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TIGHT GAPS IN $\mathcal{P}(\omega)$

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ABSTRACT. We prove some results on gaps in $\mathcal{P}(\omega)$. In particular we prove that the following statement is consistent.

($*$) Every $\subseteq^*$-increasing ω_1-sequence in $\mathcal{P}(\omega)$ is the bottom half of some tight (ω_1, ω_2^*)-gap.

1. INTRODUCTION

In this paper we prove the theorem stated in the abstract. It answers a question of P. Nyikos who derived various topological consequences of the statement ($*$), (see [N]).

Most of our set-theoretical notation is standard. Let $A = \{a_\alpha : \alpha \in \kappa\}$ be $\subseteq^*$-increasing and $B = \{b_\beta : \beta \in \lambda^*\}$ be $\subseteq^*$-decreasing sequence in $\mathcal{P}(\omega)$. We say that (A, B) is a (κ, λ^*)-pre-gap if $a_\alpha \subseteq^* b_\beta$ for $\alpha \in \kappa$ and $\beta \in \lambda$. ($a \subseteq^* b$ means that $a - b$ is finite). A pre-gap (A, B) is a gap if there is no $c \subseteq \omega$ which splits (A, B), i.e., such that $a_\alpha \subseteq^* c$ for $\alpha \in \kappa$ and $c \subseteq^* b_\beta$ for $\beta \in \lambda$. Generalizing slightly the definition in [N] we say that an infinite set $c \subseteq \omega$ is beside the pre-gap (A, B) if it is almost contained in every element of B but it is not in the ideal generated by A, i.e., $c - a_\alpha$ is infinite for all $\alpha \in \kappa$. A tight gap is a pre-gap with no set beside it. For every pre-gap (A, B) there is a natural splitting forcing associated with (A, B).

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$$\mathbb{P}_{(A,B)} = \{\langle x, y, s\rangle : x \text{ is a finite subset of } \kappa,\ y \text{ is a finite subsets of } \lambda\ s \in {}^{n}\{0,1\} \text{ and } \bigcup_{\alpha\in x}(a_\alpha - n) \subseteq \bigcap_{\beta\in y} b_\beta\}$$

Order $\mathbb{P}_{(A,B)}$ by: $\langle x_1, y_1, s_1\rangle \leq \langle x_2, y_2, s_2\rangle$ if $x_1 \supseteq x_2$, $y_1 \supseteq y_2$, $s_1 \supseteq s_2$ and for $i \in \mathrm{dom}(s_1) - \mathrm{dom}(s_2)$ if $i \in \bigcup_{\alpha\in x_2} a_\alpha$, then $s_1(i) = 1$ and if $i \notin \bigcap_{\beta\in y_2} b_\beta$, then $s_1(i) = 0$.

Note that if G is $\mathbb{P}_{(A,B)}$-generic, then $c = \bigcup\{s : \exists x, y(\langle x, y, s\rangle \in G)\}$ splits (A, B). We say that c is a generic splitting set for (A, B). We recall the following facts from [B]. If a pre-gap (A, B) is not a gap or its type is not (ω_1, ω_1^*), then $\mathbb{P}_{(A,B)}$ is ccc. There are (ω_1, ω_1^*)-gaps, e.g. the Hausdorff gap, such that the corresponding forcing is not ccc. Moreover for any (ω_1, ω_1^*)-gap (A, B) there is a ccc forcing which adds an uncountable antichain to $\mathbb{P}_{(A,B)}$. Note that if $\mathbb{P}_{(A,B)}$ is not ccc then (A, B) is an absolute gap, it can not be split without collapsing $\aleph_1$.

The paper is organised as follows. In section 2 we iterate splitting forcings and simultaneously build (ω_1, ω_2^*) gaps by induction. We use the method of Laver [L], the upper halfs of the gaps consist of the generic splitting sets. This is essential in proving that the iteration is ccc. In section 3 we prove a preservation lemma and as a result we prove that the statement $(*)$ is consistent with Martin's Axiom as well as with different values of the continuum.

2. The iteration of the splitting poset

Let $A = \{a_\eta : \eta \in \omega_1\}$ be $\subseteq^*$-increasing ω_1-sequence in $\mathcal{P}(\omega)$. In this section we prove the following theorem.

