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DENSE PERIODICITY ON FINITE TREES

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ABSTRACT. Those continuous functions from a finite tree into itself for which the set of periodic points is dense in the tree are described.

Introduction. We consider those continuous functions of a finite tree into itself for which the set of periodic points is dense in the tree. This generalizes the results of Barge and Martin [B-M] where they considered continuous mappings of an interval for which the set of periodic points is dense in the interval.

We show there is a collection (possibly finite or empty) $\{J_1, J_2, \dots\}$ of nondegenerate subcontinua of the tree and a positive integer N such that $f^N(J_i) = J_i$, and for all i, there is a point $x_i \in J_i$ such that $\{f^{Nk}(x_i) \mid k \geq 0\}$ is a dense set in J_i, and if $x \notin \cup_i J_i$ then $f^N(x) = x$. Thus, after we have taken care of the various "rotations" that may occur amongst various branches of the tree, we are left with regions where complicated dynamical behavior is occurring. In fact, in [R2, Theorem 3.4] it was shown that a mapping with a dense orbit on a finite tree also has sensitive dependence on initial conditions. Hence, $f^N|_{J_i}$ satisfies Devaney's [D] definition for being a chaotic map.

Definitions and Notation. By a finite tree we mean the connected union of finitely many arcs intersecting in their endpoints such that the space contains no loops, that is the union of the arcs is a hereditarily unicoherent continuum. A point

x is periodic of period n, where n is a positive integer, if $f^n(x) = f\big(f^{n-1}(x)\big) = x$. The point x has prime period n if for any positive integer k having the property that $f^k(x) = x$ then n divides k.

By a continuum we mean a compact connect metric space. A continuum X is decomposable if there exist proper subcontinua H and K of X such that $X = H \cup K$, otherwise X is indecomposable. Equivalently, X is decomposable if there is a proper subcontinuum of X having nonempty interior.

If X is a continuum and f is a mapping of X into X, we will denote the inverse limit space having f as its sole bonding map and X as the single base space by (X, f). That is,

$$\begin{aligned}(X, f) = \{(x_0, x_1, x_2, \dots) \mid x_i \in X \\ \text{and } f(x_{i+1}) = x_i \text{ for all } i \geq 0\}\end{aligned}$$

where the topology is generated by the metric

$$\mathrm{d}\big((x_0, x_1, \dots), (y_0, y_1, \dots)\big) = \sum_{i=0}^{\infty} |x_i - y_i|/2^i$$

and $|x_i - y_i|$ represents the metric on the continuum X. Throughout we will denote elements of the inverse limit space using underlined letters as in $\underline{x} = (x_0, x_1, \dots)$.

If $f\colon X \to X$ is a continuous surjection define the mapping $\hat{f}\colon (X, f) \to (X, f)$ by $\hat{f}\big((x_0, x_1, \dots)\big) = (f(x_0), f(x_1), \dots) = (f(x_0), x_0, x_1, \dots)$ is a homeomorphism and is sometimes referred to as the induced or shift homeomorphism. The functions $\pi_n\colon (X, f) \to X$ defined by $\pi_n\big((x_0, x_1, \dots)\big) = x_n$ are projection maps and are continuous. Finally, if H is a subset of a topological space X we denote the interior of a set H with respect to X by $\mathrm{int}(H)$ and the closure of H by $\mathrm{cl}(H)$.

Results. We begin by proving several lemmas which correspond to extreme cases. The main result, Theorem 5, is then proven by decomposing the general situation into these special cases.

Lemma 1. *Let T be a finite tree and suppose that $f : T \to T$ is continuous, the inverse limit space (T, f) is indecomposable and the set of periodic points of f is dense in T. Then there exists a point x in T such that $\{f^{nk+j}(x) | n \in \mathcal{Z}^+\}$ is dense in T for any positive integer k and any non-negative integer j.*

Proof: Let U be an open set in T and let J be the closure of a component of U. Suppose that $p \in \text{int}(J)$ is a periodic point of f having period r. Let $L = \text{cl}(\cup\{f^{nr}(J) | n \in \mathcal{Z}^+\})$. Since p is a fixed point of f^r, L is a subcontinuum of T.

