Topology Proceedings

Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings Department of Mathematics & Statistics Auburn University, Alabama 36849, USA
E-mail:	topolog@auburn.edu
ISSN:	0146-4124

Topology Proceedings
Vol. 19, 1994

A GLANCE AT COMPACT SPACES WHICH MAP "NICELY" ONTO THE METRIZABLE ONES

VLADIMIR V. TKACHUK[1]

ABSTRACT. We call a space metrizably fibered if it maps continuously and with metrizable fibers onto a metrizable space. Most of our attention is concentrated on the class $\mathcal{M}$ of metrizably fibered compact spaces. It is evident that $\mathcal{M}$ is a subclass of the class $\mathcal{FC}$ of first countable compact spaces. We prove that $\mathcal{M}$ is strictly smaller than $\mathcal{FC}$ and that $\mathcal{M}$ is invariant with respect to open maps while not being invariant under continuous mappings. It is established that if perfectly normal compact space is metrizably fibered, then so are all its continuous images. We also introduce the concept of weakly metrizably fibered space and show that any Eberlein compact space of weight less than or equal to continuum is weakly metrizably fibered, while under the negation of the Souslin hypothesis there exist perfectly normal Corson compact spaces of cardinality ω_1 which are not weakly metrizably fibered.

0. INTRODUCTION.

The topologists are very short of ZFC examples of non-metrizable perfectly normal compact spaces. The most daring hypotheses about this class persist for dozens of years without noticeable progress in their solution. In fact all perfectly normal compact spaces known in ZFC are some derivatives of the

[1] Research supported by Mexican National Council on Science and Technology (CONACYT) grant 400200-5-4874E.

double arrow space S (such as continuous images of closed subsets of $S \times I^\omega$, where I is the unit segment $[0,1]$ with its usual topology). That's why D.H.Fremlin asked if it was consistent that any perfectly normal compact space has a two-to-one continuous map onto a metrizable one. This question was cited by G.Gruenhage in [5].

It seems to be a folklore that no Souslin continuum with all its intervals non-separable admits a continuous map onto a metrizable space with the inverse images of all points metrizable. This clearly implies that the negative answer to D. H. Fremlin's question is consistent with ZFC.

If a space X can be mapped continuously and with metrizable fibers onto a metrizable space, we say that X is metrizably fibered. We take a look at the class $\mathcal{M}$ of compact metrizably fibered spaces. It is proved that $\mathcal{M}$ is strictly smaller than the class $\mathcal{FC}$ of first countable compact spaces. We show that $\mathcal{M}$ is invariant under open maps, but not under the continuous ones. We establish, however, that if X is a perfectly normal compact space from $\mathcal{M}$, then any continuous image of X belongs to $\mathcal{M}$ too. We also introduce the class of weakly metrizably fibered spaces and prove that each Eberlein compact space of weight less than or equal to continuum belongs to it.

1. Notation and terminology.

Throughout this paper "a space" means "a Tychonoff space". If X is a space, then $\mathcal{T}(X)$ is its topology and $\mathcal{T}^*(X) = \mathcal{T}(X)\backslash\{\emptyset\}$. The end of a proof of a statement will be denoted by $\square$. For a space X and $A \subset X$ we denote by $\overline{A}$ the closure of A in X. If it might not be clear in which space the closure is taken, then we write $\mathrm{cl}_X(A)$ for the closure of A in X. A cardinal number τ is identified with the smallest ordinal number having power τ. A space X is Frechet–Urysohn if for any $A \subset X$ if $x \in \overline{A}$ then there is a sequence in A converging to x. A space X has countable tightness (is sequential) if for any $A \subset X$, $\overline{A} \neq A$ we have a (convergent) sequence $B = \{a_n : n \in \omega\} \subset A$ with $\overline{B} \not\subset A$. A subset $F = \{x_\alpha : \alpha \in \tau\}$ of a compact space X is called free sequence of length τ if for all $\alpha \in \tau$ we have $\overline{\{x_\beta : \beta < \alpha\}} \cap \overline{\{x_\beta : \beta \geq \alpha\}} = \emptyset$.

All other notions are standard and can be found in [4].

