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FINITELY EQUIVALENT CONTINUA SEMI-LOCALLY-CONNECTED AT NON-CUT POINTS

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ABSTRACT. Let X be a nondegenerate continuum that is semi-locally-connected (slc) or aposyndetic at each non-cut (= non-separating) point. It is shown that if X contains only finitely many nondegenerate, mutually non-homeomorphic subcontinua, then X is a graph. Even though a continuum may be slc at each non-cut point without being slc at every point, it is shown that a continuum which is aposyndetic at each non-cut point must be aposyndetic at every point.

1. INTRODUCTION.

A *continuum* is a nonempty, compact, connected, metric space. We refer the reader to [8] and [11] for other basic definitions. To avoid confusion, we note that here, as in [8] and [11], *non-cut point* means *non-separating point* (and *cut point* means *separating point*).

A continuum, X is said to be *finitely equivalent* provided that X has only finitely many, nondegenerate, mutually non-homeomorphic subcontinua. If n is the number of such subcontinua, then X is said to be n*-equivalent.*

Interest in this notion began in 1921 when Mazurkiewicz asked if a 1-equivalent continuum must be an arc [6]. Moise [7] gave a negative answer to this question with the pseudo-arc, and Henderson [2] gave an affirmative answer for the case of decomposable continua. It is still not known if the arc and the pseudo-arc are the only 1-equivalent continua.

Mahavier [5] obtained several interesting theorems and examples concerning 2-equivalent continua. In view of the limited accessibility of [5], we state the following result since we will refer to it in the forthcoming discussion. The result may be proved using Theorem 3.2 of [9].

1.1 Mahavier's Theorem [5]. *If X is a 2-equivalent continuum that contains an arc, then X is a simple closed curve, a simple triod, or a continuum that is irreducible about two points.*

When we began the investigations that led to this paper, we were motivated by Mahavier's Theorem to try to prove the following result: Any 2-equivalent, semi-locally-connected (slc) continuum must be a simple closed curve or a simple triod. We proved this by showing that any such continuum must contain an arc and then applying Mahavier's Theorem. We can now prove much more. We summarize our results below.

We prove that finitely equivalent continua that are slc at each of their non-cut points must be graphs (3.1). We obtain a similar result for continua that are aposyndetic at each of their non-cut points (4.4). In fact, for the purpose of proving this result, we prove that continua that are aposyndetic at each of their non-cut points are actually aposyndetic at every point (4.3). In contrast, continua that are slc at each of their non-cut points need not be slc at every point. This is easily seen by considering the planar continuum that consists of the unit circle centered at (0,0) and the circles of radius $(n-1)/n$ centered at $(1/n, 0)$ for each $n = 2, 3, \dots$.

The symbol cl denotes the closure operator.

2. A PRELIMINARY RESULT.

Our purpose is to prove the result in 2.2. It will be used in the next section in the proof of our first main theorem.

The following notation is from 11.29 of [8] and is used throughout this section. If X is a continuum and if A and B are nonempty, compact subsets of X, then we write

$$C = \text{ irr } (A, B)$$

to mean that C is a subcontinuum of X and, as such, is irreducible with respect to intersecting both A and B (i.e., $C \cap A \neq \emptyset$, $C \cap B \neq \emptyset$, and no proper subcontinuum of C intersects both A and B). When A or B is a one-point set, we omit the set notation.

It is convenient to have the following lemma for use in the proof of 2.2. The lemma is probably well known, but we include a proof of it for completeness.

2.1 Lemma. *Let X be a continuum, let M be a subcontinuum of X, and let C be a component of $X - M$. Let J be a nonempty, compact subset of C, and let Q be a subcontinuum of X such that*

$$Q = \; irr \; (M, J).$$

Then, $Q \subset C \cup M$.

Proof: Let $p \in Q \cap J$, and let G denote the component of p in $Q - M$. Then $\text{cl}(G)$ is a subcontinuum of Q, $\text{cl}(G) \cap J \neq \emptyset$, and, since $Q - M$ is a nonempty, proper subset of the continuum Q,

$$\text{cl}(G) \cap M \neq \emptyset \text{ (5.6 of [8]).}$$

These properties of $\text{cl}(G)$, together with our assumption that $Q = \text{irr}(M, J)$, imply that $\text{cl}(G) = Q$. Note that $G \cap C \neq \emptyset$ since $p \in G \cap J$ and $J \subset C$. Thus, since C is a component of $X - M$ and G is a connected subset of $X - M$, we see that $G \subset C$. Hence, using the previously proved fact that $Q = \text{cl}(G)$, we now see that $Q \subset \text{cl}(C)$. Therefore, since C is closed in $X - M$, it follows immediately that $Q \subset C \cup M$. This proves 2.1.

