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REMARKS ON NORMALITY OF Σ-PRODUCTS

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ABSTRACT. We show that a Σ-product of semi-stratifiable spaces, each finite subproduct of which is paracompact, is normal if and only if the Σ-product is countably paracompact. Second, we show that a Σ-product of separable spaces is collectionwise normal if and only if it is normal.

Many results have been obtained for normality of Σ-products. The countably tight condition of Σ-products plays very important roles in this study (see [7, 11]). On the other hand, as seen in [4, 5, 12, 13], it has recently become more important to investigate Σ-products without such a condition. In particular, Yang [14] has recently proved that a Σ-product of paracompact σ-spaces is normal iff it is countably paracompact. He has used an idea in [11, 12] for his proof of the result. In Section 1, we shall extend his result to Σ-products of semi-stratifiable spaces, using an idea in [13] instead of [11, 12].

It is assumed in many cases for normality of Σ-products that the factor spaces are in some class of generalized metric spaces. In Section 2, we deal with normality of Σ-products whose factor spaces are not in such a class.

All spaces are assumed to be Hausdorff, and ω denotes the set of all non-negative integers.

1. Σ-PRODUCTS OF SEMI-STRATIFIABLE SPACES

Let $X = \prod_{\lambda\in\Lambda} X_\lambda$ be a product of spaces $X_\lambda, \lambda \in \Lambda$, where the index set Λ is assumed to be uncountable. Fix a point $s = (s_\lambda) \in X$. The subspace $\Sigma = \{x = (x_\lambda) \in X : \mathrm{Supp}(x)$

is at most countable} of X is called a *Σ-product* of spaces $X_\lambda, \lambda \in \Lambda$, where $\mathrm{Supp}(x)$ denotes $\{\lambda \in \Lambda : x_\lambda \neq s_\lambda\}$. The $s \in \Sigma$ is called a *base point* of Σ. The mention of the base point s is often omitted.

For a set Λ, we denote by $[\Lambda]^\omega$ the set of all infinite countable subsets of Λ. For each $R \in [\Lambda]^\omega$, we denote by X_R the countable subproduct $\prod_{\lambda \in R} X_\lambda$ of X, and denote by p_R the projection of Σ onto X_R.

For an $R \in [\Lambda]^\omega$, a subset S of Σ is *R-cylindrically closed* (*open*) [13] in Σ if $p_R(S)$ is closed (open) in X_R and $p_R^{-1}p_R(S) = S$. For convenience sake, we say that a subset S of Σ is *cylindrically closed* (*open*) if it is R-cylindrically closed (open) in Σ for some $R \in [\Lambda]^\omega$.

Note that R-cylindrically closed sets in Σ are also R'-cylindrically closed if $R, R' \in [\Lambda]^\omega$ with $R \subset R'$. So we have

Proposition 1. *Let Σ be a Σ-product, each countable subproduct of which is normal. Then any disjoint cylindrically closed sets in Σ are separated by disjoint (cylindrically) open sets.*

For our main result, we state the following auxiliary concept which seems to be useful for the proof.

Definition. A Σ-product Σ is said to be *cylindrically normal* if any disjoint closed sets in Σ, one of which is cylindrically closed, are separated by disjoint open sets.

A space X is *countably paracompact* if every countable open cover of X has a locally finite open refinement. For a space X, Z is a *zero-set* in X if $Z = f^{-1}(0)$ for some continuous real-valued function of X. As in [14], we will also use the following:

Lemma 2. [6] *Let X be a countably paracompact space. If Z is a zero-set and F is a closed set disjoint from Z in X, then F and Z are separated by disjoint open sets.*

A space X is *semi-stratifiable* [3] if there is a function g of $X \times \omega$ into the topology of X, satisfying

(i) $\bigcap_{n\in\omega} g(x,n) = \{x\}$ for each $x \in X$,
(ii) if $\{x_n\}$ is a sequence of points in X with $y \in \bigcap_{n\in\omega} g(x_n, n)$ for some $y \in X$, then $\{x_n\}$ converges to y.

The function g is called a *semi-stratifiable function* of X.

Note that a semi-stratifiable space is perfect, and that the class of semi-stratifiable spaces is countably productive (see [3]).

