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BORGES NORMALITY AND GENERALISED METRIC SPACES

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ABSTRACT. Borges normality is a strong normality condition abstracted from Borges' proof that stratifiable spaces have the Dugundji extension property. We use this, together with the Collins-Roscoe mechanism, to provide characterisations of Nagata, metrisable, proto-metrisable, linearly stratifiable and ω_μ-metrisable spaces.

1. INTRODUCTION AND BASIC DEFINITIONS

In his proof [2] that stratifiable spaces have the Dugundji extension property, Borges used the following property of stratifiable spaces (which, combined with hereditary paracompactness, he showed, is enough to obtain the Dugundji extension property):

Definition 1.1. *A space X is said to be* Borges Normal (BN) *if for each $x \in X$ and open U containing x there is an open set $H(x,U)$ containing x and a natural number $n(x,U)$ such that, if $H(x,U) \cap H(y,V) \neq \emptyset$ and $n(x,U) \leq n(y,V)$, then $y \in U$.*

In [25, Theorem 2.1] we proved that BN is equivalent to the condition decreasing (A) of Collins and Roscoe [6]. In this paper we use this to find new characterisations of Nagata spaces and metrisable spaces. We then consider cardinal generalisations of Borges normality and in the process find new characterisations of proto-metrisable spaces, linearly stratifiable spaces and ω_μ-metrisable spaces, in terms of Collins-Roscoe type conditions. The characterisation of proto-metrisable spaces partially answers a question of Gartside and Moody [9]. This work represents part of a thesis accepted by Oxford University in partial fulfillment of the requirements for the degree of Doctor of Philosophy. Part of this work was presented at the 1993 Spring Topology Conference in Columbia, SC. The author is very grateful to Dr. P. Moody for suggesting Borges normality as an object of study and also Dr. P. Collins for his help and support. Before proceeding we review the Collins-Roscoe mechanism. All spaces are assumed to be T_1 unless otherwise stated. Any undefined notions can be found in either [7], [10] or [15].

Recall [6], if X is a space and $\mathcal{W} = \{\mathcal{W}(x) : x \in X\}$, where $\mathcal{W}(x)$ is a collection of sets of the form $\mathcal{W}(x) = \{W(n,x) : n \in \omega\}$, and $x \in W(n,x) \subseteq X$ for all x and n, then we say that $\mathcal{W}$ satisfies (A) if,

(A) given $x \in U$ open in X, there is an open $V = V(x,U)$ containing x and a natural number $s = s(x,U)$ such that $y \in V$ implies $x \in W(s,y) \subseteq U$.

We say that X satisfies (A) if X has a family $\mathcal{W}$ satisfying (A). If, in addition, $W(n+1,x) \subseteq W(n,x)$ for all $x \in X$ and for all $n \in \omega$, then we say X satisfies decreasing (A). We say that X satisfies open [neighbourhood] (A) if each of the $W(n,x)$ is open [a neighbourhood of x].

Other variants of the mechanism have also been studied (see for instance [4, 5, 6, 8, 17, 18]). Of most importance are the following:

(G) given $x \in U$ open in X, there is an open $V = V(x,U)$ containing x such that $y \in V$ implies $x \in W(s,y) \subseteq U$ for some natural number s.

In a sense, (A) is a uniform version of (G), in that the same number s will work for all $y \in V$, whereas in (G) s may vary with y. If the $\mathcal{W}(x)$ are not countable, and each $\mathcal{W}(x)$ is merely a family of subsets of X containing x, then we have the following generalisation of (G):

(F) given $x \in U$ open in X, there is an open $V = V(x, U)$ containing x such that $y \in V$ implies $x \in W \subseteq U$ for some $W \in \mathcal{W}(y)$.

In the same way as for (A), we can put conditions on the elements of the $\mathcal{W}(x)$. We say X satisfies well-ordered (F) if each of the $\mathcal{W}(x)$ is well-ordered under reverse inclusion. Properties like open (F) have the obvious definitions. In passing, we note that all spaces satisfy (F), so, we are actually obliged to place extra restrictions in this case. It is clear that all of the mechanism conditions we have described are hereditary. We sometimes say, instead of X satisfies decreasing (A) etc., that X has decreasing (A) or X is a decreasing (A) space. The following theorems have been proved concerning the mechanism and are used in the sequel.

Theorem 1.2. ([5,6]) *The following are equivalent for a space X:*

(i) *X is metrisable;*
(ii) *X satisfies neighbourhood decreasing (A);*
(iii) *X satisfies open decreasing (A);*
(iv) *X satisfies open decreasing (G).*

Theorem 1.3. ([5]) *If the space X has $\mathcal{W}$ satisfying well-ordered (F), then X is paracompact and monotonically normal.*

A *pair-base* for a space X is a collection $\mathbb{P}$ of pairs, $\mathbb{P} = \{(P_1, P_2) : P_i \text{ open in } X \text{ and } P_1 \subseteq P_2\}$ such that, for all $x \in U$ with U open, there exists $(P_1, P_2) \in \mathbb{P}$ such that $x \in P_1 \subseteq P_2 \subseteq U$. A collection of pairs $\mathbb{P}$ as above is said to be a *rank one* collection if whenever $(P_1, P_2), (Q_1, Q_2) \in \mathbb{P}$ are such that $P_1 \cap Q_1 \neq \emptyset$, then either $P_1 \subseteq Q_2$ or $Q_1 \subseteq P_2$.

Definition 1.4. **([11])** *A space X is* proto-metrisable *if and only if X has a rank one pair-base.*

As mentioned above, the author has proved the following result.

Theorem 1.5. **([25])** *X has decreasing (A) if and only if X is BN.*

In Section 2 we shall use this result to show that metrisability is equivalent to a stronger version of Borges normality (Theorem 2.9). It is known that stratifiable spaces are precisely the decreasing (A) spaces with countable pseudocharacter. By considering cardinal generalisations of Borges normality we provide a cardinal analogue of this for linearly stratifiable spaces (Theorem 5.2) as well as a generalisation of Theorem 1.2 to ω_μ-metrisable spaces (Theorem 6.1). Gartside and Moody asked whether the proto-metrisable spaces are those spaces which satisfy well-ordered open (F). By considering an alternative cardinal generalisation of Borges normality we partially answer this question by showing that the proto-metrisable spaces are those spaces satisfying well-ordered open (Z) (Theorem 7.3) where (Z) is a uniform version of (F) (in the same sense that (A) is a uniform version of (G) as previously discussed).

By Theorem 1.3, Borges normal spaces are hereditarily paracompact (and therefore satisfy the Dugundji extension property [25]). It is clear that Borges normality is a strengthening of monotone normality. Indeed, Borges normality is merely monotone normality with natural numbers deciding which of $x \in V$ or $y \in U$ will occur when $H(x,U) \cap H(y,V) \neq \emptyset$ (see Theorem 5.19 in [10]). Furthermore, Borges normal spaces are acyclic monotonically normal (see [18]). We can therefore consider decreasing (A) as a normality condition as well as a condition on local networks in the spirit of the Collins-Roscoe mechanism.

Like monotone normality, decreasing (A) is not productive: if $X = A(\omega_1)$ is the one point compactification of the discrete space of size ω_1, then X satisfies decreasing (A) (see Example 2.14) but X^2 is not even monotonically normal since X contains a convergent sequence but is not stratifiable (Theorem 4.1, [12]). However, in comparison with monotone normality, decreasing (A) is ill-behaved. Although, like monotone normality, decreasing (A) is hereditary, it is unstable under various constructions which preserve monotone normality. Decreasing (A) is not preserved by scattering (see Section 4) since the Michael Line does not satisfy decreasing (A) [6], nor is it preserved by duplication (the duplicate of the closed unit interval contains a copy of the Michael line). It is unknown whether like monotone normality, decreasing (A) is preserved by closed maps or even perfect maps. There is one aspect, however, in which it is better behaved than monotone normality. Rudin has shown [23] that there is a locally compact, monotonically normal space the one point compactification of which is not monotonically normal. However, for decreasing (A), the author has shown:

Theorem 1.6. **([24])** *If X is a locally compact space satisfying decreasing (A), then X^*, the one point compactification of X, satisfies decreasing (A).*

2. Metrisation

Theorem 2.1. ([1], [17]) *The following are equivalent for a space X*

(1) *X is stratifiable;*
(2) *X has countable pseudocharacter and satisfies decreasing (A);*
(3) *X has countable pseudocharacter and satisfies decreasing (G).*

Since Ceder has shown [3] that Nagata spaces are exactly the first countable stratifiable spaces, we have as an immediate corollary of this result.