2.2 Proposition. *Let X be a continuum that is slc at each of its non-cut points. Let M be a subcontinuum of X such that $X - M$ has at least k distinct components for some integer $k \geq 2$. Then, there is a subcontinuum, A_k, of X such that*

A_k is irreducible about some k points and A_k is not irreducible about fewer than k points.

Proof: Let $C_1, \ldots, C_k$ denote k distinct components of $X - M$. For each $i = 1, \ldots, k$, let p_i be a non-cut point of X such that $p_i \in C_i$ (6.8 of [8]). Since X is slc at each p_i, there is an open subset, U_i, of X for each i such that $p_i \in U_i$, $\mathrm{cl}(U_i) \subset X - M$, and $X - U_i$ is a continuum (4.14 of [11, p. 50]). Now, let B_i denote the component of p_i in $\mathrm{cl}(U_i)$, and note that B_i is a continuum such that $B_i \cap Bd(U_i) \neq \emptyset$ for each i (5.4 of [8]). Hence (11.30 of [8]), there is a subcontinuum, K_i, of B_i for each i such that

(1) $K_i = \mathrm{irr}(p_i, Bd(U_i))$.

Note the following fact for use later: Since B_i is a subcontinuum of $X - M$ and $B_i \cap C_i \neq \emptyset$ (because $p_i \in B_i \cap C_i$), $B_i \subset C_i$ and, hence,

(2) $K_i \subset C_i$.

Note that $X - U_i$ is a continuum containing $M \cup Bd(U_i)$ and that, by (1), $K_i \cap Bd(U_i) \neq \emptyset$. Hence (11.30 of [8]), there is a subcontinuum, M_i, of $X - U_i$ for each i such that

(3) $M_i = \mathrm{irr}(M, K_i \cap Bd(U_i))$.

Using (3) to apply 2.1 with $J = K_i \cap Bd(U_i)$, noting that $J \subset C_i$ by (2), we have that

(4) $M_i \subset C_i \cup M$.

Next, we prove (5) and (6) below.

(5) $K_i \cup M_i = \mathrm{irr}(p_i, M)$ for each i.

(6) If H is a subcontinuum of $K_i \cup M_i \cup M$ such that $H = \mathrm{irr}(p_i, M)$ for a given i, then $H = K_i \cup M_i$.

Proof of 5: Fix i. Let L be a subcontinuum of $K_i \cup M_i$ such that $p_i \in L$ and $L \cap M \neq \emptyset$. Let S be the component of p_i in $L \cap U_i$. Since $M_i \subset X - U_i$, $S \subset L - M_i$. Hence, $\mathrm{cl}(S)$ is a subcontinuum of K_i. Also, since $L \cap U_i \neq L$ (because $L \cap M \neq \emptyset$

and $U_i \cap M = \emptyset$), we have that $\mathrm{cl}(S) \cap Bd(U_i) \neq \emptyset$ (5.6 of [8]). Thus, since $p_i \in \mathrm{cl}(S)$, we see by (1) that $\mathrm{cl}(S) = K_i$ Therefore,

(a) $L \supset K_i$.

Next, note that $K_i \cap M = \emptyset$ (since $K_i \subset B_i \subset \mathrm{cl}(U_i) \subset X - M$). Thus, since $L \cap M \neq \emptyset$, $(L - K_i) \cap M \neq \emptyset$. Hence, there is a component, T, of $L - K_i$ such that $T \cap M \neq \emptyset$. Since $L \subset K_i \cup M_i$, $T \subset M_i$. Hence, $\mathrm{cl}(T)$ is a subcontinuum of M_i. By (1), $p_i \in K_i$. Thus, since $p_i \in L$, $L - K_i \neq L$; also, $L - K_i \neq \emptyset$ (since $T \neq \emptyset$). Hence, $L - K_i$ being a nonempty, proper subset of the continuum L, we see that $\mathrm{cl}(T) \cap K_i \neq \emptyset$ (5.6 of [8]). Since