In the proof of Theorem 3 below, we use the following notaion: For a finite sequence $\xi = (\alpha_0 \cdots \alpha_{n-1}\alpha_n)$, let $\xi_- = (\alpha_0 \cdots \alpha_{n-1})$ and $\xi^\frown(\alpha) = (\alpha_0 \cdots \alpha_n \alpha)$. Let Ξ be an index set such that one can assign $R_\xi \in [\Lambda]^\omega$ for each $\xi \in \Xi$. Then X_{R_ξ} and p_{R_ξ} are abbreviated by X_ξ and p_ξ, respectively. For each ξ, $\eta \in \Xi$ with $R_\xi \subset R_\eta$, p_ξ^η denotes the projection of X_η onto X_ξ.

Now, we are ready to prove the main theorem.

Theorem 3. *Let Σ be a Σ-product of semi-stratifiable spaces, each finite subproduct of which is paracompact. Then the following are equivalent.*

(a) *Σ is normal.*
(b) *Σ is cylindrically normal.*
(c) *Σ is countably paracompact.*

Proof: Let Σ be a Σ-product of spaces $X_\lambda, \lambda \in \Lambda$, with some base point $s \in \Sigma$. It follows from [3, Theorem 2.1] and [10, Theorem 4.9] that each countable subproduct of $\prod_{\lambda\in\Lambda} X_\lambda$ is paracompact and semi-stratifiable.

(a) $\Rightarrow$ (c): When uncountably many X_λ's have at least two points, this immediately follows from [13, Proposition 3]. Otherwise, this is obvious.

(c) $\Rightarrow$ (b): Let A be an R-cylindrically closed set in Σ for some $R \in [\Lambda]^\omega$. Let B be a closed set in Σ disjoint from A. Since X_R is perfectly normal, $p_R(A)$ is a zero-set in X_R. Hence A is a zero-set in Σ. Since Σ is countably paracompact, it follows from Lemma 2 that A and B are separated by disjoint open sets in Σ.

(b) $\Rightarrow$ (a): Let A and B be any disjoint closed sets in Σ. Now, for each $n \in \omega$, we construct a collection $\mathcal{U}_n$ of open sets in Σ and an index set Ξ_n of n-tuples such that for each $\xi \in \Xi_n$ one can assign $R_\xi \in [\Lambda]^\omega, E(\xi) \subset \Sigma, H(\xi) \subset \Sigma, x_\xi \in \Sigma$ and a function g_ξ, satisfying the following conditions (1)-(7) for each $n \in \omega$:

(1) $\mathcal{U}_n$ is σ-locally finite in Σ such that $\overline{U} \cap A = \emptyset$ for each $U \in \mathcal{U}_n$ if n is odd, and $\overline{U} \cap B = \emptyset$ for each $U \in \mathcal{U}_n$ if n is even.

(2) $\xi \in \Xi_n$ implies $\xi_- \in \Xi_{n-1}$, where $\Xi_0 = \{\emptyset\}$.

(3) For each $\xi \in \Xi_n, E(\xi)$ is an R_{ξ_-}-cylindrically closed set and $H(\xi)$ is an R_{ξ_-}-cylindrically open set set in Σ with $E(\xi) \subset H(\xi)$, where $E(\emptyset) = H(\emptyset) = \Sigma$.

(4) $\{H(\xi) : \xi \in \Xi_n\}$ is σ-locally finite in Σ.

(5) For each $\mu \in \Xi_{n-1}$, $E(\mu)$ is covered by $\mathcal{U}_n \bigcup \{E(\xi) : \xi \in \Xi_n$ with $\xi_- = \mu\}$.

(6) For each $\xi \in \Xi_n$, g_ξ is a semi-stratifiable function of X_ξ such that $p^\xi_{\xi_-}(g_\xi(x,k)) \subset g_{\xi_-}(p^\xi_{\xi_-}(x), k)$ for each $x \in X_\xi$ and $k \in \omega$.

(7) For each $\xi \in \Xi_n$,
(a) $x_\xi \in E(\xi_-) \cap A$ if n is odd, and $x_\xi \in E(\xi_-) \cap B$ if n is even,
(b) $p_{\xi_-}(E(\xi)) \subset g_{\xi_-}(p_{\xi_-}(x_\xi), n)$,
(c) $R_\xi = R_{\xi_-} \cup \mathrm{Supp}(x_\xi)$.

