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NIELSEN NUMBERS AND HOMEOMORPHISMS OF GEOMETRIC 3-MANIFOLDS

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1. INTRODUCTION

Given a compact polyhedron X and a self-mapping $f : X \to X$ the Nielsen number $N(f)$ provides a lower bound for the number of fixed points of f. This number is a homotopy invariant and in a variety of settings can be realized geometrically. That is, there exists a mapping homotopic to f which has exactly $N(f)$ fixed points. For example, this occurs when X is a topological manifold and f is a homeomorphism [JG].

As the Nielsen number is a topological invariant, it would be nice to have computational tools at hand which would allow one to compute its value. But in general this seems to be a difficult task, even in very restricted settings such as for self-homeomorphisms of manifolds. In the paper [M] a number of instances where the Nielsen number can be effectively computed by an algorithmic procedure are presented. One such is the case when the fundamental group $\pi_1(X)$ is finite. In [K] an algorithm for the computation of $N(f)$ is given when X is a compact surface and f is a homeomorphism. An important step in the implementation of this algorithm is an algorithm due to Bestvina and Handel [BH].

The purpose of this paper is to describe an algorithmic procedure for computing the value of $N(f)$ when the space X is taken from a certain class of 3-manifolds and f is a homeomorphism. The first algorithm applies to Seifert fibre spaces. Here it is shown how the computation reduces to computing the Nielsen number for the induced map on the underlying orbifold. Since this orbifold is a surface, the algorithm in [K] can be applied. The second algorithm is mainly an application of a result of Jiang and Wang [JW] which gives a characterization of fixed point classes for self-homeomorphisms of manifolds taken from a class of aspherical geometric 3-manifolds.

Before moving on to the results of the paper, we briefly describe the few basic notions of Nielsen theory which will be used. A detailed treatment of the subject can be found in either [B] or [Ji]. Let $f : X \to X$ be a continuous self-mapping of a compact polyhedron. An equivalence relation is defined on the set of fixed points of f, denoted $\mathrm{Fix}(f)$, by $x \sim y$ if and only if there exists a path α in X going from x to y such that $f(\alpha)$ is homotopic to α rel endpoints. An equivalence class is called a *fixed point class* or sometimes referred to as a *Nielsen class.* Each component of $\mathrm{Fix}(f)$ has an integer valued index, and the index of a fixed point class is the sum of the indices of the components which belong to that class. A fixed point class is *essential* if it has nonzero index. Then $N(f)$ is defined to be the number of essential fixed point classes. It is well known that homotopic maps have the same Nielsen number, so when talking about a map we often mean the homotopy class of maps containing the given one.

2. NIELSEN NUMBERS AND SEIFERT FIBRE SPACES

Let M be a compact 3-dimensional Seifert fibre space. That is a space which is foliated by simple closed curves, called *fibres*, such that a fibre f has a neighborhood which is either a solid Klein bottle or a fibred solid torus T_r, where r denotes the number of times a fibre near f wraps around f. A fibre is *regular* if it has neighborhoods fibre equivalent to $S^1 \times D^2$, otherwise it is called a *critical* fibre. The reader is referred to [S] for details and other properties of Seifert fibre spaces. There is a natural quotient map $q : M \to F$ where F is a 2-dimensional orbifold, and hence, topologically a compact surface. The projection of the critical fibres, which we denote by S, consists of a finite set of points in the interior of F together with a finite subcollection of the boundary components. Any fibre-preserving homeomorphism $h : M \to M$ naturally induces a relative surface homeomorphism of the pair (F, S), which we will denote by $\hat{h}$.

The following theorem indicates how the computation of Nielsen numbers reduces to the surface data of certain Seifert fibre spaces. Observe that if M and F are both orientable, then M admits a coherent orientation of all of its fibres and the homeomorphism h either preserves fibre orientation of all the fibres or it reverses fibre orientation.