$$\mathrm{cl}(T) \subset M_i \subset X - U_i \text{ and } K_i \subset \mathrm{cl}(U_i),$$

we see that $\mathrm{cl}(T) \cap K_i \subset Bd(U_i)$. Thus, since $\mathrm{cl}(T) \cap K_i \neq \emptyset$, we have that

$$\mathrm{cl}(T) \cap [K_i \cap Bd(U_i)] \neq \emptyset.$$

Hence, recalling that $\mathrm{cl}(T) \cap M \neq \emptyset$ (since $T \cap M \neq \emptyset$) and that $\mathrm{cl}(T)$ is a subcontinuum of M_i, we see by (3) that $\mathrm{cl}(T) = M_i$. Therefore,

(b) $L \supset M_i$.

Since $L \subset K_i \cup M_i$, we see from (a) and (b) that $L = K_i \cup M_i$. Therefore, in view of how we chose L, we have proved (5).

Proof of (6): Fix i, and let H satisfy the hypothesis of (6). Noting that $p_i \in H - M$, let D be the component of p_i in $H - M$. Then, since $H - M$ is a nonempty, proper subset of the continuum H, $\mathrm{cl}(D) \cap M \neq \emptyset$ (5.6 of [8]). Also, since $p_i \in D$, $p_i \in \mathrm{cl}(D)$. Thus since $\mathrm{cl}(D)$ is a subcontinuum of H and $H = \mathrm{irr}(p_i, M)$ (by the hypothesis in (6)), we have that

$$\mathrm{cl}(D) = H;$$

also, since $D \subset H - M \subset K_i \cup M_i$ (see (6)), $\mathrm{cl}(D)$ is a subcontinuum of $K_i \cup M_i$ and, thus, by (5), we have that

$$\mathrm{cl}(D) = K_i \cup M_i.$$

Therefore, $H = K_i \cup M_i$. This proves (6).

Next, we define A_k and prove that A_k satisfies the conclusion to 2.2. First, define N and Q as follows:

$$N = M \cup [\cup_{i=1}^{k}(K_i \cup M_i)], \quad Q = M \cup (\cup_{i=1}^{k} M_i).$$

By (5), $K_i \cup M_i$ is a continuum containing p_i and intersecting M for each i. Hence, it follows easily that N is a continuum and that $p_i \in N$ for each i. Therefore, there is a subcontinuum, A_k, of N such that A_k is irreducible about $\{p_1, \ldots, p_k\}$ (4.35 (a) of [8]).

We prove (7), (8), and (9) below.

(7) $A_k \supset \cup_{i=1}^{k}(K_i \cup M_i)$.

(8) $A_k \cap Q$ is a continuum.

(9) $p_i \notin Q$ for any i.

Proof of (7): Fix $j \leq k$. We prove (7) by proving that $K_j \cup M_j \subset A_k$. Recall that $p_i \in A_k \cap C_i$ for each i. Hence, A_k is a subcontinuum of X and A_k intersects each of the k components C_i of $X - M$. Thus, since $k \geq 2$ (by assumption in 2.2), it follows immediately that $A_k \cap M \neq \emptyset$. Therefore, since $p_j \in A_k$, there is a subcontinuum, H_j, of A_k such that (11.30 of [8])

(c) $H_j = \mathrm{irr}(p_j, M)$.

Thus, since $p_j \in C_j$, we see by 2.1 that $H_j \subset C_j \cup M$. Therefore, since $H_j \subset A_k \subset N$ and $M \subset N$,

(d) $H_j \subset (C_j \cup M) \cap N = (C_j \cap N) \cup M$.

We now show that $(*)$ $C_j \cap N \subset K_j \cup M_j \cup M$. Let $x \in C_j \cap (N - M)$. Then, by the definition of N, $x \in C_j \cap (K_\ell \cup M_\ell)$ for some ℓ. If $x \in C_j \cap K_\ell$, then, by (2), $x \in C_j \cap C_\ell$; if $x \in C_j \cap M_\ell$, then, by (4) and the fact that $x \notin M$, we have again that $x \in C_j \cap C_\ell$. Thus, since the sets $C_1, \ldots, C_k$ are mutually disjoint, $\ell = j$. Hence, $x \in K_j \cup M_j$. This proves $(*)$. By (d) and $(*)$, we have that

(e) $H_j \subset K_j \cup M_j \cup M$.

