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ON ω^* AND ABSOLUTELY DIVERGENT SERIES

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ABSTRACT. In this paper we summarize some of our former results on series, ultrafilters and cardinal characteristics in a new unified manner by Galois-Tukey connections. Using some new observations about the connection between separative factorization of the comparison ordering of divergent series and ω^* we get a new insight into these older results. This gives a new type of characterization of points of ω^* and a (possibly) new sort of duality.

Using Galois-Tukey connections we rephrase some of our former results from [V1], [V2] and [CV] in the language of [V3]. We recall some basic facts and introduce notation (to be selfcontained) concerning ω^*—the reminder of the Čech-Stone compactification of natural numbers, series and cardinal characteristics. Studying nowhere dense subsets of ω^* generated by series we characterize the separative factorization of the comparison ordering of absolutely divergent series (downwards). Moreover the same structure concerned upwards gives a new type of characterization of points of ω^* (we show it on Q-points and rapid ultrafilters).

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The remainder of the Čech-Stone compactification of natural numbers.

Let ω denotes the set of natural numbers, $[\omega]^\omega$ is the system of all infinite subsets of ω, $[\omega]^{<\omega}$ is the system of all finite subsets of ω, $\mathcal{P}(\omega)/\mathrm{fin}$ is the Boolean algebra of subsets of ω modulo ideal of finite sets (sometimes seen as $[\omega]^\omega$). The Stone space of algebra $\mathcal{P}(\omega)/\mathrm{fin}$ is denoted $\omega^* = \mathrm{St}(\mathcal{P}(\omega)/\mathrm{fin})$ and equipped with the topology generated by base consisting of sets of form: for $A \subseteq \omega$ let $A^* = \{j : j \text{ is a uniform ultrafilter on } \omega \text{ and } A \in j\}$. We will often without noting switch from $j \in \omega^*$ to $j \subseteq [\omega]^\omega$ and back. For an ideal $\mathcal{I}$ on ω, $\mathcal{I}^+ = \mathcal{P}(\omega) \setminus \mathcal{I}$ and $\mathcal{F}_\mathcal{I}$ denotes the dual filter (and vice versa for a filter $\mathcal{F}$ on ω, $\mathcal{I}_\mathcal{F}$ is the dual ideal). Ideals and filters on ω can be viewed (represented) as subsets of ω^* in the following way: $\sigma(\mathcal{I}) = \bigcup\{A^* : A \in \mathcal{I}\}$ is the open set corresponding to $\mathcal{I}$ and $\delta(\mathcal{F}) = \bigcap\{A^* : A \in \mathcal{F}\}$ is the closed set corresponding to $\mathcal{F}$. For $\mathcal{F}, \mathcal{G} \subseteq \mathcal{P}(\omega)$, $\langle \mathcal{F} \cup \mathcal{G} \rangle$ denotes the smallest filter (if at all) generated by $\mathcal{F} \cup \mathcal{G}$. Note that $\sigma(\mathcal{I})$ is open dense iff $\delta(\mathcal{F}_\mathcal{I})$ is nowhere dense iff $\mathcal{I}$ is tall (i.e. $(\forall X \in [\omega]^\omega)(\exists Y \in [X]^\omega)(Y \in \mathcal{I})$). The mapping i : open$(\omega^*) \to$ ideals on ω defined by $i(G) = \{X \subseteq \omega : X^* \subseteq G\}$ is order isomorphism from $(\text{open}(\omega^*), \subseteq)$ into (not onto) (ideals on $\omega, \subseteq$) in some sense inverse to σ : ideals on $\omega \to$ open(ω^*) defined above. (Similarly for filters, δ and its inverse.) Standard reference sources in topology are [E], [vD], [vM], [W].

Series, comparison and ideals.