Assume that the above construction has been already performed for no greater than n. Let n be an even number. Pick a $\xi \in \Xi_n$ and fix it. Let

$$\mathcal{W} = \{g_\xi(p_\xi(x), n+1) \cap p_\xi(E(\xi)) : x \in E(\xi) \cap A\}.$$

Since $p_\xi(E(\xi))$ is closed in X_ξ and $\bigcup \mathcal{W}$ is an F_σ-set in X_ξ, there is a σ-locally finite collection $\mathcal{F}$ of closed sets in X_ξ such that $\mathcal{F}$ refines $\mathcal{W}$ and $\bigcup \mathcal{F} = \bigcup \mathcal{W}$. Moreover, there is a σ-locally finite collection $\{G_F : F \in \mathcal{F}\}$ of open sets in X_ξ such that $F \subset G_F \subset p_\xi(H(\xi))$ for each $F \in \mathcal{F}$. Let $\mathcal{F}_+ = \{F \in \mathcal{F} : p_\xi^{-1}(F) \cap A = \emptyset\}$. Since Σ is cylindrically normal, there is an open set $U(F)$ in Σ such that $p_\xi^{-1}(F) \subset U(F) \subset \overline{U(F)} \subset p_\xi^{-1}(G_F) \backslash A$ for each $F \in \mathcal{F}_+$. Let $F_0 = p_\xi(E(\xi)) \backslash \bigcup \mathcal{W}$. Then note that $p_\xi^{-1}(F_0) \cap A = \emptyset$. Similarly, there is an open set U_0 in Σ such that $p_\xi^{-1}(F_0) \subset U_0 \subset \overline{U_0} \subset H(\xi) \backslash A$. We put $\mathcal{U}(\xi) = \{U(F) : F \in \mathcal{F}_+\} \cup \{U_0\}$. On the other hand, we put $\mathcal{F}_- = \mathcal{F} \backslash \mathcal{F}_+$. Moreover, let Ξ_ξ denote an index set of $(n+1)$-tuples such that $\mathcal{F}_- = \{F_{\xi^\frown(\alpha)} : \xi^\frown(\alpha) \in \Xi_\xi\}$, where let $\Xi_\xi = \emptyset$ if $\mathcal{F}_- = \emptyset$. For each $\eta = \xi^\frown(\alpha) \in \Xi_\xi$, let $E(\eta) = p_\xi^{-1}(F_\eta)$ and $H(\eta) = p_\xi^{-1}(G_{F_\eta})$. Here, letting ξ range over Ξ_n, we set $\mathcal{U}_{n+1} = \bigcup\{\mathcal{U}(\xi) : \xi \in \Xi_n\}$ and $\Xi_{n+1} = \bigcup\{\Xi_\xi : \xi \in \Xi_n\}$. Then it is easily verified that the conditions (1)-(5) for $n+1$ are satisfied. Since $\mathcal{F}$ refines $\mathcal{W}$, for each $\eta = \xi^\frown(\alpha) \in \Xi_{n+1}$, we can pick an $x_\eta \in E(\xi) \cap A$ such that $F_\eta \subset g_\xi(p_\xi(x_\eta), n+1)$. Moreover, let $R_\eta = R_\xi \cup \mathrm{Supp}(x_\eta)$. Then (6) for $n+1$ is satisfied. Since p_ξ^η is continuous, we can take a semi-stratifiable function g_η of X_η, satisfying (7) for $n+1$. For the case that n is odd, we only replace A with B in the above. Thus, we have accomplished the desired construction.