Theorem 1. *There is a ccc poset $\mathbb{P}$ which forces $\subseteq^*$-decreasing ω_2-sequence B, such that (A, B) is a tight (ω_1, ω_2^*)-gap. Moreover $\mathbb{P}$ is an iteration of forcings of cardinality $\aleph_1$.*

Proof: We define a finite support ccc iteration $\langle \mathbb{P}_\alpha; \mathbb{Q}_\alpha : \alpha \leq \omega_2\rangle$. Along with the iteration we build $B = \{b_\gamma : \gamma \in \omega_2\}$. For $\alpha \in \omega_2$ let $B_\alpha = \{b_\gamma : \gamma < \alpha\}$. The induction hypothesis at stage α is

$(*_\alpha)$ $\qquad (A, B_\alpha)$ is a pre-gap.

Suppose that $\mathbb{Q}_\gamma$ and b_γ have been defined for $\gamma < \alpha$. Let $\mathbb{Q}_\alpha$ be the splitting forcing $\mathbb{P}_{(A,B_\alpha)}$ and let b_α be a generic splitting set for (A, B_α). It is obvious that the induction hypothesis is satisfied. Hence , if $\mathbb{P}_{\omega_2}$ is ccc, then (A, B) is a pre-gap. Moreover since every real is in some intermediate model, a standard density argument shows that (A, B) is a tight gap.

To finish the proof of the theorem it is enough to show that $\mathbb{P}_{\omega_2}$ is ccc. This will be done in two steps. First we will show that for every $\alpha \leq \omega_2$ and $p \in \mathbb{P}_\alpha$ there is a stronger condition $\bar{p}$ which is determined. Then we show that every uncountable collection of determined conditions contains an uncountable subcollection of pairwise compatible conditions. Thus $\mathbb{P}_{\omega_2}$ has property K.

Lemma 1. *For all $\alpha \leq \omega_2$ if $p \in \mathbb{P}_\alpha$, $k \in \omega$ and $F \subseteq \alpha$ finite, then there is $\bar{p} \leq p$ in $\mathbb{P}_\alpha$ such that*

(1) *There is an $\bar{x} \in [\omega_1]^{<\omega}$ and $n \geq k$ such that: For each $\beta \in$supp$(\bar{p})$ there exist a $\bar{y}_\beta \in [\beta]^{<\omega}$ and an $\bar{s}_\beta \in {}^n\{0,1\}$ such that $\bar{p} \restriction_\beta \Vdash \bar{p}(\beta) = \langle \bar{x}, \bar{y}_\beta, \bar{s}_\beta \rangle$.*
(2) *For $\beta \in$supp$(\bar{p})$, $\bar{y}_\beta \subseteq$supp$(\bar{p})$.*
(3) *$F \subseteq$ supp$(\bar{p})$*

Proof: We proceed by induction on α. Suppose that the lemma is true for all $\beta < \alpha$. Let $p \in \mathbb{P}_\alpha$, $F \subseteq \alpha$ finite and $k \in \omega$. Let $\beta = \max\{\mathrm{supp}(p) \cup F\}$ and $r \leq p \restriction_\beta$, $r \in \mathbb{P}_\beta$ be such that r determines $p(\beta)$, i.e. $r \Vdash p(\beta) = \langle x_\beta, y_\beta, s_\beta \rangle$ for some $x_\beta \in [\omega_1]^{<\omega}$, $y_\beta \in [\beta]^{<\omega}$ and $s_\beta \in {}^l\{0,1\}$, $l \in \omega$. Let $E = y_\beta \cup (F \cap \beta)$ and let $m = \max\{k, l\}$. Applying the induction hypothesis to r, E and m we get $q \leq r$ in $\mathbb{P}_\beta$, $n \geq m$ and $x \in [\omega_1]^{<\omega}$ such that the conditions of the lemma are satisfied. For $\xi \in \mathrm{supp}(q)$ let $q(\xi) = \langle x, y_\xi, s_\xi \rangle$. We define $\bar{p}$ in the following way: $\mathrm{supp}(\bar{p}) = \mathrm{supp}(q) \cup \{\beta\}$. For $\xi \in \mathrm{supp}(\bar{p})$ we define $\bar{x}, \bar{y}_\xi, \bar{s}_\xi$ and then put $\bar{p}(\xi) = \langle \bar{x}, \bar{y}_\xi, \bar{s}_\xi \rangle$. Let $\bar{x} = x \cup x_\beta$ and for $\xi \in \mathrm{supp}(q)$ let $\bar{y}_\xi = y_\xi$ and $\bar{s}_\xi = s_\xi$. Now we define $\bar{p}(\beta)$. Let $\bar{y}_\beta = y_\beta$. Before we will define $\bar{s}_\beta$ let us recall that $s_\beta \in {}^l\{0,1\}$, $l \leq n$ and n is the domain of every s_ξ such that $\xi \in \mathrm{supp}(q)$. For $i \in n - l$ let $\bar{s}_\beta(i) = 0$ if there is $\eta \in y_\beta$