Theorem 2.1. *Let M be a compact, orientable, aspherical, 3-dimensional Seifert fibre space such that the quotient orbifold F is orientable and all fibres have neighborhoods of type T_1. Let $h : M \to M$ be a fibre-preserving homeomorphism inducing $\hat{h} : F \to F$. If h reverses fibre orientation, then $N(h) = 2N(\hat{h})$. If h preserves fibre orientation, then $N(h) = 0$.*

Proof: First, by a small isotopy, arrange that h has a finite number of fibres which are mapped to themselves. If h preserves fibre orientation a further isotopy, which leaves $\hat{h}$ unchanged, ensures that none of these fibres contains a fixed point. Thus, $Fix(h) = \emptyset$ and so, $N(h) = 0$. If h reverses fibre orientation there are exactly 2 fixed points on each invariant fibre. Since M is aspherical Lemma 3.2 in [S] ensures that for any invariant fibre the 2 fixed points on that fibre are in distinct fixed point classes. On the other hand, the restriction on the fibre types in the hypothesis implies that a Nielsen path in F can always be lifted to a Nielsen path in M. In fact, there will be two distinct lifts of each path in F. As a result, h has two fixed point classes covering each fixed point class of $\hat{h}$. Since the index of a fixed point class of h is the same as that of its projection under q the result follows. □

As one can see, Theorem 2.1 only deals with a restricted class of Seifert fibre spaces. For the remaining ones some terminology needs to be introduced. Let X be a compact connected surface and let P be a finite set of points contained in the interior of X together with a finite set of simple closed curves in ∂X. Let $N(P)$ be an open regular neighborhood of P in X. A homeomorphism $f : X \to X$ is said to be in *fixed point standard form relative to* P if (1) $f(P) = P$; (2) $f_0 \equiv h \mid_{(X \setminus N(P))}$ has exactly $N(f_0)$ components in its fixed point set, and each such component is either a single point or a surface with negative Euler characteristic; (3) the number of fixed surfaces is maximal with respect to isotopy of the pair (X, P); (4) each component of $X \setminus (Fix(f_0) \cup P \cup \partial X)$ homotopy equivalent to S^1 is contained in $Fix(f)$.

Immediate from the results in [JG] we have

Lemma 2.2. *Any self-homeomorphism of* (X, P) *is isotopic to one in fixed point standard form relative to* P.

The following lemma indicates that the presence of critical fibers which contain essential fixed points influences the nature of the induced homeomorphism $\hat{h}$.

Lemma 2.3. *Fix a fibre orientation on* T_r *and let* $h : T_r \to T_r$ *be a fibre-preserving homeomorphism. If* $r > 1$ *and* h *reverses fibre orientation, then* $\hat{h}$ *is orientation-reversing.*

Proof: Suppose that $\hat{h}$ is orientation-preserving. Without loss we can assume that $\hat{h}$ is the identity. Let f denote the critical fibre and identify $T_r \setminus f$ with $\cup_t(S^1 \times S^1 \times \{t\}), t \in (0,1]$. For each t, $Fix(h) \cap (S^1 \times S^1 \times \{t\})$ consists of a pair of parallel simple closed curves which, when $r > 1$, represent non-trivial elements of $H_1(T_r)$. Since $Fix(h)$ is closed, this would imply that $f \subset Fix(h)$. A contradiction as on fibres near f, h is not close to the identity. □

Now suppose that β is a Z_k-action on X with the fixed point set of the action being either a finite set of points or a finite number, possiby none, of simple closed curves. The latter can only occur when $k = 2$. Let X_β denote the orbit space of the action and let $p : X \to X_\beta$ be the projection mapping. Suppose that $g : X_\beta \to X_\beta$ is a homeomorphism which is in fixed point standard form relative to some set P, and which lifts to a homeomorphism $\tilde{g}$ of X. A component C of Fix(g) is said to be *β-connected* if $p^{-1}(C)$ is a connected subset of X.