Corollary 2.2. *A space X is a Nagata space if and only if X is a first countable decreasing (A) space.*

So, given our alternative characterisation of decreasing (A), we can prove:

Theorem 2.3. *A space X is a Nagata space if and only if for each $x \in X$ there exist sequences of neighbourhoods of x, $\{U_n(x)\}_{n=1}^{\infty}$ and $\{V_n(x)\}_{n=1}^{\infty}$ and natural numbers $\{p(n,x)\}_{n=1}^{\infty}$ such that for all $x, y \in X$,*

(1) *$\{U_n(x)\}_{n=1}^{\infty}$ is a local neighbourhood base at x,*
(2) *$V_n(x) \cap V_m(y) \neq \emptyset$ and $p(n,x) \leq p(m,y)$ implies $y \in U_n(x)$.*

Proof: If X is a Nagata space, then clearly it satisfies the conditions of the Theorem. As for the converse, if X satisfies the conditions above, X is clearly first countable and, if $x \in U$ open, pick $n \in \omega$ such that $x \in U_n(x) \subseteq U$ and let $H(x, U) = V_n(x)$ and $n(x, U) = p(n, x)$. It is easy to check that these define BN operators and hence, by Corollary 2.2 and Theorem 1.5, X is Nagata. □

The original definition of Nagata spaces was abstracted from the following theorem:

Theorem 2.4. [19] *A T_1 space X is metrisable if and only if for each $x \in X$ there exist sequences of neighbourhoods of x, $\{U_n(x)\}_{n=1}^{\infty}$ and $\{V_n(x)\}_{n=1}^{\infty}$, such that for all $x, y \in X$,*

(1) *$\{U_n(x)\}_{n=1}^{\infty}$ is a local neighbourhood base at x,*
(2) *$V_n(x) \cap V_n(y) \neq \emptyset$ implies $x \in U_n(y)$,*
(3) *$y \in V_n(x)$ implies $V_n(y) \subseteq U_n(x)$.*

So, can we, in a similar fashion, add a third condition to Theorem 2.3 to get a new metrisation theorem? We have:

Theorem 2.5. *A space X is metrisable if and only if for each $x \in X$ there exist sequences of neighbourhoods of x, $\{U_n(x)\}_{n=1}^{\infty}$ and $\{V_n(x)\}_{n=1}^{\infty}$ and natural numbers $\{p(n,x)\}_{n=1}^{\infty}$ such that for all $x, y \in X$,*

(1) $\{U_n(x)\}_{n=1}^{\infty}$ *is a local neighbourhood base at* x,
(2) $V_n(x) \cap V_m(y) \neq \emptyset$ *and* $p(n,x) \leq p(m,y)$ *implies* $y \in U_n(x)$,
(3) $y \in V_n(x)$ *and* $p(n,x) \leq p(m,y)$ *implies* $V_m(y) \subseteq U_n(x)$.

Proof: For necessity, define $U_n(x) = B_n(x)$, $V_n(x) = B_{3n}(x)$ and $p(n,x) = n$ for all $x \in X$ and $n \in \omega$. (Where $B_n(x)$ is the $1/n$-ball centred at x.)

As for sufficiency, by Theorem 1.2, it is enough to show that X satisfies neighbourhood decreasing (A). Note also that without loss of generality the $V_n(x)$ form a decreasing sequence and the $p(n,x)$ an increasing sequence for each x. For $a \neq x$ define,

$$n_a(x) = \begin{cases} 0 & \text{if } A = \emptyset \\ \max A & \text{otherwise,} \end{cases}$$

where $A = \{p(n,x) : a \in V_n(x)\}$. Note that for some $n \in \omega$, $x \notin U_n(a)$, therefore A is bounded above by $p(n,a)$ by condition (2). Also, without loss of generality, $p(n,x) \geq 1$ for all x and n. Now define $p_a(x) = \max\{n_a(x)\,,\, n_x(a)\}$ for $x \neq a$ and let $\mathcal{W}(a) = \{W(n,a) : n \in \omega\}$ where

$$W(n,a) = \{a\} \cup \{y \in X : p_a(y) \geq n\}.$$

Obviously we have that $W(n+1,a) \subseteq W(n,a)$ for all $n \in \omega$ and for all $a \in X$. We claim that $\mathcal{W} = \{\mathcal{W}(a) : a \in X\}$ satisfies decreasing (A). Take $x \in U$ where U is an open set in X. Let n be minimal such that $x \in U_n(x) \subseteq U$ (by (1)) and then, let $V = V_n(x)^\circ$ and $s = p(n,x)$. Then, if $a \in V$, we need to show that $x \in W(s,a) \subseteq U$. If $x \neq a$ then $n_a(x) \geq p(n,x)$, since $a \in V_n(x)$, and hence $p_a(x) \geq p(n,x)$. Otherwise $x = a$. In either case $x \in W(p(n,x),a)$. Now $y \in W(p(n,x),a)$ implies that either $y = a \in U$, or $p_a(y) \geq p(n,x) \geq 1$ and therefore we have:

Case 1. $n_a(y) \geq p(n,x)$, in which case there exists $m \in \omega$ such that $a \in V_m(y)$ and $p(m,y) = n_a(y)$. Therefore $a \in V_n(x) \cap V_m(y)$ and $p(m,y) \geq p(n,x)$ which implies $y \in U_n(x) \subseteq U$ by (2).

Case 2. $n_y(a) \geq p(n,x)$, in which case there exists $m \in \omega$ such that $y \in V_m(a)$ and $p(m,a) = n_y(a)$. Now $a \in V_n(x)$ and $p(m,a) \geq p(n,x)$ therefore $V_m(a) \subseteq U_n(x)$ by (3) and hence $y \in U$.

It remains to show that each $W(n,a)$ is a neighbourhood of a. Now, first assume that there exists $y \in X$ such that $p_a(y) \geq n$. Choose such a y with $p_a(y)$ minimal. Now since X is T_1 there is an $m \in \omega$ such that $y \notin U_m(a)$. We claim that $a \in V_m(a) \subseteq W(n,a)$. So take $x \in V_m(a)$.

Case 1. $p_a(y) = n_a(y)$. In this case $a \in V_r(y)$ where $p_a(y) = p(r,y)$. Therefore $a \in V_m(a) \cap V_r(y)$ but $y \notin U_m(a)$ and hence $p(m,a) > p(r,y)$ by (2). However, $x \in V_m(a)$ and therefore $p_a(x) \geq p(m,a)$ and thus $p_a(x) > p_a(y) \geq n$ and hence $x \in W(n,a)$.

Case 2. $p_a(y) = n_y(a)$. So $y \in V_r(a)$ where $p_a(y) = p(r,a)$. Since $y \notin U_m(a)$, $V_r(a) \not\subseteq V_m(a)$ and therefore $m > r$. Hence, $p(m,a) \geq p(r,a) \geq n$ which again implies $p_a(x) \geq n$.

We now consider the case when $p_a(y) < n$ for all $y \in X$. We claim that, in this case, a is isolated. Firstly there is a $y \in X$ such that $p_a(y) \geq 1$ else $V_m(a) = \{a\}$ for all $m \in \omega$. So pick y such that $p_a(y)$ is maximal and pick $m \in \omega$ such that $y \notin U_m(a)$. Take $x \in V_m(a)^\circ \setminus \{a\}$.

Case 1. $p_a(y) = n_a(y)$. So $a \in V_r(y)$ where $p_a(y) = p(r,y)$ and therefore $V_m(a) \cap V_r(y) \neq \emptyset$. Hence, $p(m,a) > p(r,y)$ (else $y \in U_m(a)$ by (2)) which implies $p_a(x) > p_a(y)$, a contradiction.