Assume that $L \neq T$. If $f^r(T \backslash L) = T \backslash L$ then $f^{-r}(L) = L$. Since $f^r(L) = L$, $(L, f^r|_L)$ is a proper subcontinuum of (T, f^r) having nonempty interior. This implies that (T, f^r) is decomposable. But (T, f^r) is homeomorphic to (T, f) which was given to be indecomposable. This contradiction implies that there exists a point $y \in T \backslash L$ such that $f^r(y) \in L$. If in fact $f^r(y) \in \text{int}(L)$ then it follows then that there exists a periodic point $q \in T \backslash L$, close to y, such that $f^r(q) \in \text{int}(L)$. Suppose q has period m. Then $f^{rm}(q) = q$. But $f^{rm}(q) \in L$. This would contradict $q \in T \backslash L$. Hence, if $y \in T \backslash L$ and $f^r(y) \in L$, $f^r(y)$ must be an element of $L \backslash \text{int}(L)$. Thus f^{-r} is invariant on the interior of L. Again this implies that $(L, f^r|_L)$ is decomposable since $(L, f^r|_L)$ is a proper subcontinuum of (T, f^r) having nonempty interior. Hence we may conclude that $L = T$.

Let $A = \text{int}(T) \backslash \bigcup_{n \geq 0} \Big(f^n(\text{endpoints}(J)) \Big)$. Then A is dense in T since the set of endpoints of J is finite. From above we see that $A \subseteq \cup\{f^{nr}(J) | n \in \mathcal{Z}^+\}$. So for any $x \in A$ there exists a positive integer k such that $f^{-k}(x) \cap \text{int}(J) \neq \emptyset$. Thus given any open sets U and V of T there exists a positive integer k such that $f^{-k}(V) \cap U \neq \emptyset$. This implies that for any open set V in T, $\bigcup_{k \geq 0} f^{-k}(V)$ is dense in T. It follows then from a result of Auslander and Yorke, [A-Y, Lemma 3], that there is a point $x \in T$ such that $\{f^n(x) | n \in \mathcal{Z}\}$ is dense in T.

Since (T, f) is indecomposable, it follows from [R2, Theorem 4.1] that f^N has a dense orbit on T where N is the least common multiple of the integers less than or equal to the maximum order of any branch point in T. But then from [R2, Corollary 3.2] we have that $\{f^{nk+j}(x) | n \in \mathcal{Z}^+\}$ is dense in T for all $k > 0$ and $j \geq 0$ as claimed.

Lemma 2. *Suppose that the set of periodic points of F is dense in the finite tree T and that (T, f) contains no indecomposable subcontinua with nonempty interior. Then f^N is the identity where N is the least common multiple of the set of positive integers less than or equal to the number of endpoints of the tree T.*

Proof: If (T, f) contains no indecomposable subcontinua with nonempty interior then it follows from [R1, Theorems 2.13 and 2.17] that there is a monotone upper semicontinuous decomposition $\mathcal{G}$ of (T, f) such that the quotient space $G = (T, f)/\mathcal{G}$ is a finite tree having no more endpoints than T and every element of the decomposition has empty interior. Further, if $g_{\underline{x}}$ is the element of $\mathcal{G}$ containing the point $\underline{x}$ and $\hat{\hat{f}} : (T, f)/\mathcal{G} \to (T, f)/\mathcal{G}$ is defined by $\hat{\hat{f}}(g_{\underline{x}}) = g_{\hat{f}(\underline{x})}$, then $\hat{\hat{f}}$ is a homeomorphism.

If $\underline{x}$ is a periodic point of f having period m then

$$\underline{x} = (x, f^{m-1}(x), f^{m-2}(x), \dots, f(x), x, f^{m-1}(x), \dots)$$

is a periodic point of $\hat{f}$ in (T, f) of period m. Thus $\hat{\hat{f}}^m(g_{\underline{x}}) = g_{\hat{f}(\underline{x})} = g_{\underline{x}}$. So $g_{\underline{x}}$ is a periodic point of $\hat{\hat{f}}$ with period a divisor of m.

Let $\underline{y} = (y_1, y_2, \dots) \in (T, f)$. Given $\epsilon > 0$ choose M such that $\sum_{k=M}^{\infty} \text{diam } (T)/2^k < \epsilon/2$. Since the set of periodic points of f are dense in T there is a periodic point x, say of period n, such that $\sum_{k=1}^{M-1} |f^{M-k}(x) - y_k|/2^k < \epsilon/2$. Hence $\mathrm{d}(\underline{x}, \underline{y}) < \epsilon$ where $\underline{x} = (x, f^{n-1}(x), f^{n-2}(x), \dots)$. Thus the set of periodic points of $\hat{f}$ is dense in (T, f).