Here is a simple example which illustrates these definitions. Let X be a non-orientable surface and suppose $g : X \to X$ is in fixed point standard form (relative to the emptyset). Then g has two lifts $\tilde{g}_1$ and $\tilde{g}_2$ to the natural double cover of X by an orientable surface. The β-connected components of $Fix(g)$ are exactly those 2-dimensional components which are non-orientable. Note that if C is β-connected, then $Fix(\tilde{g}_1) \cup Fix(\tilde{g}_2)$ contains exactly one component which covers C. All other components of $Fix(g)$ are covered by two components. As components correspond to fixed point classes [K;Lemma 4.2], the notion of β-connectedness indicates the number of fixed point classes which "cover" a given fixed point class of g.

Let $p : \tilde{M} \to M$ be a regular finite covering where both $\tilde{M}$ and M are Seifert fibre spaces. We assume that p induces a covering, not necessarily regular, $p : \tilde{F} \to F$ of the associated orbifolds with the property that there exists a covering translation $\beta : \tilde{F} \to \tilde{F}$ which acts by a Z_k-action with fixed point set as described above. It follows that $p(Fix(\beta)) \subset S$.

Let $h : M \to M$ be a fibre-preserving homeomorphism which lifts to $\tilde{h} : \tilde{M} \to \tilde{M}$. We say that h is in *standard form* provided that $\hat{h}$ is in fixed point standard form relative to S, $Fix(h)$ consists of two points on each invariant fibre on which h reverses orientation and the preimage of a surface fixed by $\hat{h}$ contains two surfaces fixed by h.

To each homeomorphism $h : M \to M$ in standard form we consider three types of fixed point classes denoted by Φ_h, Ω_h and Δ_h. To define the first, suppose $x \in S \cap Fix(\hat{h})$ is such that there exists a simple closed curve l based at x with $\hat{h}(l)$ homotopic to l (rel x), and further, that the fibre over x has a neighborhood of type T_r where $r > 1$. By Lemma 2.3, it follows that $\hat{h}$ is orientation-reversing on a neighborhood of l. As a result we have, without loss, that $\hat{h}(l) = l$ and that the fixed point class of $\hat{h}$ containing x is simply $Fix(\hat{h}) \cap l$. Let Φ_h be the set of essential fixed point classes of h that project into such a fixed point class of $\hat{h}$.

To define the others, given a fixed point class σ let $F_\sigma = \{x \in \sigma \mid q(x) \in p(Fix(\beta))\}$. Define the set Ω_h of essential fixed point classes of h by $\sigma \in \Omega_h$ provided that $\sigma \notin \Phi_h$ and satisfies either of the following: (i) $q(\sigma)$ does not contain a β-connected surface and F_σ is a non-empty finite set with a total fixed point index of zero, or (ii) $q(\sigma)$ contains a β-connected surface and F_σ is a finite set, possibly empty, with a total fixed point index of zero. Define the set Δ_h of essential fixed point classes of h by $\sigma \in \Delta_h$ if and only if $\sigma \notin \Phi_h$ and satifies the equation

$$k(index(\sigma)) = (k-1)index(F_\sigma).$$

For a homeomorphism $h : M \to M$ of a Seifert fibre space let $N_0(h) = N(h) - \phi_h$ where ϕ_h is the cardinality of Φ_h.

Theorem 2.4. *Let $p : \tilde{M} \to M$ be a covering as described above and let $h : M \to M$ be a homeomorphism in standard form. Let $\tilde{h} : \tilde{M} \to \tilde{M}$ be a lift of h. Then*

$$N_0(h) = 1/k \left[\Sigma_{i=1}^{k} N_0(\beta^i \circ \tilde{h}) + (k-1)\omega_h + \delta_h\right]$$

where ω_h denotes the cardinality of Ω_h and δ_h denotes the cardinality of Δ_h.