Case 2. $p_a(y) = n_y(a)$. So $y \in V_r(a)$, where $p_a(y) = p(r,a)$, and $a \in V_m(a)$. However, $V_r(a) \not\subseteq U_m(a)$ so $p(m,a) > p(r,a)$ by (3), which is also a contradiction.

So, in either case, $V_m(a)^\circ \setminus \{a\} = \emptyset$ and hence a is isolated. We have therefore shown that each of the $W(n,a)$ is a neighbourhood of a and the proof is complete. □

Corollary 2.6. *A space X is metrisable if and only if for each $x \in X$ there exist sequences of neighbourhoods of x, $\{U_n(x)\}_{n=1}^\infty$ and $\{V_n(x)\}_{n=1}^\infty$ and natural numbers $\{p(n,x)\}_{n=1}^\infty$ such that for all $x, y \in X$,*

(1) *$\{U_n(x)\}_{n=1}^{\infty}$ is a local neighbourhood base at x,*
(2) *$V_n(x) \cap V_m(y) \neq \emptyset$ and $p(n,x) \leq p(m,y)$ implies $V_m(y) \subseteq U_n(x)$.*

Bearing in mind condition (2) in this corollary, recall:

Theorem 2.7. ([11]) *A space X is proto-metrisable if and only if, for all $x \in X$ and open U containing x, there exists an open set $H(x,U)$ containing x such that if $H(x,U) \cap H(y,V) \neq \emptyset$, then $H(y,V) \subseteq U$ or $H(x,U) \subseteq V$.*

We have seen above, with Borges normality, the notion of introducing natural numbers to decide which of various possibilities occur. It seems natural, then, to take this idea further and define:

Definition 2.8. *A space X is said to be* Proto-BN (PBN) *if for all $x \in X$ and open U containing x there is an open set $H(x,U)$ containing x and a natural number $n(x,U)$ such that if $H(x,U) \cap H(y,V) \neq \emptyset$ and $n(x,U) \leq n(y,V)$, then $H(y,V) \subseteq U$.*

We have seen that Borges normality is equivalent to decreasing (A). Similarly, proto-Borges normality is also equivalent to a mechanism type condition.

If X is a space and $\mathcal{W} = \{\mathcal{W}(x) : x \in X\}$ where $\mathcal{W}(x)$ is a collection of sets of the form $\mathcal{W}(x) = \{W(n,x) : n \in \omega\}$, and $x \in W(n,x) \subseteq X$ for all x and n, then we say that $\mathcal{W}$ satisfies (A′) if,

(A′) given $x \in U$ open in X, there is an open $V = V(x,U)$ containing x and an integer $s = s(x,U)$ such that $y \in V$ implies $V \subseteq W(s,y) \subseteq U$.

We say that a space satisfies (A′) if it has a family $\mathcal{W}$ satisfying (A′). Decreasing (A′) is defined in the obvious way and it is this condition which is equivalent to PBN. However, not only can we characterise proto-Borges normality in terms of a

Collins-Roscoe condition, but we also have a lot more, as the following theorem shows.

Theorem 2.9. *For a space X the following are equivalent:*

(a) *X is metrisable;*
(b) *X satisfies decreasing (A');*
(c) *X is PBN.*

Proof: To prove that (a) implies (c), assume that x is contained in an open set U. Pick m such that the $1/m$ ball about x, $B_m(x)$, is a subset of U. Let $H(x,U) = B_{3m}(x)$ and let $n(x,U) = m$. It is obvious that these define operators as in Definition 2.8.

To prove that (c) implies (a), it is enough to show that PBN spaces are first countable. Then, if $\{U_n(x)\}$ is a countable local base at x, let $V_n(x) = H(x, U_n(x))$ and let $p(n,x) = n(x, U_n(x))$. It can easily be seen that these satisfy the conditions of Corollary 2.6.

Now, we proceed to prove that PBN spaces are first countable. For each $x \in X$ let $\mathcal{B}(x)$ be a local base at x. For all $n \in \omega$, choose $B(n,x) \in \mathcal{B}(x)$ such that $n(x, B(n,x)) = n$, if such a set exists. We claim that $\{H(x, B(n,x)) : n \in \omega\}$ forms a countable local base at x. Let U be an open set containing x and pick $B \in \mathcal{B}(x)$ such that $x \in B \subseteq U$. Let $n = n(x,B)$, then we have that $H(x, B(n,x)) \cap H(x,B) \neq \emptyset$ and $n(x,B) \leq n(x, B(n,x))$, hence $x \in H(x, B(n,x)) \subseteq B \subseteq U$.

We now prove that (b) implies (c). Suppose that X satisfies decreasing (A'). Let x be contained in an open set U. Define $H(x,U) = V(x,U)$ and $n(x,U) = s(x,U)$. It is straightforward to check that these satisfy the conditions in Definition 2.8.

Finally, we show that (c) implies (b). Assume X is PBN with operators H and n as in Definition 2.8. Define a family $\mathcal{W}$ as follows. $\mathcal{W} = \{\mathcal{W}(x) : x \in X\}$ where $\mathcal{W}(x) = \{W(n,x) : n \in \omega\}$ and $W(n,x)$ is defined by:

$$W(n,x) = \{x\} \cup \bigcup\{H(y,T) : T \text{ open, } y \in T \text{ and } x \in H(y,T) \text{ and } n(y,T) \geq n\}.$$

If x is contained in open U then define $V = H(x,U)$ and $s = n(x,U)$. We claim that, for all $y \in V$, $V \subseteq W(s,y) \subseteq U$ and hence, X satisfies decreasing (A′) as required. Clearly, the first inclusion holds. Suppose $z \in W(s,y)$. This means that $z \in H(a,T)$ for some a and open T containing a such that $y \in H(a,T)$ and $n(a,T) \geq s = n(x,U)$. This implies $y \in H(a,T) \cap H(x,U)$. Hence, $H(a,T) \subseteq U$ which proves that $W(s,y) \subseteq U$. The proof is now complete. □

Williams and Zhou [27] have defined the following strengthening of monotone normality and have proved the subsequent result.

Definition 2.10. *A space is said to be* extremely normal (EN) *if for all $x \in X$ and open U containing x there is an open set $H(x,U)$ containing x such that if $x \neq y$ and $H(x,U) \cap H(y,V) \neq \emptyset$, then either $H(y,V) \subseteq U$ or $H(x,U) \subseteq V$.*

Theorem 2.11. **([27])** *X is proto-metrisable if and only if X is EN and has a linearly ordered base at every point (i.e. at each point there is a local base which is linearly ordered w.r.t. $\subseteq$).*

To tidy up the loose ends we therefore make the following definition.

Definition 2.12. *A space X is said to be* extremely BN (EBN) *if for all $x \in X$ and open U containing x there is an open set $H(x,U)$ containing x and a natural number $n(x,U)$ such that if $x \neq y$ and $H(x,U) \cap H(y,V) \neq \emptyset$ and $n(x,U) \leq n(y,V)$, then $H(y,V) \subseteq U$.*

Proposition 2.13. *A space X is metrisable if and only if X is first countable and EBN.*

Proof: Necessity is clear. To prove sufficiency, note that X is first countable and EN so, by Theorem 2.11, X is protometrisable and, by Corollary 2.2, X is Nagata and so, by a result of Nyikos [20], we deduce that X is metrisable. □

Example 2.14. *There exists a space X which is EBN but not metrisable.*

Proof: Let X^* be a T_1 space with all but one point isolated, say p. Take $x \in U$ open in X^* and define,

$$H(x,U) = \begin{cases} \{x\} & \text{if } x \neq p \\ U & \text{if } x = p \end{cases} \qquad n(x,U) = \begin{cases} 2 & \text{if } x \neq p \\ 1 & \text{if } x = p. \end{cases}$$

It is easy to check that the above defines an EBN operator on X^* and taking $X = \omega_1 + 1$ with its usual order topology and isolating all points other than ω_1 gives an example of such a space which is not metrisable. □

In the light of Theorem 2.11, it is natural to ask: Does EBN together with linearly ordered base at every point imply metrisability? The answer is no, by Example 2.14.

