : \overline{X}_i^d \to S^1$ in this family and for each positive integer n, $\overline{e \circ f_{\frac{r}{n}i}} : \overline{X}_i^d \to S^1$, is the the nth root of $\overline{e \circ f_{r,i}}$. Thus, the group generated by the homotopy classes of these functions will be divisible. Since the 1-dimensional Čech cohomology is always torsion free, the group generated by the homotopy classes of these functions in $\check{H}^1(\overline{X}_i^d)$ will be isomorphic to the additive reals. Since $\nu_d X = \varprojlim\{\overline{X}_i^d, \bar{e}_i\}$, for each r we get a map $\overline{e \circ f_r}|_{\nu_d X} : \nu_d X \to S^1$. Since S^1 is an ANR, these maps are all essential and no two of them are homotopic. It is also the case that for each r and each positive integer n, the map $\overline{e \circ f_{\frac{r}{n}}}|_{\nu_d X} : \nu_d X \to S^1$ will be an nth root of $\overline{e \circ f_r}|_{\nu_d X} : \nu_d X \to S^1$. Thus, the subgroup of $\check{H}^1(\nu_d X)$ generated by the homotopy classes of these functions will be isomorphic to the additive reals. This completes the proof of Theorem 2.

4. Some Topological Properties of $\overline{X}^d$ and $\nu_d X$.

In this section of the paper we show that Theorem 1 and Theorem 2 can be used to show some unusual properties of both $\overline{X}^d$ and $\nu_d X$.

Theorem 4. *Suppose that X is noncompact and connected with proper metric d. Then $\overline{X}^d$ is neither locally connected nor arcwise connected.*

Proof: Since $\check{H}^1(\overline{X}^d)$ contains a copy of the additive reals it must certainly contain a copy of the additive group of rationals. So, let Q be a copy of the rationals in $\check{H}^1(\overline{X}^d)$. Let Σ_ω denote the rational solenoid which is the Pontryagin dual of the group Q with the discrete topology. Now the inclusion homomorphism $Q \subset \check{H}^1(\overline{X}^d)$ induces a mapping $f : \overline{X}^d \to \Sigma_\omega$ such that f^* is equivalent to the above inclusion homomorphism $f^* : \check{H}^1(\Sigma_\omega) = Q \subset \check{H}^1(\overline{X}^d)$ [9, Corollary 1.6]. The map f has to be onto or f^* would be the trivial homomorphism on $\check{H}^1(\Sigma_\omega)$. This prevents $\overline{X}^d$ from being locally connected and arcwise connected. □

Theorem 5. *Suppose that X satisfies the conditions of Theorem 2. Then $\nu_d X$ is neither locally connected nor arcwise connected.*

Proof: The same proof given for Theorem 4 prevents $\nu_d X$ from being locally connected or arcwise connected. There is a more general condition given in [9] in terms of $\check{H}^1(Z)$ which prevents the local connectivity of a space Z. □

Let X be noncompact, locally compact, connected metric space with d a proper metric on X. Let us define a metric compactification C of X to be an *approximation of the Higson compactification* of X if the embedding of X in C induces a map of $\overline{X}^d$ onto C. There is interest in the metrizable approximations of the Higson compactification since these compactifications are obviously more tractable than the Higson compactification. Now if we let Q be a copy of the rationals with $Q \subset \check{H}^1(\overline{X}^d)$, then there will be a metrizable compactification C of X such that the embedding of X in C induces a map $g : \overline{X}^d \to C$ such that $\check{H}^1(C)$ contains a copy of the rationals such that the homomorphism induced by g will be nontrivial on that copy of the rationals. This implies that any compactification larger than C which is also a quotient of the Higson compactification of X will have a copy of the rationals in its 1-dimensional Čech cohomology. So, among the metrizable approximations of the Higson compactification there is a cofinal collection of metrizable compactifications of X all of which have a copy of the rationals in their 1-dimensional Čech cohomology. None of these compactifications will be locally connected or arcwise connected nor will their remainders.

In a similar fashion, suppose that X is a noncompact locally compact metric space which has the property that for every compact subset K of X, there is an open set U of X which contains K such that $X\backslash U$ is connected. Suppose that d is a proper metric on X. Then there is a compactification C of X having the property that it is an approximation of the Higson compactification of X such that $\check{H}^1(C\backslash X)$ contains a copy of the group of rationals and such that if D is any other approximation of the Higson compactification greater than C, then $\check{H}^1(D\backslash X)$ also contains a copy of the group of rationals. Thus, for any such approximation D, $D\backslash X$ cannot be locally connected or arcwise connected.

5. Some Examples.

In this section of the paper we give a few examples that help illustrate the results obtained in the previous sections. The first example is the counterexample to the Higson Conjecture.