Now, by (c) and (e), we may apply (6) to conclude that $H_j = K_j \cup M_j$. Therefore, since $H_j \subset A_k$, $K_j \cup M_j \subset A_k$. This proves (7).

Proof of (8): By (7), $M_i \subset A_k \cap Q$ for each i; hence, $A_k \cap Q \neq \emptyset$. Also, clearly, $A_k \cap Q$ is compact. Therefore, to prove (8), it remains to show that $A_k \cap Q$ is connected. We obtain some preliminary facts in (f)-(i) below for use in the proof. Since $\cup_{i=1}^k M_i \subset A_k \cap Q$ (by (7)), we see that

(f) $A_k \cap Q = (A_k \cap M) \cup (\cup_{i=1}^k M_i)$.

By (7), $\cup_{i=1}^k K_i \subset A_k$. Also, since $A_k \subset N, A_k - Q \subset \cup_{i=1}^k K_i$. Hence,

(g) $A_k = (A_k \cap Q) \cup (\cup_{i=1}^k K_i)$.

Since $C_i \subset X - M$ for each i, we have by (2) that

(h) $K_i \cap M = \emptyset$ for each i.

By (3), $K_i \cap M_i \neq \emptyset$ for each i. Also, by (4) and (h), $K_i \cap M_j \subset C_j$ and, by (2), $K_i \cap M_j \subset C_i$. Therefore, since the sets $C_1, \dots, C_k$ are mutually disjoint, we see that

(i) $K_i \cap M_j \neq \emptyset$ if and only if $i = j$.

Now, suppose that $A_k \cap Q$ is not connected. Then, there are nonempty, mutually separated sets, E and F, such that

(j) $A_k \cap Q = E \cup F$

Define two sets, P_E and P_F, as follows:

$$P_E = \cup\{K_i : K_i \cap E \neq \emptyset\},\ P_F = \cup\{K_i : K_i \cap F \neq \emptyset\}.$$

We show that the two sets $E \cup P_E$ and $F \cup P_F$ form a separation of A_k (thereby obtaining a contradiction). By (i), $K_i \cap M_i \neq \emptyset$ for each i and, by (f) and (j), $M_i \subset E \cup F$ for each i. Hence, $K_i \cap (E \cup F) \neq \emptyset$ for each i. Thus,

$$P_E \cup P_F = \cup_{i=1}^k K_i.$$

Therefore, recalling (g) and (j), we see that

$$\text{(k) } A_k = (E \cup P_E) \cup (F \cup P_F).$$

We prove that $(E \cup P_E) \cap (F \cup P_F) = \emptyset$. Suppose that $P_E \cap P_F \neq \emptyset$. Then, since the sets $K_1, \ldots, K_k$ are mutually disjoint (by (2)), there exists a single index, ℓ, such that $K_\ell \cap E \neq \emptyset$ and $K_\ell \cap F \neq \emptyset$. By (j) and (h),

$$K_\ell \cap E \subset (A_k \cap Q) - M.$$

Hence, by (f) and (i), $K_\ell \cap E \subset M_\ell$. Thus, since $K_\ell \cap E \neq \emptyset$, $M_\ell \cap E \neq \emptyset$. Similarly, $M_\ell \cap F \neq \emptyset$. Therefore, since $M_\ell \subset E \cup F$ (by (f) and (j)) and since E and F are mutually separated, we have a contradiction to the connectedness of M_ℓ. Thus, we have proved that

$$(\ell) \ P_E \cap P_F = \emptyset.$$

By (ℓ), $K_i \not\subset P_E \cap P_F$ for any i. In other words, if $K_i \subset P_E$ then $K_i \cap F = \emptyset$, and if $K_i \subset P_F$ then $K_i \cap E = \emptyset$. Therefore,

$$\text{(m) } P_E \cap F = \emptyset \text{ and } P_F \cap E = \emptyset.$$

By (ℓ), (m), and the fact that $E \cap F = \emptyset$, we have that

$$\text{(n) } (E \cup P_E) \cap (F \cup P_F) = \emptyset.$$

It is easy to see that $E \cup P_E$ and $F \cup P_F$ are nonempty and closed in X. Hence, by (k) and (n), we have a contradiction to the connectedness of A_k. Thus, our supposition that $A_k \cap Q$ is not connected is false. Therefore, $A_k \cap Q$ is connected. This completes the proof of (8).