3. γ-SPACES

Regarding Nagata's double sequence theorem (Theorem 2.4): the spaces satisfying conditions 1-3 are the metric spaces, those satisfying conditions 1 and 2 are the Nagata spaces and it has been proved in [16] that those satisfying conditions 1 and 3 are precisely Hodel's γ-spaces [13]. The question therefore arises, since the first two statements have been weakened by introducing natural numbers which decide where certain points lie (Theorems 2.3 and 2.5), can we do the same for the last statement? Specifically, is it true that the class of spaces which satisfy conditions 1 and 3 of Theorem 2.5 is the class of γ-spaces? We answer this question in the affirmative. Before we prove this, we rephrase the definition of γ-space in terms of the

Collins-Roscoe mechanism. If $\mathcal{W} = \{\mathcal{W}(x) : x \in X\}$ is a family such that $\mathcal{W}(x) = \{W(n,x) : n \in \omega\}$ and $x \in W(n,x) \subseteq X$ for all n and x, then we say $\mathcal{W}$ satisfies (E) if,

(E) given $x \in U$ open in X, there is an open $V = V(x,U)$ containing x and an integer $s = s(x,U)$ such that $y \in V$ implies $W(s,y) \subseteq U$.

Theorem 3.1. ([6]) *A T_1 space X is a γ-space if and only if X satisfies neighbourhood (E).*

Theorem 3.2. *A space X is a γ-space if and only if for each $x \in X$ there exist sequences of neighbourhoods of x, $\{U_n(x)\}_{n=1}^{\infty}$ and $\{V_n(x)\}_{n=1}^{\infty}$ and natural numbers $\{p(n,x)\}_{n=1}^{\infty}$ such that for all $x, y \in X$,*

(1) *$\{U_n(x)\}_{n=1}^{\infty}$ is a local neighbourhood base at x,*
(2) *$y \in V_n(x)$ and $p(n,x) \leq p(m,y)$ implies $V_m(y) \subseteq U_n(x)$.*

Proof: That X satisfies conditions (1) and (2) if X is a γ-space is obvious. As for the converse, without loss of generality we may assume that for all $n \in \omega$ and $x \in X$, $V_{n+1}(x) \subseteq V_n(x)$ and $p(n+1,x) \geq p(n,x)$. For $y \neq a$, the set of integers $A = \{p(n,a) : y \in V_n(a)\}$ is bounded above since $y \notin U_m(a)$ for some m, therefore, if $y \in V_n(a)$ then $n < m$ and $p(n,a) \leq p(m,a)$. Hence we make the following definition:

$$n_y(a) = \begin{cases} 0 & \text{if } A = \emptyset \\ \max A & \text{otherwise.} \end{cases}$$

Letting $W(n,a) = \{a\} \cup \{y \in X : n_y(a) \geq n\}$ we check that X satisfies neighbourhood (E). We first check that each $W(n,a)$ is a neighbourhood of a.

Case 1. $\exists y$ such that $n_y(a) \geq n$. In this case take $y \in X$ such that $n_y(a) \geq n$ is minimal. Since X is T_1 there is an m such that $y \notin U_m(a)$. We claim that $a \in V_m(a) \subseteq W(n,a)$. If $x \in V_m(a)$ then this implies that $n_x(a) \geq p(m,a)$. By choice of y, for some integer r, $y \in V_r(a)$, with $p(r,a) = n_y(a) \geq n$. Since $y \notin U_m(a)$, this implies $V_r(a) \not\subseteq V_m(a)$ and hence, $m > r$.

This means that $p(m,a) \geq p(r,a) \geq n$ and therefore, $n_x(a) \geq n$ which implies that $x \in W(n,a)$.

Case 2. $n_y(a) < n$ for all $y \neq a$. In this case we claim that a is isolated. First, there exists $y \in X$ such that $n_y(a) \geq 1$ else $V_m(a) = \{a\}$. So, pick y such that $n_y(a)$ is maximal and pick m such that $y \notin U_m(a)$. Assume, for a contradiction, that there exists a point $x \in V_m(a)^\circ \setminus \{a\}$. This implies that $n_x(a) \geq p(m,a)$. For some r, $y \in V_r(a)$ where $n_y(a) = p(r,a)$. Now, $a \in V_m(a)$ but $V_r(a) \not\subseteq U_m(a)$. Hence $p(m,a) > p(r,a)$ by (2). Consequently, $n_x(a) > n_y(a)$, which, by choice of y, is a contradiction.

It remains to check that $\mathcal{W}$ satisfies (E). Take a point $x \in X$ and an open set U containing x. For some n, $x \in V_n(x) \subseteq U_n(x) \subseteq U$. Let $V = V_n(x)$ and $s = p(n,x)$. We claim that, for all $a \in V$, $W(s,a) \subseteq U$ and hence $\mathcal{W}$ satisfies (E). So, take $y \in W(p(n,x),a)$. Either, $y = a \in U$ or $n_y(a) \geq p(n,x)$ in which case there exists an integer m such that $y \in V_m(a)$ and $p(m,a) = n_y(a) \geq p(n,x)$. Now, $a \in V_n(x)$, therefore, by condition (3), $V_m(a) \subseteq U_n(x)$ and consequently, $y \in U$. The proof is complete. □

4. α-BORGES NORMALITY

In the remaining sections of this paper, α denotes an infinite ordinal unless otherwise stated.

Definition 4.1. *A space X is said to be α-*Borges normal (α-BN) *if for all $x \in X$ and open U containing x there is an open set $H(x,U)$ containing x and an ordinal $\tau(x,U) < \alpha$ such that if $H(x,U) \cap H(y,V) \neq \emptyset$ and $\tau(x,U) \leq \tau(y,V)$, then $y \in U$.*

If X is a space and $\mathcal{W} = \{\mathcal{W}(x) : x \in X\}$ where $\mathcal{W}(x)$ is a collection of sets of the form $\mathcal{W}(x) = \{W(\beta,x) : \beta < \alpha\}$ and $x \in W(\beta,x) \subseteq X$ for all $\beta < \alpha$, then we say that $\mathcal{W}$ satisfies (αA) if,

(αA) given x contained in an open set U, there exists open $V = V(x,U)$ containing x and an ordinal $\beta = \beta(x,U) < \alpha$, such that $y \in V$ implies $x \in W(\beta, y) \subseteq U$.

We say X satisfies (αA) if X has a family $\mathcal{W}$ satisfying (αA). If, in addition, $W(\beta, x) \subseteq W(\gamma, x)$ whenever $\beta \geq \gamma$, then we say that X satisfies well-ordered (αA). Open (αA) and neighbourhood (αA) are defined similarly. We note also that X satisfies (A) if and only if X satisfies (ωA).

The following result was proved in [25]. We do, however, note here that the proof is not a direct cardinal generalisation of Theorem 1.5.

Theorem 4.2. *For all ordinals $\alpha \geq \omega$, X satisfies well-ordered (αA) if and only if X is α-BN.*

Theorem 4.3. *If a space X is stratifiable over α, for some cardinal $\alpha \geq \omega$, then X is α-BN.*

Proof: First note that stratifiability over α is a cardinal generalisation of stratifiability [26]. The proof of our result is a direct cardinal generalisation of the corresponding result for stratifiable spaces proved by Borges [2]. □

Now let $\mathcal{C}$ denote the class consisting of those spaces which satisfy well-ordered (αA) for some ordinal α (i.e. we let α vary). The next two theorems show that the class $\mathcal{C}$ is stable under two constructions which are important in the theory of generalised metric spaces, duplication and scattering, and hence $\mathcal{C}$ is a fairly large class of spaces. The proofs of these two are based on corresponding results in [8]. Given a space X, the *Alexandroff duplicate* $\mathcal{D}(X)$ is the set $X \times \{0,1\}$ topologised by isolating all the points in $X \times \{1\}$ and letting basic neighbourhoods around the point $(x,0)$ be of the form $(B \times \{0\}) \cup ((B \setminus \{x\}) \times \{1\})$, where B is a basic neighbourhood of x in X. If X is compact then $\mathcal{D}(X)$ will also be compact. Given a class $\mathcal{C}$ of topological spaces, the scattering process is defined as follows: given a space X_0 in $\mathcal{C}$ and a subspace Y_0 of X_0, then, for each $y \in Y_0$, we replace y by a clopen copy of some element of $\mathcal{C}$. The resulting space is denoted by X_1. We then take $Y_1 \subseteq X_1$ and replace points of Y_1 by clopen copies of elements of $\mathcal{C}$. This process is continued transfinitely, taking some subspace of the inverse limit at limit ordinals and stopping at some particular stage. The class of spaces gained from the class $\mathcal{C}$ in this way is denoted by $S(\mathcal{C})$. Nyikos has proved [20] that the class of proto-metrisable spaces is precisely the class $S(\mathcal{M})$ where $\mathcal{M}$ is the class of metrisable spaces. We shall use this fact, together with the second of the following two results to prove Theorem 7.3.