2. Mapping compact spaces onto the metrizable ones with inverse images of all points metrizable.

We shall need some simple inner characterizations of metrizably fibered spaces.

2.1. Proposition. *The following are equivalent for every space X;*
(1) X admits a continuous map p onto a second countable space M such that $p^{-1}(y)$ is metrizable for all $y \in M$;
(2) X has a countable family γ of cozero open sets such that $\cup\gamma = X$ and the set $\gamma(x) = \cap\{U \in \gamma : x \in U\}$ is metrizable for any $x \in X$;
(3) X has a family $\gamma \subset \mathcal{T}^*(X)$ as in (2) with the additional property that it is closed with respect to finite intersections and for any $x \in X$ and $U \in \gamma$ with $U \supset \gamma(x)$ there exists some $V \in \gamma$ such that $\gamma(x) \subset V \subset \overline{V} \subset U$;
(4) X has a countable family γ of zero sets such that $\cup\gamma = X$ and the set $\gamma(x) = \cap\{U \in \gamma : x \in U\}$ is metrizable for any $x \in X$.

Proof: It is evident that (3) $\Longrightarrow$ (2). Assume that $p : X \to M$ is a map like in (1). We shall prove simultaneously that (1) $\Longrightarrow$ (3) and (1) $\Longrightarrow$ (4). Fix a countable base $\mathcal{B}$ in M closed with respect to finite intersections and let $\gamma = \{p^{-1}(W) : W \in \mathcal{B}\}$ (or $\gamma = \{p^{-1}(\overline{W}) : W \in \mathcal{B}\}$ respectively). It straightforward that γ is like in (3) (or in (4) respectively), so that we proved (1) $\Longrightarrow$ (3) and (1) $\Longrightarrow$ (4).

Let us prove simultaneously that (2) $\Longrightarrow$ (1) and (4) $\Longrightarrow$ (1). If we have a family γ like in (2) (or in (4) respectively), pick a continuous map $p_U : X \to I$ with $p_U^{-1}((0,1]) = U$ (or $p_U^{-1}((0,1]) = X \backslash U$ respectively) for all $U \in \gamma$. We claim that the diagonal product p of the mappings p_U is what we need to prove (1). Clearly, the image $M = p(X)$ is second countable. If $y \in p^{-1}p(x)$ for some $x \in X$, then $p_U(y) = p_U(x)$ for all $U \in \gamma$. Thus $y \in \cap\{U : x \in U\} = \gamma(x)$ and therefore $p^{-1}p(x) \subset \gamma(x)$ for all $x \in X$. All sets $\gamma(x)$ being metrizable we established the metrizability of all fibers of p. □

2.2. Theorem. *Let X be a compact space which admits a continuous map with metrizable fibers onto a metrizable space (i.e. $X \in \mathcal{M}$). If $f : X \xrightarrow{onto} Y$ is an open map, then $Y \in \mathcal{M}$.*

Proof: By Proposition 2.1 X has a family γ as in 2.1(3). Let us prove that the family $\delta = \{f(U) : U \in \gamma\}$ satisfies 2.1(2). Every $U \in \gamma$ is σ-compact so that the set $f(U)$ is open and σ-compact in Y. Hence the family δ consists of cozero sets of Y. Clearly, $\cup\delta = Y$ so let us only check that $\delta(y)$ is metrizable for any $y \in Y$.

Pick an $x \in f^{-1}(y)$. Suppose that $z \in Y$ and $f^{-1}(z) \cap \gamma(x) = \emptyset$. The set $\gamma(x)$ is compact and γ is closed under finite intersections, so there is a $U \in \gamma$ such that $x \in U$ and $U \cap f^{-1}(z) = \emptyset$. Therefore $f(U) \not\ni z$ and $z \notin \delta(y)$.