such that $s_\eta(i) = 0$. Otherwise let $\bar{s}_\beta(i) = 1$. This completes the definition of $\bar{p}$. It follows from the construction that the conditions of the lemma are satisfied.

To finish the proof of the lemma we have to show that $\bar{p} \in \mathbb{P}_\alpha$. Let $\{\xi_0, \dots, \xi_t\}$ be an increasing enumeration of $\operatorname{supp}(\bar{p})$. We show, by induction on $j \leq t$, that $\bar{p} \restriction_{\xi_j} \Vdash \bar{p}(\xi_j) \in \mathbb{Q}_{\xi_j}$. Let $j = 0$. Recall that $\bar{p}(\xi_0) = \langle \bar{x}, \bar{y}_{\xi_0}, \bar{s}_{\xi_0} \rangle$. We claim that $\bar{y}_{\xi_0} = \emptyset$. Note that $\bar{y}_{\xi_0} \subseteq \operatorname{supp}(\bar{p})$ by property (2). Now the claim follows from the minimality of ξ_0. By the definition of the splitting poset $\langle x, \emptyset, s \rangle$ is a condition for all x and s.

Suppose now that $\bar{p} \restriction_{\xi_j} \Vdash \bar{p}(\xi_j) \in \mathbb{Q}_{\xi_j}$. We prove the induction step. Assume first that $j + 1 < t$. We have to show that

$$\bar{p} \restriction_{\xi_{j+1}} \Vdash \bigcup_{\eta \in \bar{x}} (a_\eta - n) \subseteq \bigcap_{\gamma \in \bar{y}_{\xi_{j+1}}} b_\gamma$$

where a_η, b_γ are the names for the elements of the pre-gap we consider (by our construction b_γ is generic over $\mathbb{Q}_\gamma$). Note that $\bar{y}_{\xi_{j+1}} \subseteq \{\xi_0, \dots, \xi_j\}$ and by the induction hypothesis $\bar{p}(\xi_i) = \langle \bar{x}, \bar{y}_{\xi_i}, \bar{s}_{\xi_i} \rangle$ is an element of $\mathbb{Q}_{\xi_i}$. Hence it forces that $\bigcup_{\eta \in \bar{x}} (a_\eta - n) \subseteq b_{\xi_i}$. Thus $\bar{p} \restriction_{\xi_{j+1}}$ forces required property.

Finally we show that $\bar{p} \restriction_\beta \Vdash \bar{p}(\beta) \in \mathbb{Q}_\beta$ and $\bar{p}(\beta) \leq \langle x_\beta, y_\beta, s_\beta \rangle$. The proof that $\bar{p}(\beta)$ is an element of $\mathbb{Q}_\beta$ is similar to the one in the induction step from j to $j + 1$. Therefore we only show that $\bar{p}(\beta)$ is an extension of $\langle x_\beta, y_\beta, s_\beta \rangle$. It is enough to check that $\bar{p} \restriction_\beta$ forces that for $i \in n - l$ if $i \notin \bigcap_{\gamma \in y_\beta} b_\gamma$, then $\bar{s}_\beta(i) = 0$ and if $i \in \bigcup_{\gamma \in x_\beta} a_\gamma$, then $\bar{s}_\beta(i) = 1$. The first case easily follows from the definition of $\bar{s}_\beta$. Note that for $\eta \in \bar{y}_\beta$, s_η is the characteristic function of an initial part of b_η. As for the second case, $\bar{p} \restriction_\beta$ forces that $\langle x_\beta, y_\beta, s_\beta \rangle$ is an element of $\mathbb{Q}_\beta$, i.e. the following holds

$$\bigcup_{\eta \in x_\beta} (a_\eta - l) \subseteq \bigcap_{\gamma \in y_\beta} b_\gamma.$$

Hence $\bar{p}$ forces that if $i \in n - l$ and $i \in \bigcup_{\eta \in x_\beta} a_\eta$, then $s_\gamma(i) = 1$ for every $\gamma \in y_\beta$. It follows by the definition of $\bar{s}_\beta$, that $\bar{s}_\beta(i) = 1$ in this case. This completes the proof of the lemma.