Theorem 4.4. *$\mathcal{C}$ is closed under taking the Alexandroff duplicate. Indeed, if X satisfies well-ordered (αA), then $\mathcal{D}(X)$ satisfies well-ordered $((\alpha+1)A)$.*

Proof: If $\mathcal{W} = \{\mathcal{W}(x) : x \in X\}$ where $\mathcal{W}(x) = \{W(\beta, x) : \beta < \alpha\}$, satisfies well-ordered (αA) for the space X then define,

$$\begin{aligned} W_D(\beta,(x,1)) &= \{(x,1)\} \cup (W(\beta,x)\times\{0\}) && \text{for } \beta<\alpha,\\ W_D(\alpha,(x,1)) &= \{(x,1)\},\\ W_D(\beta,(x,0)) &= W(\beta,x)\times\{0\} && \text{for } \beta<\alpha,\\ W_D(\alpha,(x,0)) &= \{(x,0)\}. \end{aligned}$$

It is routine to check that $\mathcal{W}_D = \{\mathcal{W}_D(p) : p \in \mathcal{D}(X)\}$, where $\mathcal{W}_D(p) = \{W_D(\beta,p) : \beta \le \alpha\}$, satisfies well-ordered ((α+1)A) for $\mathcal{D}(X)$. $\square$

We note, in passing, that this theorem is the best possible. The duplicate of $[0,1]$ contains a copy of the Michael Line and hence, does not satisfy decreasing (A) since the Michael Line doesn't.

Theorem 4.5. *$\mathcal{C}$ is closed under the scattering process, i.e. $\mathcal{C} = S(\mathcal{C})$.*

Proof: Suppose that for some ordinal δ we are given:

(1) Topological spaces $(X_\alpha : \alpha \le \delta)$ such that $X_0 = \{\emptyset\}$.

(2) For each $\alpha < \delta$, a subset $A_{\alpha+1}$ of X_α, and for each $a \in A_{\alpha+1}$, a space $X_{\alpha+1}(a)$ which has well-ordered $((\lambda_{a,\alpha})\mathrm{A})$ for some ordinal $\lambda_{a,\alpha}$.

(3) For each $\beta \leq \alpha \leq \delta$ a continuous surjection $j_{\alpha\to\beta} : X_\alpha \to X_\beta$.

In addition, we assume that, for each $\alpha < \delta$, the space $X_{\alpha+1}$ is obtained from X_α by replacing each point a of $A_{\alpha+1}$ by a clopen copy of $X_{\alpha+1}(a)$. Also,

$$\begin{aligned} j_{\alpha+1\to\alpha+1} &= \mathrm{id}_{X_{\alpha+1}} \\ j_{\alpha+1\to\alpha} &= \begin{cases} x & \text{if } x \in X_\alpha \setminus A_{\alpha+1} \\ a & \text{if } x \in X_{\alpha+1}(a) \end{cases} \\ j_{\alpha+1\to\beta} &= j_{\alpha\to\beta} \circ j_{\alpha+1\to\alpha} \qquad (\beta < \alpha). \end{aligned}$$

Finally, for each $\lambda \leq \delta$ which is a limit, $X_\lambda = \{(x_\alpha)_{\alpha<\lambda} : x_\alpha \in X_\alpha, \beta \leq \alpha < \lambda \Rightarrow j_{\alpha\to\beta}(x_\alpha) = x_\beta\}$. The set X_λ is endowed with the subspace topology induced by the product space $\prod_{\alpha<\lambda} X_\alpha$, and

$$\begin{aligned} j_{\lambda\to\lambda} &= \mathrm{id}_{X_\lambda}, \\ j_{\lambda\to\alpha} &= \pi_\alpha \restriction X_\lambda \qquad (\alpha < \lambda) \end{aligned}$$

where $\pi_\alpha : (\prod_{\beta<\lambda} X_\beta) \to X_\alpha$ is the projection map.

Since a subspace of a well-ordered $(\alpha\mathrm{A})$ space has well-ordered $(\alpha\mathrm{A})$, it suffices to prove that X_δ has well-ordered $(\alpha\mathrm{A})$. We shall recursively define, for $\alpha \leq \delta$, $\mathcal{W}_\alpha = \{\mathcal{W}_\alpha(x) : x \in X_\alpha\}$ where $\mathcal{W}_\alpha(x) = \{W_\alpha(\tau, x) : \tau < \lambda_\alpha\}$ for some ordinal λ_α. Assume that this has been done for each $\alpha < \gamma \leq \delta$ and that the following three conditions are satisfied.

Inductive hypotheses:

$(I_\gamma 1)$ For each $\alpha < \gamma$, $\mathcal{W}_\alpha$ satisfies well-ordered $(\lambda_\alpha\mathrm{A})$ for the space X_α

$(I_\gamma 2)$ If $\beta < \alpha < \gamma$, then $\lambda_\beta < \lambda_\alpha$

$(I_\gamma 3)$ If $\beta \leq \alpha < \gamma$, $x \in X_\alpha$ and $y = j_{\alpha\to\beta}(x)$, then $j^{-1}_{\alpha\to\beta}(W_\beta(\tau, y)) = W_\alpha(\tau, x)$ for all $\tau < \lambda_\beta$.

We now define $\mathcal{W}_\gamma$ and check that $(I_{\gamma+1}1)$ - $(I_{\gamma+1}3)$ hold. First consider the case when γ is a successor, $\alpha + 1$ say. For each $a \in A_{\alpha+1}$, let $\mathcal{W}_{X_{\alpha+1}(a)} = \{\mathcal{W}_{X_{\alpha+1}(a)}(x) : x \in X_{\alpha+1}(a)\}$, where $\mathcal{W}_{X_{\alpha+1}(a)}(x) = \{W_{X_{\alpha+1}(a)}(\tau, x) : \tau < \lambda_{a,\alpha}\}$, satisfy well ordered $(\lambda_{a,\alpha}\mathrm{A})$ for the space $X_{\alpha+1}(a)$. Let κ be an ordinal such that $\lambda_{a,\alpha} \leq \kappa$ for all $a \in A_{\alpha+1}$. Let $\lambda_\gamma = \lambda_\alpha + \kappa$. If $x \in X_{\alpha+1}(a)$ then define,

$$\begin{array}{lcll} W_{\alpha+1}(\tau, x) & = & j^{-1}_{\alpha+1\to\alpha}(W_\alpha(\tau, a)) & \text{if } \tau < \lambda_\alpha \\ W_{\alpha+1}(\lambda_\alpha + \tau, x) & = & W_{X_{\alpha+1}(a)}(\tau, x) & \text{if } \tau < \lambda_{a,\alpha} \\ W_{\alpha+1}(\tau, x) & = & \{x\} & \text{otherwise for } \tau < \lambda_\gamma. \end{array}$$

If $x \notin X_{\alpha+1}(a)$ for any $a \in A_{\alpha+1}$, but $x \in X_{\alpha+1}$, then recall that $x \in X_\alpha$ and define

$$W_{\alpha+1}(\tau, x) = \begin{cases} j^{-1}_{\alpha+1\to\alpha}(W_\alpha(\tau, x)) & \text{if } \tau < \lambda_\alpha \\ \{x\} & \text{if } \lambda_\alpha \leq \tau < \lambda_\gamma. \end{cases}$$