Consequently, $f^{-1}(z) \cap \gamma(x) \neq \emptyset$ for all $z \in \delta(y)$ and this means exactly $\delta(y) \subset f(\gamma(x))$. Any continuous image of the metrizable compact space $\gamma(x)$ is metrizable so $\delta(y)$ is metrizable too. □

2.3. Example. *The class $\mathcal{M}$ is not invariant under continuous maps.*

Proof: Note first, that every metrizably fibered compact space Y is first countable. Indeed, let $y \in Y$. If the family γ is as in 2.1(3), then $\gamma(y)$ is a closed G_δ-set in Y and y is a G_δ-set in $\gamma(y)$. Therefore $\{y\}$ is a G_δ-set in Y. In compact spaces any G_δ-point is a point of countable character, so $Y \in \mathcal{FC}$.

Let X be the Alexandroff duplicate of I, i.e. $X = I_0 \cup I_1$ where I_i are disjoint copies of I. All points of I_1 are isolated in X and the base at a point $t_0 \in I_0$ consists of the sets $U_0 \cup (U_1 \setminus \{t_1\})$ where U_0 is an open interval in I_0 containing t and U_1, t_1 are the respective copies of U_0 and t_0 in I_1. The space

X admits a two-to-one continuous map onto I. If we identify the points of I_0 we will obtain a one-point compactification Y of the discrete space of power $\mathfrak{c}$. The space Y does not belong to $\mathcal{M}$ because it is not first countable. $\square$

2.4. Example. *There are first countable compact spaces which are not metrizably fibered, i.e. the class $\mathcal{M}$ does not coincide with the class $\mathcal{FC}$ of first countable compact spaces.*

Proof: Let $X = I \times I \times I$ be the lexicographic cube. Recall that its topology is generated by the following order: $(x, y, z) < (x_1, y_1, z_1)$ iff $x < x_1$ or $x = x_1$ and $y < y_1$; or $x = x_1$, $y = y_1$ and $z < z_1$.

It is well known that X is a first countable compact space. We shall prove that $X \notin \mathcal{M}$. Take any continuous map $f : X \to M$ where M is a second countable space with a metric ρ. Let us prove that there are at most countably many $t \in I$ such that the image of the set $I_t = \{t\} \times I \times I$ contains more than one point.

If it is not so, then the set $A = \{t \in I : |f(I_t)| > 1\}$ is uncountable. Pick the points $a_t, b_t \in I_t$ such that $f(a_t) \neq f(b_t)$ for each $t \in A$. There is an $\varepsilon > 0$ and an uncountable $B \subset A$ such that $\rho(f(a_t), f(b_t)) \geq \varepsilon$ for all $t \in B$. The set B can not be scattered, so there is a nontrivial sequence $S = \{t_n : n \in \omega\} \subset B$ converging to a point $z \in B$. Any convergent sequence in I contains a monotone convergent subsequence so we may assume that S is monotone.

Case 1. If S is increasing, then both sequences $\{a_{t_n} : n \in \omega\}$ and $\{b_{t_n} : n \in \omega\}$ converge to the point $(z, 0, 0) \in X$ which is a contradiction because the oscillation of f at this point would be $\geq \varepsilon$.

Case 2. If S is decreasing, then both sequences $\{a_{t_n} : n \in \omega\}$ and $\{b_{t_n} : n \in \omega\}$ converge to the point $(z, 1, 1) \in X$ which is a contradiction because the oscillation of f at this point would be $\geq \varepsilon$.

From what we established it follows that there are continuum many points $z \in M$ such that $f^{-1}(z)$ contains the set $\{t\} \times I \times I$

so all these fibers are not metrizable. □

We saw that a continuous image of a metrizably fibered space is not necessarily metrizably fibered. However, it is so if the image is perfectly normal or even perfectly κ-normal. Recall that a space is perfectly κ-normal if the closure of any open set in this space is a zero set.

2.5. Theorem. *Let $X \in \mathcal{M}$. Then any continuous perfectly κ-normal image of X also belongs to $\mathcal{M}$.*

Proof: Let $f : X \to Y$ be a continuous onto map. We may assume f to be irreducible. Fix a family γ in X as in 2.1(3). We assert that the family $\eta = \{\overline{f(U)} : U \in \gamma\}$ satisfies the condition 2.1(4) for Y.