Proof of (9): Fix i. Recall from above (1) that $p_i \in U_i \subset X - M$ and, from above (3), that $M_i \subset X - U_i$. Hence,

$$\text{(o) } p_i \notin M \cup M_i.$$

By (1), $p_i \in K_i$. Thus, by (i) in the proof of (8),

$$\text{(p) } p_i \notin M_j \text{ for any } j \neq i.$$

Combining (o) and (p) proves (9).

Finally, we complete the proof of 2.2 as follows.

By the definition of A_k (above (7)), A_k is a subcontinuum of X such that A_k is irreducible about the k points $p_1, \ldots, p_k (p_i \neq p_j$ when $i \neq j$ since $p_i \in C_i$ for each i). Therefore, it only remains to prove that A_k is not irreducible about any fewer than k points. Let $x_1, \ldots, x_m$ be any m points of A_k where $m < k$. Then, since the k sets $C_1, \ldots, C_k$ are mutually disjoint, there exists $t \leq k$ such that $x_i \notin C_t$ for any $i \leq m$. Let

$$Z = (A_k \cap Q) \cup (\cup_{i \neq t} K_i).$$

By (7), $A_k \cap Q \supset M_i$ for each i and, by (3), $M_i \cap K_i \neq \emptyset$ for each i. Hence,

$$(A_k \cap Q) \cap K_i \neq \emptyset \text{ for each } i.$$

Therefore, using (8), we see that Z is a continuum. Furthermore, by (7), $Z \subset A_k$. Next, note that the sets $K_1, \ldots, K_k$ are mutually disjoint (by (2)) and that $p_t \in K_t$ (by (1)). Hence,

$$p_t \notin \cup_{i \neq t} K_i.$$

Thus, by (9), $p_t \notin Z$. Therefore, since $p_t \in A_k$, $Z \neq A_k$. We have now proved that Z is a proper subcontinuum of A_k. Finally, we prove that $x_i \in Z$ for each $i \leq m$. Fix $j \leq m$. If $x_j \in A_k \cap Q$, then $x_j \in Z$. So, assume for the proof that $x_j \notin A_k \cap Q$. Then, since $x_j \in A_k$ (by the way x_j was chosen), we see from (g) in the proof of (8) that

$$x_j \in \cup_{i=1}^k K_i.$$

Also, by our choice of t, $x_j \notin C_t$ and, hence, by (2), $x_j \notin K_t$. Thus,

$$x_j \in \cup_{i \neq t} K_i.$$

Hence, $x_j \in Z$. Therefore, we have proved that $x_i \in Z$ for each $i \leq m$. This completes the proof of 2.2.

3. Finitely equivalent continua slc at each non-cut point

We prove the result in 3.1. Recall that a continuum is called a *θ_n-continuum* (some $n = 1, 2, \dots$) provided that no subcontinuum of it separates it into more than n components ([1], [10]).

3.1 Theorem. *Let X be a continuum that is slc at each of its non-cut points. It X is finitely equivalent, then X is a graph (and conversely). Furthermore, if X is n-equivalent, every subcontinuum of X is a θ_{n+1}-continuum.*