If $g_{\underline{y}}$ is a point in G and U is any open neighborhood of $g_{\underline{y}}$ then $p^{-1}(U)$ is open in (T, f), where p is the projection map onto the quotient space. Hence there is a periodic point $\underline{x}$ of $\hat{f}$

in $p^{-1}(U)$. Thus g_y is a periodic point of $\hat{\hat{f}}$ in U. So the set of periodic points of $\hat{\hat{f}}$ is dense in G. But G is a finite tree and $\hat{\hat{f}}$ is a homeomorphism. Thus $\hat{\hat{f}}^N$, where N is the integer given in the lemma's hypothesis, must fix all of the endpoints of the tree. It follows then that each edge of the tree is invariant under $\hat{\hat{f}}^N$ and the endpoints of each edge are fixed points of $\hat{\hat{f}}^N$. Since the periodic points of $\hat{\hat{f}}^N$ are dense in G, $\hat{\hat{f}}^N$ must be the identity mapping.

Let $g \in \mathcal{G}$ and $g_0 = \pi_0(g)$. If $x \in g_0$ then $f^N(x) \in g_0$ and $f^{-N}(x) \subseteq g_0$. This follows, since, if $y \notin g_0$ but $f^N(y) \in g_0$ then there is $H \in \mathcal{G}$ such that $\hat{f}^N(H) \neq H$. But $\hat{\hat{f}}^N(H) = H$ for all $H \in \mathcal{G}$ which implies that $\hat{f}^N(H) = H$. Thus $g = \{(x_0, x_1, \dots) \,|\, x_{Nk} \in g_0\}$. Now if g_0 is nondegenerate it must have nonempty interior being a subcontinuum of a finite tree. But g_0 cannot have nonempty interior for otherwise g would be an element of the decomposition $\mathcal{G}$ with nonempty interior contradicting [R1, Theorem 2.13]. Finally, $f^N(g_0) = f^N(\pi_N(g)) = \pi_0(\hat{f}^N(g)) = \pi_0(g) = g_0$. Thus f^N is the identity map as claimed.

In the previous lemma, if the number of endpoints of the tree is even moderately large then N will be quite large. In fact for a particular fixed mapping f, the endpoints of G form a permutation group under $\hat{\hat{f}}$. Thus N needs to be order of this permutation group. But if k is a positive integer less than or equal to the number of endpoints, E, of T then $\hat{\hat{f}}$ may fix $E-k$ of the endpoints and form a k cycle on the remaining. The order of this group would then be k. Thus the least common multiple of the integers less than or equal to E is the smallest number for which $\hat{\hat{f}}^N$ would fix all of the endpoints of G for any mapping f.

A finite collection C of subcontinua of a finite tree T is said to be non-overlapping provided $\text{int}(H) \cap \text{int}(K) = \emptyset$ for each two (distinct) members H and K of C. An element H of C is interior to T with respect to C if there are subcontinua K_1 and K_2 in C such that if M is the irreducible subcontinuum between K_1 and K_2 then $M \cap H$ has nonempty interior.

Lemma 3. *If T is a finite tree having n endpoints and C is a collection of at least $n+1$ non-overlapping subcontinua of T, then there is an element of C which is interior to T with respect to C.*

Proof: Let $\hat{T}$ be the minimal subtree of T containing C obtained by taking the intersection of all subcontinua of T which contain C. The tree $\hat{T}$ has at most n endpoints. Each of these endpoints is contained in exactly one of the elements of C. Since the number of elements of C is greater than n, at least one element of C does not contain an endpoint of $\hat{T}$. Let H be such an element. Then the closure of $\hat{T} \setminus H$ contains at least two components. Choose elements of C from two different components. The intersection of the irreducible subcontinuum between these elements of C and H has nonempty interior. Hence H is interior to T with respect to C.