We are ready to prove that $\mathbb{P}_{\omega_2}$ is ccc. Let $\{p_\alpha : \alpha \in \omega_1\} \subseteq \mathbb{P}_{\omega_2}$. For every $\alpha \in \omega_1$ let $\bar{p}_\alpha \leq p_\alpha$ be a condition satisfying conditions (1),(2) and (3) of the previous lemma. For $\xi \in \operatorname{supp}(\bar{p}_\alpha)$, put $\bar{p}_\alpha(\xi) = \langle \bar{x}^\alpha, \bar{y}^\alpha_\xi, \bar{s}^\alpha_\xi \rangle$ and let $\bar{n}^\alpha$ be the domain of $\bar{s}^\alpha$.

Using the Δ-lemma and thinning out if necessary we can assume that:

(1) There is $n \in \omega$ such that $n = \bar{n}^\alpha$ for all $\alpha \in \omega_1$.
(2) The collection $\{\operatorname{supp}(\bar{p}_\alpha) : \alpha \in \omega_1\}$ forms a Δ-system with the root Δ.
(3) $\bar{s}^\alpha_\xi = \bar{s}^\beta_\xi$ for $\xi \in \Delta$ and $\alpha, \beta \in \omega_1$.

We claim that any two conditions are compatible. Let $\alpha, \beta \in \omega_1$. We define p and then show by induction that $p \restriction_\xi \in \mathbb{P}_\xi$ and $p \restriction_\xi$ extends both $\bar{p}_\alpha \restriction_\xi$ and $\bar{p}_\beta \restriction_\xi$. Let $\operatorname{supp}(p) = \operatorname{supp}(\bar{p}_\alpha) \cup \operatorname{supp}(\bar{p}_\beta)$. For $\xi \in \operatorname{supp}(p)$ we define $p(\xi) = \langle x, y_\xi, s_\xi \rangle$. Let $x = \bar{x}^\alpha \cup \bar{x}^\beta$. To define y_ξ and s_ξ we have to distinguish three cases depending on whether ξ is in Δ or not.

(a) $\xi \in \Delta$. Let $y_\xi = \bar{y}^\alpha_\xi \cup \bar{y}^\beta_\xi$ and $s_\xi = \bar{s}^\alpha_\xi (= \bar{s}^\beta_\xi)$.
(b) $\xi \in \operatorname{supp}(\bar{p}_\alpha) - \Delta$. Let $y_\xi = \bar{y}^\alpha_\xi$ and $s_\xi = \bar{s}^\alpha_\xi$.
(c) $\xi \in \operatorname{supp}(\bar{p}_\beta) - \Delta$. Let $y_\xi = \bar{y}^\beta_\xi$ and $s_\xi = \bar{s}^\beta_\xi$.

This completes the definition of p. Let $\{\xi_i : i \in m\}$ be an increasing enumeration of $\operatorname{supp}(p)$. We show by induction on $i \in m$ that $p \restriction_{\xi_i} \Vdash p(\xi_i) \in \mathbb{Q}_{\xi_i}$ and $p \restriction_{\xi_i} \leq \bar{p}_\alpha \restriction_{\xi_i}, \bar{p}_\beta \restriction_{\xi_i}$. Notice that the latter is obvious since the functions s_ξ appearing in p are equal to the corresponding functions in $\bar{p}_\alpha$ or $\bar{p}_\beta$ and we extended only x_ξ and y_ξ for $\xi \in \operatorname{supp}(p)$.

We prove that $p \restriction_{\xi_i} \Vdash p(\xi_i) \in \mathbb{Q}_{\xi_i}$. Assume that $i = 0$. Since ξ_0 is minimal it follows that $y_{\xi_0} = \emptyset$ and thus $\langle x, \emptyset, s_{\xi_0} \rangle$ is a condition in $\mathbb{Q}_{\xi_0}$. Suppose now that the claim is true for $i < m$. We assume that $\xi_{i+1} \in \Delta$, the other two cases are similar. We have to show that

$$\bigcup_{\eta \in x_{\xi_{i+1}}} (a_\eta - n) \subseteq \bigcap_{\gamma \in y_{\xi_{i+1}}} b_\gamma$$

We proceed as in the proof of the lemma. Recall that $y_{\xi_{i+1}} \subseteq \{\xi_0, \ldots, \xi_i\}$ and for $l \leq i$, $p(\xi_l) = \langle x, y_{\xi_l}, s_{\xi_l} \rangle$ forces that $(a_\eta -$

$n) \subseteq b_{\xi_l}$ for all $\eta \in x$. Hence $p \upharpoonright_{\xi_{i+1}} \Vdash p(\xi_l) \in \mathbb{Q}_{\xi_l}$. This completes the proof.