Proof: We first show that X is locally connected. Suppose that X is not locally connected. Then, X contains a convergence continuum, K, such that X is not connected im kleinen (cik) at any point of K (5.22(b) and 5.12 of [8]). There is a non-cut point, p, of X such that $p \in K$ (6.29(b) of [8]). Since X is not cik at p, there exists $\epsilon > 0$ such that any neighborhood of p of diameter $< \epsilon$ has infinitely many components. We choose a subcontinuum, M, of X as follows. Since p is a non-cut point of X, X is slc at p (by an assumption in our theorem). Therefore, there is an open neighborhood, V, of p of diameter $< \epsilon$ such that $X - V$ is a continuum (4.14 of [11, p. 50]). Let $M = X - V$. Note that, since V is a neighborhood of p of diameter $< \epsilon$, $X - M$ has infinitely many components. Hence, the hypothesis of 2.2 are satisfied for *each* $k \geq 2$. Thus, on applying 2.2 for each such k, we see that X is not finitely equivalent. This contradicts an assumption in our theorem. Therefore, we have proved that X is locally connected. Since X is n-equivalent for some n (finite), we see using 2.2 again that X is a θ_{n+1}-continuum. Thus, since X is locally connected, X is a graph (4.7 of [1, p. 156]). We prove the second part of 3.1 as follows. Let Y be a subcontinuum of X. Since X is n-equivalent, clearly Y is m-equivalent for some $m \leq n$. Since X is a graph, Y is locally connected (9.4 of [8]); hence, Y is slc (8.44(d) of [8]). Thus, we may now apply 2.2 to Y to see that Y is a θ_{m+1}-continuum. Therefore, Y is a θ_{n+1}-continuum. This completes the proof of 3.1.

Recalling our original interest from the discussion following 1.1, let us note some special cases of 3.1. It follows immediately from 3.1 that the only 2-equivalent, slc continua are a simple closed curve and a simple triod. Also, by 3.1, an arc is the only 1-equivalent, slc continuum (though this can be proved with simpler methods). Continuing in this vein, we see using 3.1 that the only 3-equivalent, slc continua are a figure X and a figure H (since it follows from 3.1 that such a continuum must be a tree, otherwise it would contain a noose, which is 4-equivalent). It would be of interest to determine which graphs, or at least how many, are n-equivalent for each n.

Regarding the last part of 3.1, we mention that θ_{n+1} is not always best possible (e.g., consider a simple closed curve).

4. Continua that are aposyndetic at each non-cut point

We prove that such continua are aposyndetic at every point. As a corollary, we obtain a result similar to 3.1. First, we prove two lemmas. Recall that non-cut point means non-separating point.

4.1 Lemma. *Let X be a continuum that is aposyndetic at each of its non-cut points. Then, for each $c \in X$, the collection of all of the components of $X - \{c\}$ is a null sequence (i.e., for each $\epsilon > 0$, only finitely many components of $X - \{c\}$ have diameter $\geq \epsilon$).*

Proof: Suppose that, for some $c \in X$ and some $\epsilon > 0$, there are infinitely many distinct components, $C_i (i = 1, 2, \dots)$, of $X - \{c\}$ such that diameter $(C_i) \geq \epsilon$ for each i. Note that $C_i \cup \{c\}$ is a continuum for each i (5.9 of [8]). Let d denote the metric for X. Let B denote the open d-ball in X with center c and radius $\epsilon/4$. Let $x_i \in C_i$ for each i such that $d(x_i, c) \geq \epsilon/2$ (x_i exists since $C_i \cup \{c\}$ is compact and has diameter $\geq \epsilon$). Finally, let K_i denote the component of $C_i - B$

containing x_i for each i. Then, by applying 5.6 of [8] to the continuum $C_i \cup \{c\}$ for each i, we see that $K_i \cap$ cl$(B) \neq \emptyset$ for each i; thus,

$$\text{diameter } (K_i) \geq \epsilon/4 \text{ for each } i.$$

Also, by 4.18 of [8], there is a subsequence, $\{K_{i(j)}\}_{j=1}^{\infty}$, of the sequence $\{K_i\}_{i=1}^{\infty}$ such that $\{K_{i(j)}\}_{j=1}^{\infty}$ converges,

$$\lim_{j\to\infty} K_{i(j)} = K, \text{ where } K \text{ is a continuum.}$$

Hence, it follows easily that K is a convergence continuum (since K is nondegenerate, its diameter being $\geq \epsilon/4$, and since $K \cap K_{i(j)} = \emptyset$ for all but at most one j – see 5.11 of [8]). Therefore, K contains a non-cut point, p, of X (6.29(b) of [8]). Now, let A be a subcontinuum of X such that A contains p in its interior (in X). Then, A must intersect $K_{i(j)}$, hence $C_{i(j)}$, for infinitely many j. Thus, since the sets $C_{i(j)}$ are distinct components of $X - \{c\}$, it must be that $c \in A$. This argument with A proves that X is not aposyndetic at p (with respect to c). Thus, since p is a non-cut point of X, we have a contradiction to the assumption in 4.1. Therefore, we have proved 4.1.