Observe that $(I_{\gamma+1}2)$ and $(I_{\gamma+1}3)$ are both satisfied and each $\mathcal{W}_{\alpha+1}(x)$ is well-ordered in the appropriate manner. Hence it suffices to show that, if U is an open neighbourhood of $x \in X_{\alpha+1}$, then there is an open neighbourhood V of x and an ordinal $\tau < \lambda_{\alpha+1}$ such that $x \in W_{\alpha+1}(\tau, y) \subseteq U$ whenever $y \in V$. Since $X_{\alpha+1}(a)$ is open in $X_{\alpha+1}$ and $\mathcal{W}_{X_{\alpha+1}(a)}$ satisfies $(\lambda_{a,\alpha}\mathrm{A})$ for $X_{\alpha+1}(a)$, we need only consider the case when $x \notin X_{\alpha+1}(a)$ for any $a \in A_{\alpha+1}$. There is an open set O in X_α such that $x \in O$ and $j^{-1}_{\alpha+1\to\alpha}(O) \subseteq U$. $\mathcal{W}_\alpha$ satisfies $(\lambda_\alpha\mathrm{A})$ for X_α and hence there is an open set V in X_α which contains x and $\tau < \lambda_\alpha$ such that, $x \in W_\alpha(\tau, z) \subseteq O$ whenever $z \in V$. Notice that $j^{-1}_{\alpha+1\to\alpha}(V)$ is an open neighbourhood of x in $X_{\alpha+1}$. Suppose $y \in j^{-1}_{\alpha+1\to\alpha}(V)$ and let $z = j_{\alpha+1\to\alpha}(y)$. Since $z \in V$, $x \in W_\alpha(\tau, z) \subseteq O$. But then $x \in j^{-1}_{\alpha+1\to\alpha}(W_\alpha(\tau, z)) \subseteq j^{-1}_{\alpha+1\to\alpha}(O) \subseteq U$. That is,

$$x \in W_{\alpha+1}(\tau, y) \subseteq U \qquad \text{since } \tau < \lambda_\alpha.$$

Now consider the case when γ is a limit, μ say. If $x = (x_\alpha)_{\alpha<\mu}$ is an element of X_μ then we first claim that if $\beta \leq \alpha < \mu$ and $\tau < \lambda_\beta$, then

$$(1) \qquad j^{-1}_{\mu\to\alpha}(W_\alpha(\tau, x_\alpha)) = j^{-1}_{\mu\to\beta}(W_\beta(\tau, x_\beta)).$$

The equality follows from $(I_\mu 3)$. Let λ_μ be an ordinal such that $\lambda_\alpha < \lambda_\mu$ for all $\alpha < \mu$ and define

$$W_\mu(\tau, x) = j_{\mu\to\alpha}^{-1}(W_\alpha(\tau, x_\alpha)) \text{ if } \tau < \lambda_\alpha \qquad (\alpha < \mu)$$
$$W_\mu(\tau, x) = \{x\} \qquad \text{if } \tau \geq \lambda_\alpha \text{ for all } \alpha < \mu \text{ and } \tau < \lambda_\mu.$$

By (1) and by definition we have that if $\tau_1 \leq \tau_2 < \lambda_\mu$ then $W_\mu(\tau_2, x) \subseteq W_\mu(\tau_1, x)$.

Clearly $(I_{\gamma+1}2)$ and $(I_{\gamma+1}3)$ are satisfied and by a similar argument to the successor case, $\mathcal{W}_\mu$ satisfies $(\lambda_\mu\text{A})$ for X_μ. The proof is therefore complete. □

As a particular instance of the above two theorems, we have that the Michael Line has $((\omega+1)\text{A})$. However, the Michael Line has countable pseudocharacter, but is not linearly stratifiable. This shows that countable pseudocharacter in Theorem 2.1 cannot be generalised to α-pseudocharacter. However, if we strengthen pseudocharacter, we can recover α-stratifiability. This is dealt with in the next section. We also note that the Michael Line is a special case of enlarging the topology on the real line. This suggests the question: does enlarging the topology on a space X which satisfies well-ordered (αA) for some α result in a space satisfying well-ordered (βA) for some β? The answer is no, as the Sorgenfrey line does not satisfy well-ordered (αA) for any ordinal α [18].

5. Linear stratifiability

A space is said to be linearly stratifiable if it is stratifiable over α for some regular cardinal α. As was mentioned at the end of the previous section, Theorem 2.1 cannot be generalised directly up to cardinals, because of the Michael Line. In this section we show that a property slightly stronger than 'pseudocharacter equals α', when added to well-ordered (αA), characterises stratifiability over α. This property, which we shall call a G_α-diagonal, is defined as follows.

Definition 5.1. *For a cardinal α we say a space X has G_α-*diagonal *if there exists an α-sequence of open covers of X,*

$(\mathcal{H}_\beta)_{\beta<\alpha}$, such that if $\beta < \gamma < \alpha$, then $\mathcal{H}_\gamma$ refines $\mathcal{H}_\beta$ and for all $x \in X$ we have $\bigcap_{\beta<\alpha} St(x, \mathcal{H}_\beta) = \{x\}$.

Theorem 5.2. *The following are equivalent for a space X and a regular cardinal α:*

(a) *X is stratifiable over α;*
(b) *X satisfies well-ordered (αA) and X has G_α-diagonal;*
(c) *X has G_α-diagonal and X has $\mathcal{W}$ satisfying well-ordered (F) such that for all $x \in X$, $|\mathcal{W}(x)| \leq \alpha$.*

Before we prove this theorem, we shall first recall the notion of semi-stratifiability over α and also examine more closely the idea of well-ordered (F).

Definition 5.3. *A space X is* semi-stratifiable over α *if there exists a function $g : \alpha \times X \to \tau X$ such that,*

(1) *$x \in g(\beta, x)$ for all x,*
(2) *$\gamma \leq \beta$ implies that $g(\beta, x) \subseteq g(\gamma, x)$,*
(3) *$y \in g(\beta, x_\beta)$ for all $\beta < \alpha$ implies that y is a cluster point of the net (x_β).*

Now, assume a space X has a family $\mathcal{W}$ satisfying well-ordered (F) and each $\mathcal{W}(x)$ satisfies $|\mathcal{W}(x)| \leq \alpha$. It is immediate that, in this case, X has a family $\mathcal{W}'$, also satisfying well-ordered (F) and such that $|\mathcal{W}'(x)| = \alpha$ for all x. One simply adds the set $\{x\}$ to $\mathcal{W}(x)$ the requisite number of times.

If $\mathcal{W}$ is a family satisfying well-ordered (F) and $|\mathcal{W}(x)| = \alpha$ for all $x \in X$, then $\mathcal{A}$ is a local network at x whenever $\mathcal{A} \subseteq \mathcal{W}(x)$ and $|\mathcal{A}| = \alpha$. That is, for all neighbourhoods U of x, there exists $A \in \mathcal{A}$ such that $x \in A \subseteq U$. To see this, let U be a neighbourhood of x. By (F), since $x \in V(x, U)$, $x \in W \subseteq U$ for some $W \in \mathcal{W}(x)$ and then, since $|\mathcal{A}| = \alpha$ and $\mathcal{W}(x)$ is well-ordered by reverse inclusion, there exists $A \in \mathcal{A}$ such that $x \in A \subseteq W$.

Furthermore, if V is the operator given by (F), then, without loss of generality, we can assume that V is monotone, i.e. that if $x \in U \subseteq U'$ with U and U' open, then $V(x, U) \subseteq V(x, U')$. If V

is a given (F) operator then defining $V'(x,U) = \bigcup\{V(x,T) : T$ open and $x \in T \subseteq U\}$ gives a monotone (F) operator.

We are now in a position to prove our Theorem.

Proof of Theorem 5.2. Assume that X is stratifiable over α. We have seen (Theorem 4.3) that this implies that X is α-Borges normal and hence satisfies well-ordered (αA). That X has a G_α-diagonal is a direct generalisation of the result that stratifiable spaces have a G_δ-diagonal. We therefore have that (a) implies (b). That (b) implies (c) is obvious. It remains to check that (c) implies (a).