Indeed, Y is perfectly κ-normal, and f irreducible so all elements of η are closures of open set and therefore are zero sets in Y. Clearly, $\cup\eta = Y$, so let us only check that $\eta(y)$ is metrizable for any $y \in Y$. Fix an $x \in f^{-1}(y)$.

Suppose that $z \in Y$ and $f^{-1}(z) \cap \gamma(x) = \emptyset$. The set $\gamma(x)$ is compact and γ is closed under finite intersections, so there is an $U \in \gamma$ such that $x \in U$ and $\overline{U} \cap f^{-1}(z) = \emptyset$. Therefore $\overline{f(U)} \not\ni z$ and $z \notin \eta(y)$.

Consequently, $f^{-1}(z) \cap \gamma(x) \neq \emptyset$ for all $z \in \eta(y)$ and this means exactly $\eta(y) \subset f(\gamma(x))$. Any continuous image of the metrizable compact space $\gamma(x)$ is metrizable so $\eta(y)$ is metrizable too. □

2.6. Corollary. *(1) Any perfectly normal image of a metrizably fibered compact space is metrizably fibered;*
(2) if a perfectly normal compact space is metrizably fibered then so is every continuous image of X;
(3) if a perfectly normal compact space is obtained from the Hilbert cube and the double arrow space using closed subspaces, countable products and continuous images, then it is metrizably fibered.

Proof: The item (1) is clear. To prove (2) one must only observe that any continuous image of a perfectly normal compact space is perfectly normal. All operations mentioned in (3) preserve being metrizably fibered, so applying (1) we settle (3). □

2.7. Remark. *It is a folklore that any Souslin continuum with no intervals separable is not metrizably fibered. Hence it is consistent with ZFC that not every perfectly normal compact space is metrizably fibered. In an e-mail letter G.Gruenhage communicated to the author a proof that for any continuous map of such a Souslin continuum onto a metrizable space, there is an inverse image of a point which contains a non-empty interval.*

The last thing we'd like to look at is the property defined in 2.1(4) without requiring the closed sets of the relevant family to be G_δ-sets. It defines a new class of spaces, which is invariant under countable products, and closed subspaces. All Lindelöf spaces, belonging to this class have the cardinality less than or equal to continuum but not all compact spaces from this class are continuous images of first countable compact spaces.

2.8. Definition. *Let us call a space X weakly metrizably fibered if there is a countable family γ of closed subsets of X such that $\cup\gamma = X$ and $\gamma(x) = \bigcap\{F \in \gamma : x \in F\}$ is metrizable for every $x \in X$. In this case we shall say that γ metrizably fibers X.*

2.9. Proposition. *(1) The cardinality of a weakly metrizably fibered Lindelöf space does not exceed 2^ω;*
(2) any closed subspace of a weakly metrizably fibered space is weakly metrizably fibered;
(3) a countable product of weakly metrizably fibered spaces is weakly metrizably fibered;
(4) if a space is a countable union of its closed weakly metrizably fibered subspaces, then it is weakly metrizably fibered;
(5) any continuous image of a weakly metrizably fibered compact space is weakly metrizably fibered;

(6) every perfectly normal weakly metrizably fibered compact space is metrizably fibered.

Proof: The properties (1)-(4) are straightforward from the definiton. Let X be a weakly metrizably fibered space with the family γ as in 2.8. Let $f : X \overset{onto}{\longrightarrow} Y$ be a continuous map. Evidently, the family γ may be assumed to be closed with respect to finite intersections. Let $\eta = \{f(F) : F \in \gamma\}$. The proof that $\eta(y)$ is metrizable for any $y \in Y$ goes exactly like in 2.5. The equality $\cup\eta = Y$ being clear we established (5). To prove (6) use 2.1.(4). $\square$

2.10. Corollary. *Any continuous image of a metrizably fibered compact space is weakly metrizably fibered.*

It follows from 2.3 and 2.9(5) that not every weakly metrizably fibered space is first countable — because the Alexandroff compactification of a discrete space of power $\leq 2^\omega$ is weakly metrizably fibered. It turns out, however, that all such spaces have countable tightness.