Recall that t is the minimal cardinality of an increasing tower in $\mathcal{P}(\omega)$.

Theorem 2. *The statement $(*)$ is consistent with $t = \aleph_2$.*

Proof: We show how to adapt the proof of the previous theorem. We start with $V \models 2^{\aleph_1} = \aleph_2$ and define a finite support ccc iteration $\langle \mathbb{P}_\alpha; \mathbb{Q}_\alpha : \alpha \leq \omega_2 \rangle$. For $\beta \in \omega_2$ let $\{A^\beta_\rho : \rho \in \omega_2\}$ be a list of all nice $\mathbb{P}_\beta$-names for a $\subseteq^*$-increasing ω_1-sequence. Let a function $f : \omega_2 \times \omega_2 \to \mathcal{P}(\omega_2)$ be such that $\min f(\beta, \rho) \geq \beta$, $f(\beta, \rho)$ is cofinal in ω_2 and $f(\beta, \rho) \cap f(\alpha, \gamma) = \emptyset$ for $(\beta, \rho) \neq (\alpha, \gamma)$. Note that the order type of $f(\beta, \rho)$ is ω_2. We shall use $f(\beta, \rho)$ as an index set enumerating B^β_ρ, a $\subseteq^*$-decreasing ω_2-sequence corresponding to A^β_ρ. We define an auxiliary function $h : \omega_2 \to (\omega_2 \times \omega_2) \cup \{0\}$. At stage α of the iteration $h(\alpha)$ will tell which increasing sequence we should take care of. Let $h(\alpha) = (\beta, \rho)$ if there are β, ρ such that $\alpha \in f(\beta, \rho)$, otherwise let $h(\alpha) = 0$.

The induction hypothesis at stage $\alpha \in \omega_2$ is

$(*_\alpha)$ For $\beta, \rho \in \omega_2 (A^\beta_\rho, B^\beta_{\rho,\alpha})$ is a pre-gap in $V^{\mathbb{P}_\alpha}$, where $B^\beta_{\rho,\alpha} = \{b_\xi : \xi \in f(\beta, \rho) \cap \alpha\}$ and b_ξ is generic over $\mathbb{Q}_\xi$.

We define $\mathbb{Q}_\alpha$ depending on $h(\alpha)$. If $h(\alpha) = (\beta, \rho)$, then let $\mathbb{Q}_\alpha$ be $\mathbb{P}_{(A^\beta_\rho, B^\beta_{\rho,\alpha})}$. If $h(\alpha) = 0$ let $\mathbb{Q}_\alpha$ be a trivial forcing. Suppose that $(*_\alpha)$ holds for $\alpha < \lambda$. It is easy to see that $(*_\lambda)$ holds too. Hence it is enough to show that $\mathbb{P}_{\omega_2}$ is ccc. The proof of this fact is similar to the respective part of the proof of Theorem 1. Therefore we only state the key lemma and leave its proof to the reader.

Lemma 2. *For all $\alpha \leq \omega_2$ if $p \in \mathbb{P}_\alpha$, $k \in \omega$ and $F \subseteq \alpha$ finite, then there is $\bar{p} \leq p$ in $\mathbb{P}_\alpha$ and $n \geq k$ such that*

(1) *For all* $\beta \in supp(\bar{p})$ *if* $h(\beta) = (\alpha, \rho)$, *then* $\bar{p} \restriction_\beta \Vdash \bar{p}(\beta) = \langle \bar{x}_\beta, \bar{y}_\beta, \bar{s}_\beta \rangle$ *for some* $\bar{x}_\beta \in [\omega_1]^{<\omega}$, $\bar{y}_\beta \in [f(\alpha, \rho) \cap \beta]^{<\omega}$ *and* $\bar{s}_\beta \in^n \{0, 1\}$.
(2) *For* $\beta \in supp(\bar{p})$ *if* $\bar{p}(\beta) = \langle \bar{x}_\beta, \bar{y}_\beta, \bar{s}_\beta \rangle$, *then* $\bar{y}_\beta \subseteq supp(\bar{p})$.
(3) $F \subseteq supp(\bar{p})$
(4) *If* $\beta_1, \beta_2 \in supp(\bar{p})$ *are such that* $h(\beta_1) = h(\beta_2)$, *then* $\bar{x}_{\beta_1} = \bar{x}_{\beta_2}$.