We set $\mathcal{U} = \bigcup_{n\in\omega} \mathcal{U}_n$. By (1), $\mathcal{U}$ is a σ-locally finite collection of open sets in Σ such that $\overline{U} \cap A = \emptyset$ or $\overline{U} \cap B = \emptyset$ for each $U \in \mathcal{U}$. So it suffices to show that $\mathcal{U}$ covers Σ. Now, assume that there is some $y \in \Sigma \backslash \bigcup \mathcal{U}$. By (5), we can inductively choose a sequence $\{\xi^n\}$ of finite sequences such that $\xi^n \in \Xi_n, \xi^{n+1}_- = \xi^n$ and $y \in E(\xi^n)$ for each $n \in \omega$. By (6) and (7b), the sequence $\{p_{\xi^{m-1}}(x_{\xi^n}) : n \geq m\}$ converges to $p_{\xi^{m-1}}(y)$ (see the proof of Claim 2 in that of [13, Theorem 1]). Let $R = \bigcup_{n\in\omega} R_{\xi^n}$. We can take the point $z \in \Sigma$ defined by $p_R(z) = p_R(y)$ and $p_{\Lambda\backslash R}(z) = p_{\Lambda\backslash R}(s)$. Then it follows from (7c) that $\{x_{\xi^n} : n \in \omega\}$ converges to z. It follows from (7a) that $x_{\xi^{2n-1}} \in A$ and $x_{\xi^{2n}} \in B$ for each $n \in \omega$. This implies $z \in A \cap B$, which is a contradiction. □

Theorem 3 immediately yields

Corollary 4. [14] *A Σ-product of paracompact σ-spaces is normal if and only if it is countably paracompact.*

2. Σ-PRODUCTS OF SEPARABLE SPACES

In this section, each factor space X_λ of infinite products and Σ-products is assumed to have at least two points. Let κ be an infinite cardinal.

Recall that a space X is *κ-collectionwise normal* if every discrete collection of closed sets in X with cardinality $\leq \kappa$ can be separated by disjoint open sets.

Lemma 5. [2, p. 80] *Assume that an infinite product $X = \prod_{\lambda\in\Lambda} X_\lambda$ is normal. If each finite subproduct of X is κ-collectionwise normal, then X is κ-collectionwise normal.*

Proposition 6. *Let Σ be Σ-product of κ many spaces. Then Σ is normal if and only if it is κ-collectionwise normal.*

Proof: Let Σ be a normal Σ-product of spaces $X_\lambda, \lambda \in \Lambda$, with a base point $s \in \Sigma$, where $|\Lambda| = \kappa$. We may assume $\kappa > \omega$. Let $\{\Lambda_n : n \in \omega\}$ be a partition of Λ such that $|\Lambda_n| = \kappa$ for each $n \in \omega$. Let Σ_n be the Σ-product of the spaces $X_\lambda, \lambda \in \Lambda_n$, with the base point $p_{\Lambda_n}(s)$. Let $A(\kappa)$ be the one-point compactification of a discrete space of cardinality of κ. Then note that $A(\kappa)$ is embedded in Σ_n. Since $\prod_{i<n} \Sigma_i \times A(\kappa)$ is closed in Σ, it is normal. So it follows from [1, Theorem 2] that $\prod_{i<n} \Sigma_i$ is κ-collectionwise normal for each $n \in \omega$. Hence, by Lemma 5, $\Sigma = \prod_{i\in\omega} \Sigma_i$ is κ-collectionwise normal. □

Recall that a space X is *ccc* if every disjoint collection of open sets in X is at most countable.

The following is an extention of [8, Proposition 3].

Proposition 7. *Let Σ be a Σ-product, each finite subproduct of which is ccc. If Σ is normal, then each closed discrete subset of Σ is at most countable.*

Proof: Let Σ be a Σ-product of spaces $X_\lambda, \lambda \in \Lambda$. Let $X = \prod_{\lambda\in\Lambda} X_\lambda$. Since each finite subproduct of X is ccc, it follows that X is ccc (see [9, Theorem 2.1.9]). Since Σ is dense in X, Σ is ccc. By Proposition 6, Σ is $|\Lambda|$-collectionwise normal. Hence Σ is ω_1-collectionwise normal. Thus the ccc of Σ does not allow the existence of an uncountable discrete closed sets in Σ. □

Proposition 7 immediately yields

Corollary 8. *A Σ-product of separable spaces is collectionwise normal if and only if it is normal.*

Remark. In Proposition 7, we cannot exclude the normality of Σ in ZFC. In fact, it was shown in the proof of [5, Theorem 2.1] that if a space X is left-separated in type ω_1, first countable and ccc, then $X \times \Sigma\omega^{\omega_1}$ has an uncountable closed discrete subset. Since the product of a ccc space and a separable space is ccc, each finite subproduct of this Σ-product is ccc.

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