2.11. Theorem. *Any weakly metrizably fibered compact space has countable tightness.*

Proof: Suppose not. Fix a space X witnessing that. Then X contains a free sequence F of length ω_1 [1]. The subspace $\overline{F}$ maps continuously onto the space $\omega_1 + 1$ with its natural order topology. Using 2.9(2) and 2.9(5) conclude that $\omega_1 + 1$ is weakly metrizably fibered.

Let us prove that it is not so, thus obtaining the necessary contradiction. Suppose that a countable family γ of closed subsets of $\omega_1 + 1$ metrizably fibers $\omega_1 + 1$. Let $\mu = \{F \in \gamma : \omega_1 \notin F\}$. There is an $\alpha \in \omega_1$ such that $\cup\mu \subset \alpha$. Let $\eta = \{F \in \gamma\backslash\mu : F \cap \omega_1 \text{ is bounded in } \omega_1\}$.

There exists an ordinal $\beta \in \omega_1$ such that $(\cup\mu \bigcup \cup\eta) \cap \omega_1 \subset \beta$. All elements of $\{F \cap \omega_1 : F \in \gamma\backslash(\mu \cup \eta)\}$ are closed and unbounded in ω_1, so their intersection contains a closed unbounded subset of ω_1. For any point x of this subset $\gamma(x)$ is not metrizable — a contradiction. $\square$

Once we have proved that each weakly metrizably fibered compact space X has countable tightness, two important questions about such an X arise. First of all one wonders whether X is sequential or not. Well, it is known that under the proper forcing axiom every compact countably tight space is sequential [3] as well as there exist countably tight non-sequential compact spaces under Jensen hypothesis [8]. The author did not succeed to determine whether any weakly metrizably fibered space is sequential in ZFC.

Another important question is whether such an X has points of countable character. Evidently, the Čech-Pospišil theorem implies that under CH any space of power continuum has such points. On the other hand, in [6] V.I.Malyhin constructed by forcing an example of a Frechet–Urysohn compact space without points of countable character. It turned out that any weakly metrizably fibered compact space has sufficiently many points of countable character in ZFC.

2.12. Theorem. *Let X be a weakly metrizably fibered compact space. Then X has a point of countable character.*

Proof: Let $\gamma = \{F_n : n \in \omega\}$ be the family that metrizably fibers X. Without loss of generality we can assume all F_n's to be non-empty. The space X is compact and $\bigcup\{F_n : n \in \omega\} = X$ so one of the sets F_n has a non-empty interior.

Let n_0 be the minimal $n \in \omega$ such that $Int(F_n) \neq \emptyset$. Let U_0 be a non-empty open set with $\overline{U_0} \subset F_{n_0} \setminus \bigcup\{F_m : m < n_0\}$.

Suppose that we have natural numbers $n_0, \dots, n_k$ and non-empty open sets $U_0, \dots, U_k$ with the following properties:

(1) $n_0 < n_1 < \dots < n_k$;

(2) $\overline{U_{l+1}} \subset U_l$ for all $l < k$;

(3) $\overline{U_l} \subset Int(F_{n_l}) \cap U_{l-1}$ for all $l \leq k$;

(4) the number n_l is the smallest among $\{m > n_{l-1}: Int(F_m \cap U_{l-1}) \neq \emptyset\}$;

(5) $U_l \cap F_m = \emptyset$ for all $m \in \{1, \dots, n_l\} \setminus \{n_0, \dots, n_l\}$.

Let us consider two cases.

Case 1. There is no $m > n_k$ with $Int(F_m \cap U_k) \neq \emptyset$. The set U_k has the Baire property, so that there is a point $x \in U_k$ such that $x \notin \bigcup\{F_m : m > n_k\}$. Therefore $\gamma(x) = \bigcap\{F_{n_i} : i \leq k\}$ and $U_k \subset \gamma(x)$. The set $\gamma(x)$ being metrizable and U_k open in X all points of U_k have countable character in X so our theorem is proved.