To finish the proof of the theorem note that $2^{\aleph_0} = \aleph_2$ and $(*)$ imply that $t = \aleph_2$.

3. Preservation of gaps by forcing

In this section we prove that the statement $(*)$ is consistent with Martin's Axiom. We start with the following easy observation. Let $A = \{a_\alpha : \alpha \in \omega_1\}$ and $B = \{b_\alpha : \alpha \in \omega_2\}$ be such that (A, B) is a pre-gap. Then (A, B) is a tight gap if and only if for all $E \subseteq \omega_2$ of cardinality $\aleph_2$ the set $\bigcap_{\alpha \in E} b_\alpha$ is not beside the pre-gap (A, B).

Lemma 3. *Thight* (ω_1, ω_2^*)*-gaps are preserved by a finite support ccc iteration of posets of cardinality* $\aleph_1$.

Proof: Let $\langle \mathbb{P}_\alpha; \mathbb{Q}_\alpha : \alpha \in \kappa \rangle$ be a ccc iteration of posets of cardinality $\aleph_1$ and let (A, B) be a tight gap in the ground model V. We proceed by induction on κ. For successor stage, suppose that $\kappa = \gamma + 1$. We use the above characterization of tight gaps. Note that since $\mathbb{Q}_\gamma$ is a ccc forcing of cardinality $\aleph_1$ then for every $E \subseteq \omega_2$ of cardinality $\aleph_2$ in $V^{\mathbb{P}_\kappa}$ there is a set F of cardinality $\aleph_2$ in $V^{\mathbb{P}_\gamma}$ such that $F \subseteq E$. Obviously $\bigcap_{\alpha \in E} b_\alpha \subseteq \bigcap_{\alpha \in F} b_\alpha$ so by induction hypothesis we are done. Similar argument works for κ of cofinality ω, we use the following well-known fact.

Fact. *Let* $\langle \mathbb{R}_n : n \in \omega \rangle$ *be a finite support ccc iteration of length* ω. *In* $V^{\mathbb{R}_\omega}$ *let* $E \subseteq \omega_2$ *be a set of cardinality* $\aleph_2$. *Then there is* $n \in \omega$ *and a set* $F \in V^{\mathbb{R}_n}$ *such that* F *has cardinality* $\aleph_2$ *and* $F \subseteq E$.

Finally note that no new reals are added at stages of uncountable cofinality.

Theorem 3. *The statement* $(*)$ *is consistent with any of the following two conditions:*

(a) *MA*

(b) *continuum arbitrary large and* $t = \omega_2$.

Proof: To prove (a) we start with a model of $2^{\aleph_1} = \aleph_2$ and define a finite support ccc iteration of length ω_2 as follows. At even stages we consider a $\subseteq^*$-increasing ω_1-sequence A (given by some bookkeeping function) and force with $\mathbb{P}_{\omega_2}$ from Theorem 2. At odd stages we force with a ccc forcing of cardinality $\aleph_1$ (again given by some bookkeeping function). It is easy to see that $2^{\aleph_0} = \aleph_2$ and that MA holds in the resulting model. To show that the $(*)$ holds note that every ω_1-increasing sequence A appears at some intermediate stage. Therefore the next time we force with forcing from Theorem 2 we introduce an upper half B such that (A, B) is a tight gap. By Lemma 3 it follows that (A, B) remains to be a tight gap to the end of the iteration.

The proof of (b) is similar. We start with a model of $t = \aleph_2$ and define ccc iteration as above, i.e. at odd stages we force with arbitrary ccc forcing of cardinality $\aleph_1$ and at even stages we make sure that the condition $(*)$ holds in the resulting model. Now a tower of size $\aleph_2$ is preserved by the iteration, so $t \leq \aleph_2$ in the extension. On the other hand $(*)$ implies that $t \geq \aleph_2$. This finishes the proof since the iteration can be as long as we want.

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