Lemma 4. *Suppose that T is a finite tree with n endpoints, $f : T \to T$ is a continuous surjection, $C = \{H_0, H_1, \ldots, H_k\}$ is a collection of non-overlapping subcontinua of T, $k \geq n$, and $f(H_i) = H_{i+1}(mod\ k)$, then there are integers, $0 \leq i, j, s \leq k$ such that the irreducible subcontinuum M between H_i and H_j does not intersect any other element of C and $f(M) \cap H_s$ has nonempty interior.*

Proof: Let $\hat{T}$ be the minimal subtree of T containing C, as in the proof of Lemma 3. Suppose H_e is an element of C which contains an endpoint of $\hat{T}$. Let C_1 be the subset of C containing H_e and any $H_i \in C$ with the properties that, (i) if M is an irreducible subcontinuum between H_e and H_i then $\text{int}(M \cap H_j) = \emptyset$ for all j and (ii) H_i is not interior to $\hat{T}$ with respect to C. It follows from the unicoherence of T that there is a unique H_s in C such that if M is the irreducible subcontinuum between H_s and H_e then $\text{int}(M \cap H_j) = \emptyset$ for all j and H_s is interior to $\hat{T}$ with respect to C. Finally, let $C_2 = C \backslash (C_1 \cup \{H_s\})$.

Suppose that there is an $H_i \in C_1$ such that $f(H_i) = H_s$. C_2 is not f invariant. Thus there is an $H_j \in C_2$ such that $f(H_j) \in C_1$. If there is an $H_k \in C_2$ such that $f(H_k) \in C_2$ then let K be the irreducible subcontinuum between H_j and H_k. The image of K contains H_s. By assumption none of the elements of C_2 are mapped onto H_s. Thus there is a subcontinuum between elements of C_2 having the desired property and the lemma follows. Otherwise, $f(H_j) \in C_1$ for all $H_j \in C_2$.

Suppose that there does not exist an element of C_1 which f maps into C_2. Then C_2 has only one element and $f(H_s) \in C_2$. The subcontinuum between H_s and the element of C_2 is the desired subcontinuum M and the lemma follows. Otherwise there exists an element of C_1 which is mapped into C_2 by f.

If there is an $H_k \in C_1$ such that $f(H_k) \in C_1$, then there is a subcontinuum between elements of C_1 which f maps as desired and the lemma follows. Otherwise $f(H_k) \in C_2$ for all $H_k \in C_1 \backslash \{H_i\}$.

Suppose that $f(H_s) \in C_1$. Then the subcontinuum between H_s and one of the elements of C_1 is the desired subcontinuum M. If, on the other hand, $f(H_s) \in C_2$, then the subcontinuum between H_s and an element of C_2 is the one required. Thus the lemma is true if an element of C_1 maps onto H_s.

The case where there is an $H_j \in C_2$ such that $f(H_j) = H_s$ may be argued similiarly.

Theorem 5. *Suppose that $f: T \to T$ is continuous, that the set of periodic points of f is dense in T and N is the least common multiple of the positive integers less than or equal to the number of endpoints of the tree T. Then there is a collection (perhaps finite or empty) $\{J_1, J_2, \dots\}$ of subcontinua of T having mutually disjoint interiors such that (i) $f^N(J_i) = J_i$, (ii) for each i, there is a point $x_i \in J_i$ such that $\{f^{Nk}(x_i) \mid k \geq 0\}$ is dense in J_i, and (iii) if $x \in T \backslash \cup J_i$, then $f^N(x) = x$*

Proof: Suppose H is an indecomposable subcontinuum of (T, f) having nonempty interior. Since the periodic points of f are dense in T, there is a point $\underline{x}$ in $\text{int}(H)$ which is a periodic point of the shift homeomorphism $\hat{f}$. Suppose that the period of $\underline{x}$ is k. The set $H \cup \hat{f}^k(H)$ is a subcontinuum of H. If it were a proper subcontinuum, then H would be decomposable contrary to our assumption. Hence $\hat{f}^k(H) = H$. Let $H_i = \pi_i(H)$, the projection of H onto the i^{th} coordinate. Then $f^k(H_0) = f^k\big(\pi_0(H)\big) = \pi_0\big(\hat{f}^k(H)\big) = \pi_0(H) = H_0$.

Consider the collection of subcontinua $C = \{H_0, H_1, \ldots, H_{k-1}\}$. We have that $f(H_i) = H_{i+1}(\text{mod } k)$. If for $i \neq j$, $\text{int}(H_i) \cap \text{int}(H_j) \neq \emptyset$ then $\hat{f}^i(H) \cap \hat{f}^j(H)$ is a subcontinuum of $\hat{f}^i(H)$ having nonempty interior. This contradicts H being indecomposable. Hence, the elements of C are non-overlapping.