Example 1. Let $X = R^n$ with the usual Euclidean metric. Then the metric is proper and X is uniformly contractible. Using Theorem 1 and Theorem 2 one can show that $\check{H}^1(\overline{X}^d)$ and $\check{H}^1(\nu_d X)$ each have a subgroup isomorphic to the additive reals. We can say more. For $\overline{X}^d$ the group $\check{H}^1(\overline{X}^d) \cong C_d(X)/C_d^*(X)$ for $n > 1$. For the corona, $C_d(X)/C_d^*(X) \oplus Z \cong \check{H}^1(\nu_d R^2)$ and $C_d(X)/C_d^*(X) \cong \check{H}^1(\nu_d R^n)$ for all $n > 2$. At any rate, there is a mapping $f : \overline{X}^d \to \Sigma_\omega$ onto the rational solenoid, Σ_ω, such that the restriction of f to $\nu_d X$ is also onto Σ_ω. Since $\nu_d X$ is compact, this map gives us a metrizable compactification $C = X \cup \Sigma_\omega = R^n \cup \Sigma_\omega$ which approximates the Higson compactification having the property that the $\check{H}^1(C) \cong Q \cong \check{H}^1(C\backslash X)$. Furthermore, for any metrizable approximation of the Higson compactification D of X greater than C, the 1-dimensional Čech cohomology of D and of $D\backslash X$ will contain a subgroup isomorphic to the group of rationals.

Example 2. In this example we show that the group $\check{H}^1_d(X)$ may not be equal to the group $\check{H}^1(X)$. Let $X = S^1 \times [0,\infty)$ with the usual product metric. Clearly, $\check{H}^1(X) \cong Z$. On the other hand we will endeavor to show that $\check{H}^1_d(X) \cong 0$. Suppose that $f : S^1 \times [0,\infty) \to S^1$ satisfies $(*)$. Then as $x \to \infty$ in $[0,\infty)$, $diam(f(S^1 \times \{x\})) \to 0$. This implies that for some $x \in [0,\infty), f|_{S^1\times\{x\}}$ is null-homotopic. However, for each $x \in [0,\infty), S^1 \times \{x\}$ is a deformation retract of the space $S^1 \times [0,\infty)$ since $[0,\infty)$ is contractible. Thus f must also be null-homotopic. This proves that $\check{H}^1_d(X) \cong 0$. In this case the Higson corona has the property that $C_d(X)/C^*_d(X) \cong \check{H}^1(\nu_d X)$. On the other hand, $S^1 \times [0,\infty)$ is homeomorphic to $R^2\backslash B$ where B is the open ball of radius one centered at the origin in R^2. However, the usual metric on $R^2\backslash B$ gives a quite different Higson corona with $C_d(X)/C^*_d(X) \oplus Z \cong \check{H}^1(\nu_d R^2) \cong \check{H}^1(\nu_d(R^2\backslash B))$.

Example 3. In the proof of Theorem 3, we showed that for any noncompact X with a proper metric d, there is a copy of the integers N embedded in X such that the closure of N in $\overline{X}^d$ is equivalent to $\beta N \subset \overline{X}^d$. This cannot be true for embeddings of the reals or other subspaces which are not discrete at infinity. Suppose that X is any noncompact metric space with proper metric d. Suppose also that Y is a closed subset of X which is noncompact and which has the property that for every compact set K in Y, there is a point y in $Y \backslash K$ which is not an isolated point in Y. Then the closure of Y in $\overline{X}^d$ cannot be equivalent to βY. The reason is that no matter what metric Y may inherit from the proper metric on X, there must be two disjoint sequences of points in Y, $x_i \to \infty$ and $y_i \to \infty$ with $d(x_i, y_i) \to 0$. Since the sequences are disjoint and closed in Y, there is a continuous function $f : Y \to [0,1]$ such that $f|_{[x_i]} \equiv 0$ and $f|_{[y_i]} \equiv 1.$. There cannot be an extension of this function to $C_d^*(X)$, since f on Y does not satisfy condition $(*)$. Since f does not extend to $\overline{X}^d$, the closure of Y in $\overline{X}^d$ cannot be equivalent to βY.

Example 4. Let $X = Z^n$ with the usual Euclidean metric. Then one can show that the Higson corona of X is the same as that of R^n with the usual Euclidean metric. In fact the Higson compactification of Z^n is equivalent to the closure of Z^n in $\overline{R^n}^d$. So, the results of this paper can be extended to spaces which are not connected as well. We do not do this in this paper since it would detract from the topological simplicity of the arguments presented here.

6. Final Remarks.

For $n > 1$ the n-dimensional Čech cohomology of the Higson compactification and its corona is likely to be closely related to the n-dimensional Čech cohomology of X using techniques similar to those developed by Calder and Siegel in [1] and [2]. However, using the techniques of [5] we have been able to show that there are maps g of the Higson compactification of R^n which are onto the n-dimensional torus T^n such that any map homotopic to g is onto. This is also true of g restricted to $\nu_d R^n$ as well. Thus there are higher dimensional properties of $\overline{X}^d$ and $\nu_d X$ that can be elucidated by the techniques of this paper which may not be captured by the higher dimensional Čech cohomology.

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