Proof: Let σ be an essential fixed point class of h. First suppose that $\sigma \in \Omega_h$. If $F_\sigma = \emptyset$, then $q(\sigma)$ contains a β-connected component and so σ is covered by a single fixed point class from exactly one of the k covering homeomorphisms $\tilde{h}, \beta \circ \tilde{h}, \ldots, \beta^{k-1} \circ \tilde{h}$. Since the cover is regular this class is essential. If $F_\sigma \neq \emptyset$ only one of the covering homeomorphisms has a fixed point class which covers all of σ. All of the remaining covering homeomorphisms contain fixed point classes projecting onto F_σ, but by the local invariance of the fixed point index under p, these classes are inessential.

Now suppose that $\sigma \notin (\Omega_h \cup \Phi_h)$ and $F_\sigma = \emptyset$. Among the k covering homeomorphisms there is a total of k sets of fixed points each of which projects (under p) homeomorphically onto σ. Suppose that $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ are two such sets which are fixed by the same covering homeomorphism. Since $\hat{h}$ is in fixed point standard form and since $q(\sigma)$ does not contain any connected components, any invariant loop based at a point in $q(\sigma)$ lifts to a loop, and not a path, in $\tilde{F}$. Likewise for invariant loops based at a point in σ, and hence, $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ are in different Nielsen classes.

In the case when F_σ is non-empty and has nonzero fixed point index each of the covering homeomorphisms has an essential fixed point class that projects into σ, except in the case that $\sigma \in \Delta_h$. In this case exactly one of the covering homeomorphisms has a fixed point class which projects onto σ and the defining formula for Δ_h ensures that it will have zero index.

To finish the proof observe that fixed point classes in Φ_h lift to classes of the same type for the covering homeomorphisms, and vice versa. Thus, after ignoring such fixed point classes, a straightforward counting argument comparing the number of remaining essential fixed point classes of h to those of the k covering homeomorphisms gives the formula stated in the theorem. □

3. EXAMPLES

In this section we give two examples that illustrate how the results of the previous section are used to reduce the computation of Nielsen numbers (for homeomorphisms of Seifert fibre spaces) to the case of surface homeomorphism. These examples also indicate how critical fibres effect Nielsen classes.

Example 3.1. Let F be a compact, connected surface which contains a two-sided separating curve C. Let Y be an open regular neighborhood of C. We assume each component of $F \setminus Y$ has negative Euler characteristic. Let x_1, x_2 be distinct points on C and let X be an open regular neighborhood of

$\{x_1, x_2\}$ in F. For each integer $r \geq 1$ we define a Seifert fibre space

$$M_r = ((F \setminus X) \times S^1) \cup_{\partial X \times S^1} \{two\ copies\ of\ T_r\}.$$

A homeomorphism $\hat{h} : F \to F$ homotopic to the identity, and in fixed point standard form, is defined by sending x_1 to x_2, x_2 to x_1 and such that $Fix(\hat{h}) = F \setminus Y$. For each r, there exists a fibre-preserving homeomorphism $h : M_r \to M_r$ which induces $\hat{h}$ and reverses fibre orientation over the fibres in $Fix(\hat{h})$. If $r = 1$, then by Theorem 1.1 we see that $N(h) = 2N(\hat{h}) = 2$, as $\hat{h}$ is homotopic to the identity. When $r > 1$ we use Theorem 2.4 to compute $N(h)$. First note that $\Phi_h = \emptyset$ so $N_0(h) = N(h)$. Let $p : \tilde{M} \to M$ be an r-fold cover which induces $p : \tilde{F} \to F$ with singularities x_1 and x_2. Let g be a lift of h. By showing that $\hat{g}$ when restricted to the $2r$-punctured sphere $p^{-1}(Y)$ essentially acts as a rotation by $2\pi/r$, one gets $N(\hat{g}) = 2$. Thus, $N(g) = 4$ and since each of Δ_h and Ω_h is empty, $N(h) = 1/r[4r] = 4$.