Case 2. There exists an $m > n_k$ such that $F_m \cap U_k$ has a non-empty interior. Choose n_{k+1} to be the smallest such m. It is clear that

$$W = Int(F_{n_{k+1}} \cap U_k) \backslash (F_{n_k+1} \cup \ldots \cup F_{n_{k+1}-1}) \neq \emptyset.$$

Choose a non-empty open set $U_{k+1} \subset \overline{U}_{k+1} \subset W$. It is straightforward that the properties (1)-(5) are fulfilled for $k+1$ as well. It follows from what we did in Case 1, that we may assume the inductive construction to go on for all natural k. Let $H = \bigcap\{U_k : k \in \omega\}$. Then H is a non-empty G_δ subset in X. Our proof will be finished if we establish that H is metrizable.

Indeed, let $x \in H$. Then $x \in \bigcap\{F_{n_k} : k \in \omega\}$ and $x \notin F_m$ if $m \neq n_k$ for all $k \in \omega$. But this means exactly that $\gamma(x) = \bigcap\{F_{n_k} : k \in \omega\}$. This set is metrizable and contains H, so H is metrizable. □

2.13. Example. *There exists a compact sequential non Frechet–Urysohn weakly metrizably fibered space.*

Proof: Let X be a Mrówka space [7]. We only need to know that X is compact, sequential, non Frechet–Urysohn space such that $X = Y \cup A$ where Y is one point compactification of the discrete space of power 2^ω and A is countable.

Fix a family μ which metrizably fibers Y and add to μ all points of A each one considered as a one-point subset. It is easy to see that the resulting family γ metrizably fibers X. □

2.14. Corollary. *Not every compact weakly metrizably fibered space is a continuous image of a first countable compact space.*

Proof: Indeed, any continuous image of a first countable compact space is a Frechet–Urysohn space. □

2.15. Theorem. *Let X be an Eberlein compact space of cardinality not exceeding continuum. Then X is weakly metrizably fibered.*

Proof: Any Eberlein compact space is a continuous image of a zero-dimensional Eberlein compact space [2, Ch. 4, §8]. It is clear from the definition of Eberlein compact space, that its cardinality is $\leq 2^\omega$ if and only if its weight is $\leq 2^\omega$. Therefore X can be represented as a continuous image of a zero-dimensional Eberlein compact space Y with $w(Y) \leq 2^\omega$.

Let $\mathcal{U} = \bigcup\{\mathcal{U}_n : n \in \omega\}$, where $\mathcal{U}_n$ is a point finite family of cozero open subsets of Y and the family $\mathcal{U}$ is T_0-separating in the sense that for any different $x, y \in Y$ there is a $U \in \mathcal{U}$ such that $|U \cap \{x,y\}| = 1$. Such a family exists in any Eberlein compact space by a Rosenthal's criterion [2, Ch.4, §4].

Any $U \in \mathcal{U}_n$ is Lindelöf and hence can be represented as a disjoint union $\bigcup\{V(U,k) : k \in \omega\}$ of clopen subsets of Y. Let $\mathcal{U}_{nk} = \{V(U,k) : U \in \mathcal{U}_n\}$ for all $n, k \in \omega$. The families $\mathcal{U}_{nk}$ are point finite and consist of clopen sets in Y. Of course, their union $\mathcal{V}$ T_0-separates the points of Y. Hence the family $\{\chi_U : U \in \mathcal{V}\}$ separates the points of Y. The map $\chi = \Delta\{\chi_U : U \in \mathcal{V}\}$ embeds Y into the Cantor cube $2^{\mathcal{V}}$. Let $\chi_{nk} = \Delta\{\chi_U : U \in \mathcal{U}_{nk}\}$ and $Y_{nk} = \chi_{nk}(Y)$. Then each Y_{nk} lies in the σ-product $S = \{f \in 2^{\mathcal{V}} : |\{f^{-1}(1)| < \omega\}$ of the cube $2^{\mathcal{V}}$. The space Y embeds as a closed subset into the product of Y_{nk}, so by 2.9(2) and 2.9(3) it suffices to establish that Y_{nk} is weakly metrizably fibered for any $n, k \in \omega$. So our proof is finished by the following