So, assume $\mathcal{W}$ is a family satisfying well-ordered (F) such that for each x, $|\mathcal{W}(x)| = \alpha$ and such that the operator V is monotone. Let $(\mathcal{H}_\beta)_{\beta<\alpha}$ be a G_α-diagonal sequence. Since well-ordered (F) spaces are monotonically normal it is sufficient to prove that X is semi-stratifiable over α (c.f. Theorem 5.16 [10]). Define

$$g(\beta,x) = \bigcup\{V(x,H) : H \in \mathcal{H}_\beta \text{ and } x \in H\}.$$

Clearly $x \in g(\beta,x)$ for all x. If $\beta \geq \gamma$ and $y \in g(\beta,x)$, then $y \in V(x,H)$ for some $H \in \mathcal{H}_\beta$. Now $\mathcal{H}_\beta$ refines $\mathcal{H}_\gamma$, hence for some $H' \in \mathcal{H}_\gamma$, $H \subseteq H'$ which implies $V(x,H) \subseteq V(x,H')$ by monotonicity of V and therefore $y \in g(\gamma,x)$. It remains to check condition (3).

Suppose $x \in g(\beta,x_\beta)$ for all $\beta < \alpha$. Pick $H_\beta \in \mathcal{H}_\beta$ such that $x_\beta \in H_\beta$ and $x \in V(x_\beta,H_\beta)$. Assume, for a contradiction, that x is not a cluster point of (x_β). This means that there is an open set U containing x such that, if $I = \{\beta < \alpha : x_\beta \notin U\}$, then $|I| = \alpha$. For all $\beta \in I$, $x \in V(x_\beta,H_\beta)$. Hence, for each $\beta \in I$, there exists $W_\beta \in \mathcal{W}(x)$ such that,

$$x_\beta \in W_\beta \subseteq H_\beta \subseteq St(x,\mathcal{H}_\beta).$$

For all $\beta \in I$, $W_\beta \not\subseteq U$. Consequently, $\{W_\beta : \beta \in I\}$ is not a local network at x and therefore, $|\{W_\beta : \beta \in I\}| < \alpha$. Without loss of generality, $W_\beta = W_\gamma$ for all $\beta,\gamma \in I$ which means that for all $\beta, \gamma \in I$, $x_\beta \in St(x, \mathcal{H}_\gamma)$. This gives us that

$$x_\beta \in \bigcap_{\gamma \in I} St(x, \mathcal{H}_\gamma) = \{x\} \subseteq U$$

which is a contradiction. The proof is therefore complete. □

6. ω_μ-METRISABILITY

The notion of ω_μ-metrisability is a cardinal generalisation of metrisability which has been the subject of many papers (see for instance [21]). Many of the standard metrisation theorems have been shown to have cardinal generalisations to ω_μ-metrisable spaces. In this section we generalise Theorems 1.2 and 2.9 in this way. The statement of this theorem is as follows (recall that a space X is said to be *ω_μ-additive* if every intersection of fewer than ω_μ open sets is open):

Theorem 6.1. *The following are equivalent for a space X and a regular cardinal ω_μ:*

(1) *X is ω_μ-metrisable;*
(2) *X is ω_μ-PBN and ω_μ-additive;*
(3) *X satisfies open well-ordered $(\omega_\mu A)$;*
(4) *X satisfies neighbourhood well-ordered $(\omega_\mu A)$.*

It remains for us to actually define ω_μ-PBN.

Definition 6.2. *For a cardinal α, a space X is said to be α-*proto-BN *(α-*PBN*) if for all $x \in X$ and open U containing x there is an open set $H(x, U)$ containing x and an ordinal $\beta(x, U) < \alpha$ such that if $H(x, U) \cap H(y, V) \neq \emptyset$ and $\beta(x, U) \leq \beta(y, V)$, then $H(y, V) \subseteq U$.*

If X is a space, α is an infinite cardinal and $\mathcal{W} = \{\mathcal{W}(x) : x \in X\}$ where $\mathcal{W}(x)$ is a collection of sets of the form $\mathcal{W}(x) = \{W(\beta, x) : \beta < \alpha\}$ and where $x \in W(\beta, x) \subseteq X$ for all x and β, then we say that $\mathcal{W}$ satisfies (αA$'$) if,

(αA$'$)	given $x \in U$ open in X, there is an open $V = V(x,U)$ containing x and an ordinal $\beta = \beta(x,U) < \alpha$ such that $y \in V$ implies $V \subseteq W(\beta, y) \subseteq U$.

We say that a space satisfies (αA$'$) if it has a family $\mathcal{W}$ satisfying (αA$'$). Well-ordered (αA$'$) is defined as before. One shows that a space X is α-PBN if and only if it satisfies well-ordered (αA$'$) in exactly the same way as the countable analogue was proved in Theorem 2.9.

Before we prove our theorem, recall one result, due to Reichel, concerning ω_μ-metrisable spaces.

Theorem 6.3. [22] *For $\omega_\mu > \omega_0$, a space X is ω_μ-metrisable if and only if X is ω_μ-additive, X is collectionwise normal and $dv(X) \leq \omega_\mu$.*

Proof of Theorem 6.1. The implications, (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (4) are both straightforward.

(2) $\Rightarrow$ (3). Since X is ω_μ-PBN, X has a family $\mathcal{W}$ satisfying well-ordered (ω_μA$'$). If x is contained in an open set U, then there exists an open set V and an ordinal $\beta(x,U)$ such that for all $y \in V$, $V \subseteq W(\beta,y) \subseteq U$. In particular, for all x and U, $x \in W(\beta(x,U),x)^\circ$. We wish to show that x lies in such interiors for sufficiently many $\beta(x,U)$. We claim that for fixed x, $\{\beta(x,U) : x \in U,\ U \text{ open}\}$ is cofinal in ω_μ. If not, there exists $\beta_0 < \omega_\mu$ such that $\beta(x,U) < \beta_0$ for all U. Since X is ω_μ-additive, then $T = \bigcap_{\beta(x,U)} W(\beta(x,U),x)^\circ$ is open and contains x. If x is isolated, our claim is obvious. Otherwise, there exists $y \in T \setminus \{x\}$. Hence, $W(\beta(x, T \setminus \{y\}), x)^\circ \subseteq T \setminus \{y\}$ which contradicts $y \in T$. So we have that the set of ordinals, $\{\beta(x,U) : x \in U,\ U \text{ open}\}$, is cofinal in ω_μ. Now define $\mathcal{W}_0(x) = \{W(\beta,x)^\circ : \beta < \omega_\mu\}$. It is easy to prove that $\mathcal{W}_0$ defined in this way satisfies well-ordered open (ω_μA).

(4) $\Rightarrow$ (1). First, it is clear that, if X satisfies well-ordered neighbourhood (ω_μA), then X satisfies well-ordered (F) and

consequently, X is monotonically normal and hence collectionwise normal.

Suppose that $\{U_\lambda : \lambda < \alpha\}$ is a collection of open sets in X with non-empty intersection for some $\alpha < \omega_\mu$. If $x \in U_\lambda$ for each λ then, $x \in W(\lambda', x) \subseteq U_\lambda$ for some $\lambda' < \omega_\mu$. Let $\lambda_0 = \bigcup_{\lambda<\alpha} \lambda'$. Since ω_μ is regular, $\lambda_0 < \omega_\mu$ and $x \in W(\lambda_0, x) \subseteq W(\lambda', x) \subseteq U_\lambda$. Therefore $x \in W(\lambda_0, x) \subseteq \bigcap_{\lambda<\alpha} U_\lambda$. Hence, X is ω_μ-additive, since $W(\lambda_0, x)$ is a neighbourhood of x.

We finally show that X has $dv(X) \leq \omega_\mu$. Since, for all $\alpha < \omega_\mu$, $W(\alpha, x)$ is a neighbourhood of x, there exists an open set V and an ordinal $\gamma = \gamma(\alpha, x) > \alpha$ such that, for all $y \in V$, $x \in W(\gamma, y) \subseteq W(\alpha, x)$. Choose $\beta = \beta(\alpha, x) > \gamma$ such that $W(\beta, x) \subseteq V$. We therefore have the following combinatorial principle:

(2) for all $y \in W(\beta(\alpha, x), x)$, $x \in W(\gamma(\alpha, x), y) \subseteq W(\alpha, x)$.