The next example illustrates the computation of $N(h)$ when $\Phi_h \neq \emptyset$, or more generally, when $\hat{h}$ has an invariant simple closed curve based at a point in S whose corresponding fibre is of type T_r with $r > 1$. This example also illustrates how the index of a member of Φ_h is computed and, as a result, how one computes the value of ϵ_h (and also $\epsilon_{\beta^i \circ \tilde{h}}$) in Theorem 2.4.

Example 3.2. Let F be a nonorientable surface having negative Euler characteristic. Let ℓ be a one-sided simple closed curve in F and let x be a point on ℓ. Define a Seifert fibre space M_r just as in Example 3.1 (except here $S = \{x\}$).

Define a homeomorphism $\hat{g} : F \to F$ as follows: Let N_1, N_2 be closed regular neighborhoods of ℓ with $N_1 \subset int(N_2)$. On N_1, $\hat{g}$ is a reflection through ℓ and on $F \setminus N_2, \hat{g}$ is the identity. Extend to the remaining annulus so that the resulting $\hat{g}$ is isotopic to the identity (but not rel $\{x\}$). By a small isotopy we get $\hat{h}$, which is in fixed point standard form relative to

S, such that $Fix(\hat{h}) = \{x\} \cup cl(F \setminus N_2)$ and $\hat{h}(\ell) = \ell$. As in the previous example, $\hat{h}$ is covered by a homeomorphism $h : M_r \longrightarrow M_r$ which reverses fibre orientation on all fibres over $Fix(\hat{h})$. By lifting to the double cover of M_r corresponding to the cover of F by an orientable surface one sees that all four components of $Fix(h)$ are in distinct fixed point classes. We need only decide which components are essential. This will be done by a direct calculation, without using Theorem 2.4.

The two surfaces fixed by h each have negative Euler characteristic and hence are essential. The indices for the two fixed points in $q^{-1}(x)$ can be computed in the following manner. Let $p : \tilde{M} \longrightarrow M_r$ be an r-fold covering inducing $p : \tilde{F} \longrightarrow F$ with one singularity occurring at x. Let f be a lift of $\hat{h}$. Let D be a disk containing x meeting ℓ in an arc. By construction, $\hat{h}$ contracts one component of $(D \cap \ell) \setminus x$ towards x, expands the other outwards along ℓ and acts by a reflection on the complement of ℓ. Now, let $\tau_1, \ldots, \tau_{2r}$ denote the components of $p^{-1}(D \cap \ell)$ given in cyclic order around $p^{-1}(D)$. The lift f also acts by a reflection; fixing the point $p^{-1}(x)$ and leaving two components τ_i, τ_j invariant. It follows that $\mid i - j \mid = r$. Also, if r is even then either f is contracting on both τ_i and τ_j, or expanding on both. In this case $index(f, p^{-1}(x)) = \pm 1$. If r is odd, then f is contracting on one and expanding on the other, resulting in $index(f, p^{-1}(x)) = 0$. Thus, $N(h) = 2$ if r is odd, and 4 if r is even.

4. ALGORITHMS FOR COMPUTING NIELSEN NUMBERS

In this section we describe two algorithms for the computation of Nielsen numbers for homeomorphisms on certain 3-manifolds. The first algorithm uses the results of §2 for the Seifert fibre space setting while the latter generalizes the computation to a class of geometric 3-manifolds. For a treatment of the topology of the geometric 3-manifolds discussed here the reader is referred to [S].