Suppose that k is greater than the number of endpoints of the tree T. By Lemma 4, there are distinct integers i, j, s, $0 \leq i, j, s \leq k$ such that the irreducible subcontinuum M between H_i and H_j has the properties that $\text{int}(M \cap H_i) = \emptyset$ for $0 \leq i \leq k$ and $\text{int}\big(f(M) \cap H_s\big) \neq \emptyset$. Hence there is a periodic point y, $y \notin \cup C$ such that $f(y) \in H_s$. This contradicts f being forward invariant on C. Thus the number of elements of C is at most the number of endpoints of T and there is a positive integer l less than the number of endpoints of T such that $f^{lk}(H_i) = H_i$ for $0 \leq i \leq l$.

Next let $\{H^1, H^2, \ldots\}$ be the collection of all indecomposable subcontinua of (T, f) which have nonempty interiors. For each i, let $J_i = \pi_0(H^i)$. It follows as before that $\text{int}(J_i) \cap \text{int}(J_j) = \emptyset$ whenever $i \neq j$.

From the previous argument there is a positive integer $n(i)$ which is less than the number of endpoints of the tree T such that $f^{n(i)}(J_i) = J_i$. Let N be the least common multiple of the positive integers less than or equal to the number of endpoints of T. Then $f^N(J_i) = J_i$ for all i. Since $(J_i, f|_{J_i})$ is homeomorphic to H^i, we may apply Lemma 1 to obtain the existence of a point $x_i \in J_i$ such that $\{f^{Nk}(x_i) \mid k \in \mathcal{Z}^+\}$ is dense in J_i.

Let J be the closure of $\cup J_i$. If $x \in J \backslash \cup J_i$ then there is a sequence of subcontinua J_i converging to x and each J_i is forward invariant under f^N. Hence $f^N(x) = x$. Therefore, if $J = T$ the proof is complete.

Suppose that $J \neq T$. Let K_α be the collection of components of J. Define the decomposition $\mathcal{D}$ of T such that the only nondegenerate points of the decomposition are the components K_α. Let T^* be the decomposition space $T \backslash \mathcal{D}$ with the quotient topology. Then T^* is a finite tree with no more endpoints than T. Define $g\colon T^* \to T^*$ by $g([x]) = [f(x)]$, where $[x]$ denotes the point in T^* which corresponds to the element in $\mathcal{D}$ that contains x.

Let $P\colon T \to T^*$ be the projection map $P(x) = [x]$. Then P is a continuous monotone closed surjective map and $g \circ P = P \circ f$. Define $\hat{P}\colon (T, f) \to (T^*, g)$ by $\hat{P}(x_0, x_1, x_2, \dots) = \Big(P(x_0), P(x_1), P(x_2), \dots\Big)$. Then $\hat{P}$ is easily seen to be a continuous monotone closed surjective map.

If (T^*, g) contains no indecomposable subcontinua with nonempty interior then the theorem follows from Lemma 2. Suppose that (T^*, g) contains an indecomposable subcontinuum H having nonempty interior. Then $\hat{P}^{-1}(H)$ is a subcontinuum of (T, f). By the Brouwer Reduction Theorem (a version of Zorn's Lemma that does not require the Axiom of Choice because of the compactness of (T, f)) there is a minimal subcontinuum K of (T, f) such that $\hat{P}(K) = H$.

Suppose K is decompoable. Then there exists proper subcontinua A and B of K such that $K = A \cup B$. Thus $H = \hat{P}(A) \cup \hat{P}(B)$. But $\hat{P}(A)$ and $\hat{P}(B)$ are subcontinua of H and H is indecomposable. This implies that either $\hat{P}(A)$ or $\hat{P}(B)$ is H, contradicting K being a minimal subcontinuum of $\hat{P}^{-1}(H)$ having this property. Thus K must be indecomposable.