Let $\mathcal{G}_\alpha = \{W(\beta(\alpha, x), x)^\circ : x \in X\}$ for $\alpha < \omega_\mu$. We claim that $\mathrm{St}(x, \mathcal{G}_{\beta(\alpha,x)}) \subseteq W(\alpha, x)$ and hence $dv(X) \leq \omega_\mu$. Assume $z \in St(x, \mathcal{G}_{\beta(\alpha,x)})$. Then $x, z \in W(\beta(\beta(\alpha, x), y), y)$ for some y. We claim, $W(\beta(\beta(\alpha, x), y), y) \subseteq W(\alpha, x)$ which is enough to prove our original claim. Since $x \in W(\beta(\beta(\alpha, x), y), y)$, by Equation 2 and the well-ordering of $\mathcal{W}(x)$, $y \in W(\gamma(\beta(\alpha, x), y), x) \subseteq W(\beta(\alpha, x), x)$. Hence, again by Equation 2, $W(\gamma(\alpha, x), y) \subseteq W(\alpha, x)$. Since $\beta(\beta(\alpha, x), y) > \beta(\alpha, x) > \gamma(\alpha, x)$ our claim follows.

We have thus shown that X satisfies all the hypotheses of Reichel's Theorem and hence X is ω_μ-metrisable. $\square$

7. Proto-metrisability

Gartside and Moody [9] have recently given several interesting characterisations of proto-metrisability. They have shown that a space X is proto-metrisable if and only if X is monotonically paracompact (full normality which respects refinements of covers). They have also characterised proto-metrisability in terms of a Collins-Roscoe mechanism type condition. If

$\mathcal{W} = \{\mathcal{W}(x) : x \in X\}$ is a family such that $x \in W \subseteq X$ for all $W \in \mathcal{W}(x)$ then we say $\mathcal{W}$ satisfies (F″) if,

(F″) given x contained in an open set U, there exists open $V = V(x, U)$ containing x such that $y \in V$ implies $V \subseteq W \subseteq U$, for some $W \in \mathcal{W}(y)$.

Gartside and Moody have proved that a space X is protometrisable if and only if X has a family $\mathcal{W}$ satisfying well-ordered open (F″). In [9], they asked whether (F″) could be replaced by the more standard mechanism condition (F) in this result. In this section, we provide a partial solution to this question, replacing (F″) by a uniform version of (F) which we shall call (Z). This condition (Z) is a uniform version of (F) in the same way that (A) is a uniform version of (G).

We saw in Section 4 a generalisation of decreasing (A) up to ordinals. However, this is not the only possible generalisation. If X is a space and $\mathcal{W} = \{\mathcal{W}(x) : x \in X\}$ where $\mathcal{W}(x)$ is a collection of sets of the form $\mathcal{W}(x) = \{W(\beta, x) : \beta < \alpha_x\}$ for some ordinal α_x and where $x \in W(\beta, x) \subseteq X$ for all $\beta < \alpha_x$ then we say that $\mathcal{W}$ satisfies (Z) if,

(Z) given x contained in an open set U, there exists open $V = V(x, U)$ containing x and an ordinal $\beta = \beta(x, U)$ such that $y \in V$ implies $\beta < \alpha_y$ and $x \in W(\beta, y) \subseteq U$.

We say X satisfies (Z) if X has a family $\mathcal{W}$ satisfying (Z). If, in addition, $W(\beta, x) \subseteq W(\gamma, x)$ whenever $\gamma \leq \beta < \alpha_x$, then we say X satisfies well-ordered (Z). We say that X satisfies open (neighbourhood) (Z) if every element of $\mathcal{W}(x)$ is open (a neighbourhood of x) for each $x \in X$.

The following two lemmas are cardinal generalisations of Lemmas 1 and 2 of [6]. Since the proofs of the countable case do not appear in that paper, we include proofs here for completeness. The first lemma merely translates the topological condition (Z) into a combinatorial one. The second is the key to our characterisation of proto-metrisability. The reader will notice the idea of a rank one collection immediately in the statement of this lemma.

Lemma 7.1. *If X satisfies well-ordered neighbourhood (Z) then for all $x \in X$ and $\alpha < \alpha_x$, there exist ordinals $\beta = \beta(\alpha, x)$ and $\gamma = \gamma(\alpha, x)$ with $\alpha < \gamma < \beta < \alpha_x$ and such that $y \in W(\beta, x)$ implies $\gamma < \alpha_y$ and $x \in W(\gamma, y) \subseteq W(\alpha, x)$.*

Proof: Take $x \in X$ and $\alpha < \alpha_x$. Let $U = W(\alpha, x)$. From the definition of (Z), we have an open set V containing x and an ordinal γ such that for all $y \in V$, $\gamma < \alpha_y$ and $x \in W(\gamma, y) \subseteq U$. [Note: by making U smaller, if necessary, we can insist that $\gamma > \alpha$.] Now, pick β such that $W(\beta, x) \subseteq V$ and $\gamma < \beta < \alpha_x$. The claim is therefore proved. □

Lemma 7.2. *Let $\beta_2(\alpha, x) = \beta(\beta(\alpha, x), x)$. If $W(\beta_2(\alpha, x), x)$ meets $W(\beta_2(\alpha', y), y)$ then, either, $W(\beta_2(\alpha, x), x) \subseteq W(\alpha', y)$ or $W(\beta_2(\alpha', y), y) \subseteq W(\alpha, x)$.*

Proof: First, note that, without loss of generality,

$$\gamma(\beta(\alpha', y), y) \geq \gamma(\beta(\alpha, x), x). \tag{2}$$

If $z \in W(\beta(\beta(\alpha, x), x), x) \cap W(\beta(\beta(\alpha', y), y), y)$, then, by the previous lemma,

$$x \in W(\gamma(\beta(\alpha, x), x), z) \subseteq W(\beta(\alpha, x), x) \text{ and}$$

$$y \in W(\gamma(\beta(\alpha', y), y), z) \subseteq W(\beta(\alpha', y), y).$$

By Equation 3, $y \in W(\beta(\alpha, x), x)$ and, again, by the previous lemma, $W(\gamma(\alpha, x), y) \subseteq W(\alpha, x)$. However, $\gamma(\alpha, x) < \beta(\alpha, x) < \gamma(\beta(\alpha, x), x) \leq \gamma(\beta(\alpha', y), y) < \beta(\beta(\alpha', y), y) = \beta_2(\alpha', y)$. Putting all this together, we have that, $W(\beta_2(\alpha', y), y) \subseteq W(\gamma(\alpha, x), y) \subseteq W(\alpha, x)$, as required. □

So, by Lemma 7.2, we see that, $\mathbb{P} = \{\langle W(\beta_2(\alpha, x), x)^\circ, W(\alpha, x)^\circ\rangle : x \in X, \alpha < \alpha_x\}$ forms a rank one pair base and hence, well-ordered neighbourhood (Z) spaces are proto-metrisable. However, it is clear that Theorem 4.5 still holds if we consider the class $\mathcal{C}$ of well-ordered open (Z) spaces. That is, that scattering preserves well-ordered open (Z). At each stage of the construction, when we are constructing the next family $\mathcal{W}_\alpha$, we simply do not add the singletons $\{x\}$ into the collection $\mathcal{W}_\alpha(x)$. The families $\mathcal{W}_\alpha(x)$, therefore, only contain open sets. The singletons were not added in the proof of Theorem 4.5 to make the Collins-Roscoe mechanism work but rather to make the families $\mathcal{W}_\alpha(x)$ of the right cardinality. So, since metric spaces satisfy well-ordered open (Z) (in fact, decreasing open (A)), we have that proto-metrisable spaces satisfy well-ordered open (Z) by the stability of well-ordered open (Z) under scattering. Therefore:

Theorem 7.3. *For a space X the following are equivalent:*

(1) *X is proto-metrisable;*
(2) *X satisfies well-ordered open (Z);*
(3) *X satisfies well-ordered neighbourhood (Z).*

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