Algorithm 4.1. *(Nielsen numbers of Siefert fibre space homeomorphisms.) Given a 3-dimensional Seifert fibre space M and a homeomorphism $h : M \to M$ the value of $N(h)$ can be computed by the procedure outlined below.*
CASE 1: h is not homotopic to a fibre-preserving map. Then M is either $S^1 \times D^2$, or a I-bundle over the torus or Klein bottle or M is covered by one of S^3, $S^2 \times R$, or $S^1 \times S^1 \times S^1$ [S;Theorem 3.9]. In these cases one can compute $N(h)$ directly from the definition and the given covering. See, for example, [M] for algorithms of this type.
CASE 2: M is not aspherical. Then the universal cover of M is either S^3 or $S^2 \times R$. As in Case 1, $N(h)$ can be computed directly.
CASE 3: M is aspherical and h is fibre-preserving. By Lemma 2.2 we can assume that h is in standard form. If the hypothesis of Theorem 2.1 are satisfied, then the algorithm for computing Nielsen numbers for surface homeomorphisms given in [K] can be used to compute $N(\hat{h})$ and hence $N(h)$. If not, then M is finitely covered by a Seifert fibre space that does satisfy the hypothesis [JW;Lemma 2.2]. Apply Theorem 2.4, repeatedly if necessary, until the hypothesis of Theorem 2.1 are satisfied.

Remarks: (1) In order to apply Theorem 2.4 one needs to first identify the covering Seifert fibre space $\tilde{M}$, together with its corresponding Z_k-action β, and then find the k covering homeomorphisms of h. The fixed point classes in Ω_h can be found by first identifying the β-connected components of $Fix(\hat{h})$ using the algorithm in [BH]. One then determines the number of essential classes of h that project into these components. It may be necessary to appeal to further lifts in order to determine exactly which fixed points are in such a class. The value of ω_h is obtained by counting these classes together with the essential classes for which F_σ is non-empty but with zero index.
(2) The value of δ_h is determined by finding the fixed point classes which intersect $p(Fix(\beta))$ and satisfy the defining equation for Δ_h. Typically, one gets that $\delta_h = 0$.

(3) Finally, the value of ϕ_h and the analogous values for each of the covering homeomorphisms can be computed as done in Example 3.2. As illustrated in that example, the index of a Φ-class depends on the local structure at a critical fibre.

For the second algorithm let $\mathcal{L}$ denote the class of compact, aspherical 3-manifolds consisting of all Seifert manifolds, all hyberbolic manifolds and those which admit a non-trivial torus decomposition in the sense of Jaco-Shalen-Johannson [J]. The following algorithm outlines a method for computing Nielsen numbers for self-homeomorphisms of members of $\mathcal{L}$.

Algorithm 4.2. *Let $M \in \mathcal{L}$ and let $h : M \to M$ be given. Then $N(h)$ can be computed by the following procedure:*
STEP 1. Given a triangulation of M use normal surface theory (see, for example, [H] chapters 7-9) to find the Jaco-Shalen-Johannson torus decomposition.
STEP 2. As h must preserve this splitting, first compute the Nielsen number for each invariant component. On a hyperbolic component this is just $\mid L \mid$ where L is the Leftshetz number of the homeomorphism restricted to this component. On a Seifert fibre piece use the Algorithm 4.1 above.
STEP 3. Use the results in [JW] to decide when fixed points from distinct components are Nielsen equivalent.

Under the assumption that Thurston's Geometrization Conjecture (see [S,§6]) holds true, the above Algorithm 4.2 indicates that the Nielsen number can be effectively computed for self-homeomorphisms of many compact 3-manifolds. For instance, if M is a 3-manifold with finite fundamental group then $N(h)$ can be computed directly from the information given by the induced map on $\pi_1(M)$ (see [M]). If $\pi_1(M)$ is infinite and M is also prime and irreducible [J], then Thurston's conjecture implies that, with a few exceptions, after taking a finite cover one gets a manifold in $\mathcal{L}$. Thus one can effectively compute $N(h)$. (The exceptional cases are covered by Algorithm 4.1.) On the other hand, if M is either not prime or reducible it is not clear as to how one would get an algorithm. In these settings the structure of the mapping class group is much more complicated. The reader is referred to [Mc] for a discussion of homeomorphisms of reducible 3-manifolds.

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