4.2 Lemma. *Let X be a continuum that is aposyndetic at each of its non-cut points. Let $c \in X$, let C be a component of $X - \{c\}$, and let $M = C \cup \{c\}$ (M is a continuum by 5.9 of [8]). Then, for $p, q \in M$, X is aposyndetic at p with respect to q if and only if M is aposyndetic at p with respect to q.*

Proof: Assume that X is aposyndetic at p with respect to q. Then there is a subcontinuum, A, of X containing p in its interior (in X) such that $q \notin A$. It follows easily from 4.1 that $A \cap M$ is connected (M is the A-set in 3.1 of [11, p.67]). Hence, $A \cap M$ is a subcontinuum of M containing p in its interior in M such that $q \notin A \cap M$. Therefore, M is aposyndetic at p with respect to q. Conversely, assume that M is aposyndetic at p with respect to q. Then there is a subcontinuum, B, of M containing p in its interior in M such that $q \notin B$. First, assume that $p \neq c$. Then, by 4.1, p is in the interior of B in X;

thus, X is aposyndetic at p with respect to q. Next, assume that $p = c$. Then, let

$$Z = B \cup [\cup\{G : G \text{ is a component of } X - \{c\} \text{ and } G \neq C\}].$$

Note that $c \in B$ (since $c = p$). Thus, for each component G of $X - \{c\}$, $G \cup B$ is a continuum (5.9 of [8]). Hence, Z is connected. Also, it follows from 4.1 that Z is compact. Therefore, Z is a continuum. Furthermore, since $p = c$ is in the interior of B in M, clearly p is in the interior of Z in X. Also, $q \notin Z$ (since $q \in M - B$). Therefore, we have proved that X is aposyndetic at p with respect to q (whether $p = c$ or $p \neq c$). This completes the proof of 4.2.

4.3 Theorem. *If a continuum, X, is aposyndetic at each of its non-cut points, then X is aposyndetic (at every point).*

Proof: Let $c \in X$, and let M be as in 4.2. Let $q \in M$ such that $q \neq c$. Then, according to 4.2, it suffices to prove that M is aposyndetic at c with respect to q. Suppose that M is not aposyndetic at c with respect to q. We make use of Jones's L set,

$$L_q = \{x \in M - \{q\} : M \text{ is not aposyndetic at } x \text{ with respect to } q\}.$$

By Theorem 3 of [4]. $L_q \cup \{q\}$ is a continuum. Thus, since $c \in L_q$ (recall that $c \neq q$), we see that L_q is uncountable. Note that, by 4.2, X is not aposyndetic at any point of L_q. Hence, by the assumption in our theorem, each point of L_q is a cut point of X. Thus, since L_q is uncountable, there are three points

$$r, \quad s, \quad t \in L_q$$

such that r and t are separated in X by s (6.29 (a) of [8]). This means that there are two mutually separated sets, E and F, such that

$$X - \{s\} = E \cup F \text{ with } r \in E \text{ and } t \in F.$$

Recall from the definition of L_q that $q \notin L_q$. Thus, since $s \in L_q$, $q \neq s$. Hence, $q \in E$ or $q \in F$, say $q \in E$. Now, note the following facts: $F \cup \{s\}$ is a continuum (6.3 of [8]); $F \cup \{s\}$ contains t in its interior in X (since E and F are mutually separated sets whose union is $X - \{s\}$ and $t \in F$); and $q \notin F \cup \{s\}$ (since $q \in E$ and $E \cap F = \emptyset$). These facts show that X is aposyndetic at t with respect to q. Thus, by 4.2, M is aposyndetic at t with respect to q. This contradicts that $t \in L_q$. Therefore, we have proved that M is aposyndetic at c with respect to q. As noted above, this proves 4.3.

As a consequence of 4.3, 3.1, and a result of Jones, we have the following result.

4.4 Corollary. *Let X be a continuum that is aposyndetic at each of its non-cut points. If X is finitely equivalent, then X is a graph. Furthermore, if X is n-equivalent, every subcontinuum of X is a Θ_{n+1}-continuum.*

Proof: By 4.3, X is aposyndetic at every point. Hence, by Theorem 4 of [3], X is slc at every point. Therefore, 4.4 now follows from 3.1.

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