2.16. Lemma. *Let Z be a compact subset of S. Then Z is weakly metrizably fibered.*

Proof of the lemma. As $Z = \bigcup\{Z_n : n \in \omega\}$, where $Z_n = \{f \in Z : |f^{-1}(1)| = n\}$ and Z_n it suffices to prove the lemma for each Z_n. But every Z_n is a finite union of continuous images of closed subspaces of A^n where A is the Alexandroff compactification of the discrete space of power continuum. Now A is weakly metrizably fibered being a continuous image of the Alexandroff duplicate of the unit segment. Therefore A^n is weakly metrizably fibered and we are done. □

2.17. Example. *Under the negation of the Souslin hypothesis, there exists a perfectly normal Corson compact space which is not weakly metrizably fibered.*

Proof: Take any Souslin continuum S with all of its intervals non-separable. G.Gruenhage proved (see Remark 2.7) that for any continuous map of S onto a metrizable space the inverse image of some point contains a non-trivial interval. There exists an irreducible map f of S onto a Corson compact space X [9]. Being perfectly normal, the space X is weakly metrizably fibered iff it is metrizably fibered. Let g be a map of X onto a metrizable space M. The set $f^{-1}(g^{-1}(z))$ contains an open interval U for some $z \in M$. The map f is irreducible so there is an open non-empty subset $V \subset X$ such that $f^{-1}(V) \subset U$. It is clear that $V \subset g^{-1}(z)$. Hence $g^{-1}(z)$ can not be metrizable, because V is not separable. □

3. Unsolved problems.

Of course the most intriguing unsolved questions on the topic of this paper are the ones related to perfectly normal compact spaces. Before stating them the author would like to make it clear that he in no way pretends to be the first one who invented these questions.

3.1. Problem. *Is it consistent that any perfectly normal compact space is metrizably fibered?*

3.2. Problem. *Is it consistent that any perfectly normal compact space is obtained from the double arrow space using continuous images, closed subspaces and the products with second countable spaces?*

3.3. Problem. *Suppose that X is a metrizably fibered compact space. Is it true that every first countable continuous image of X is metrizably fibered?*

3.4. Problem. *Is any first countable weakly metrizably fibered compact space a continuous image of a metrizably fibered space?*

3.5. Problem. *Suppose that each continuous first countable image of a compact space X is metrizably fibered. Must X be perfectly normal?*

3.6. Problem. *Is any continuous image of the lexicographic square metrizably fibered?*

3.7. Problem. *Is it true in ZFC that any weakly metrizably fibered compact space is sequential?*

3.8. Problem. *Is the Helley space (i.e. the subspace of I^I with the topology of pointwise convergence which consists of monotone functions) (weakly) metrizably fibered?*

3.9. Problem. *Does there exist in ZFC a Corson compact space of power continuum which is not weakly metrizably fibered?*

References

[1] A. V. Arhangel'skii, *Structure and classification of topological spaces and cardinal invariants* (in Russian), Uspehi Mat. Nauk, **33 N 6** (1978), 29-84.

[2] A. V. Arhangel'skii, Topological function spaces (in Russian), Moscow State Univ. P. H., Moscow, 1989.

[3] Z. Balogh, *On compact Hausdorff spaces of countable tightness*, Proc. Amer. Math. Soc., **105** (1989), 755-764.

[4] R. Engelking, General Topology, PWN, Warszawa, 1977.

[5] G. Gruenhage, *Perfectly normal compacta, cosmic spaces and some partiton problems*, in: Open Problems in Topology, ed. J. van Mill and G. M. Reed, North Holland P. H., 1990, 86-95.

[6] V. I. Malykhin, *A Frechet–Urysohn bicompactum without points of countable character*, Math. Notes, **41** (1987), 210-216.

[7] S. Mrówka, *On completely regular spaces*, Fund. Math., **41 N 1** (1954), 105-106.

[8] A. Ostaszewski, *On countably compact perfectly normal spaces*, J. London Math. Soc., **14** (1976), 505-516.

[9] B. E. Shapirovsky, *On tightness, π-weight and related notions* (in Russian), Notes of Riga University, **N 3** 1976, 88-89.

Universidad Autónoma Metropolitana,
Av. Michoacán y La Purísima,
Iztapalapa, A.P.55-534, C.P.09340,
México, D.F.
e-mail: vova@xanum.uam.mx