Let $C = \{\underline{x} \in (T^*, g) \,|\, x_n \in P(J) \text{ for all } n \geq 0\}$. The set C is closed. To see this suppose that $\underline{x} \in \overline{C} \backslash C$. Then there is a positive integer n such that $\pi_n(\underline{x}) \notin P(J)$. The set $P(J)$ is closed so there exists $\epsilon > 0$ such that the open ball, $B_\epsilon\Big(\pi_n(\underline{x})\Big)$, does not intersect $P(J)$. But for all $\delta > 0$, $B_\delta(\underline{x}) \cap C \neq \emptyset$. We may choose δ so that if $\underline{y} \in B_\delta(\underline{x}) \cap C$ then $\mathrm{d}\Big(\pi_n(\underline{x}), \pi_n(\underline{y})\Big) < \epsilon$. This contradiction implies that $\overline{C} \backslash C = \emptyset$. Thus C is closed. Next let $\underline{x} \in C$. Since $P(J)$ has empty interior, for any $\epsilon > 0$ and for all positive integers n and for all $\delta_n > 0$ there is a point $y_n \in T^* \backslash P(J)$ such that $\mathrm{d}\Big(\pi_n(\underline{x}), y_n\Big) < \delta_n$. Choose N such that $\sum_{i>N} \mathrm{diam}(T^*)/2^i < \epsilon/2$ and choose δ_N such that if $\mathrm{d}\Big(\pi_N(\underline{x}), y_N\Big) < \delta_N$ then $\mathrm{d}\Big(\pi_i(\underline{x}), f^{N-i}(y_N)\Big) < \epsilon/4$ if $0 \leq i \leq N$. Let $\underline{y} \in \pi_N^{-1}(y_N)$ then $\mathrm{d}(\underline{x}, \underline{y}) < \epsilon$. Thus C has empty interior.

Then $\mathrm{int}(H) \cap \Big((T^*, g) \backslash C\Big)$ is a nonempty open subset of (T^*, g). But each point in $T^* \backslash P(J)$ has a unique preimage under P and it follows that each point in $(T^*, g) \backslash C$ has a unique preimage under $\hat{P}$. Therefore $\hat{P}^{-1}\Big(\mathrm{int}(H) \cap (T^*, g) \backslash C\Big)$ is an open subset of K.

Thus K is an indecomposable subcontinuum of (T, f) with nonempty interior. So $K = H^i$ for some i. It follows that $\hat{P}(K) = H$ is a single point, contradicting H having nonempty interior and the theorem is proven.

Results similar to Theorem 5, obtained using different methods, appear in [B1] and [B2].

The following lemma follows from [R, Lemma 3.3]. As previously in this paper, N will denote the least common multiple of the integers less than or equal to the number of endpoints of T.

Lemma 6. *The function f^N has a dense orbit on T if and only if for any subcontinua J and K of T with int$(J) \neq \emptyset$ and $K \subseteq$ int(T) there is a positive integer M such that for all $n \geq M$, $K \subseteq f^n(J)$.*

Lemma 7. *If f^N has a dense orbit, then f has a periodic point of period p, where p is any prime larger then the integer M of Lemma 6.*

Proof: Let J and K be nondegenerate intervals containing no branch points of T which are contained in the interior of T and such that $J \cap K = \emptyset$. It follows from Lemma 6 that there is a positive integer M such that if $n \geq M$, then $f^n(J) \supset K$ and $f^n(K) \supset J$. Let p be a prime that is larger than $2M + 2$, $r = (p-1)/2$, and $s = (p+1)/2$ then $r, s > M$ and $r + s = p$.

There is a subinterval J_1 of J such that $f^r(J_1) = K$. Then $f^{r+s}(J_1) \supset J$. It follows from [A-Y, Lemma 2] that there is a periodic point of f in J_1 having period $r + s = p$. Since $J \cap K = \emptyset$ this point has prime period p.

We now have the following corollary to Theorem 5.

Corollary 8. *Suppose that the periodic points of f are dense and if x is a periodic point of f then the prime period of x is a divisor of N. Then f is a periodic homeomorphism with $f^N(x) = x$ for all x in T.*

Proof: Suppose (T, f) contains an indecomposable subcontinuum H with nonempty interior. Then there is an integer $n \leq N$ such that $f^n(H) = H$. Let $g = f^N|_H$, it follows that (H, g) is indecomposable and the set of periodic point of g is dense in H. By Lemma 1 there is a point x in H which has a dense orbit under g. By Lemma 7, g has a periodic point with period $p > N$ where p is prime. But then p does not divide N. Therefore (T, f) contains no indecomposable subcontinua with nonempty interior and thus the set of subcontinua $\{J_1, J_2, \dots\}$ of Theorem 5 is empty and the corollary follows.

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