In the whole paper we deal only with absolute convergence and divergence, hence our basic object is ${}^\omega\langle 0, +\infty)$, the space of all sequences of nonnegative reals. Elements of ${}^\omega\langle 0, +\infty)$ are usually denoted a, b, c; the n-th entry is $a(n)$ or sometimes a_n.

$$\ell^1 = \left\{ a \in {}^\omega\langle 0, +\infty) : \sum_{n=0}^{\infty} a_n < +\infty \right\},$$

$$\ell^\infty = \left\{ a \in {}^\omega\langle 0, +\infty) : \overline{\lim_{n\to\infty}} a_n < +\infty \right\},$$

$$h_0 = \left\{ a \in {}^\omega\langle 0, +\infty) : \underline{\lim_{n\to\infty}} a_n = 0 \right\},$$

$$c_0 = \left\{ a \in {}^\omega\langle 0, +\infty) : \lim_{n\to\infty} a_n = 0 \right\},$$

and for $a, b \in {}^\omega\langle 0, +\infty)$ we say that a is eventually dominated by b, denoted $a \leq^* b$, if there is a n_0 such that for all $n \geq n_0$ is $a_n \leq b_n$. For $a \in {}^\omega\langle 0, +\infty)$ define

$$\mathcal{I}_a = \left\{ X \subseteq \omega : \sum_{n \in X} a_n < +\infty \right\},$$

denote $\mathcal{F}_a = \mathcal{F}_{\mathcal{I}_a}$. Observe that $a \in c_0$ iff $\delta(\mathcal{F}_a)$ is nowhere dense iff $\sigma(\mathcal{I}_a)$ is open dense. Moreover
$a \in \ell^1$ iff $\delta(\mathcal{F}_a) = \emptyset$ iff $\sigma(\mathcal{I}_a) = \omega^*$,
$a \in h_0 \setminus c_0$ iff $\mathrm{Int}(\delta(\mathcal{F}_a)) \neq \emptyset$ and $\delta(\mathcal{F}_a) \neq \omega^*$,
$a \in {}^\omega\langle 0, +\infty) \setminus h_0$ iff $\delta(\mathcal{F}_a) = \omega^*$ iff $\sigma(\mathcal{I}_a) = \emptyset$ (because $\mathcal{I}_a = [\omega]^{<\omega}$).
Hence $a \in c_0 \setminus \ell^1$ iff $\emptyset \neq \delta(\mathcal{F}_a)$ is closed nowhere dense subset of ω^*, and those we are interested in. Standard reference source for real analysis is [F].

Cardinal characteristics, Galois-Tukey connections.

There is a large variety of cardinal characteristics studied in applications of set theory in real analysis, topology, algebra etc. (see [vD], [vM], [V]). An attempt of a unifying approach was given in [V3] (we follow it here). For arbitrary binary relation R we say that $D \subseteq \mathrm{rng}(R)$ is R-dominating if $(\forall x \in \mathrm{dom}(R))(\exists y \in D)((x,y) \in R)$ and $B \subseteq \mathrm{dom}(R)$ is R-unbounded if $(\forall y \in \mathrm{rng}(R))(\exists x \in B)((x,y) \notin R)$. Define

$$\mathfrak{b}(R) = \min\{|B| : B \subseteq \mathrm{dom}(R) \text{ and } B \text{ is } R\text{-unbounded } \},$$

$$\mathfrak{d}(R) = \min\{|D| : D \subseteq \mathrm{rng}(R) \text{ and } D \text{ is } R\text{-dominating } \}.$$

Lot of cardinal invariants studied in [vD], [vM], [V] are of this form. To prove inequalities between cardinal characteristics we introduced in [V3] the following machinery: A pair of functions (E, F) is called a Galois-Tukey connection from R to S if E : dom$(R) \to$ dom(S) and F : rng$(S) \to$ rng(R) and $(E(x), v) \in S$ implies $(x, F(v)) \in R$. Note that if there is a Galois-Tukey connection from R to S then $\mathfrak{b}(S) \leq \mathfrak{b}(R)$ and $\mathfrak{d}(S) \geq \mathfrak{d}(R)$. The fact that there is a connection from R to S will be denoted by $R \to S$.

Nowhere dense sets of ω^* generated by series.

Observe that $a \leq^* b$ implies $\delta(\mathcal{F}_a) \subseteq \delta(\mathcal{F}_b)$.

Lemma. *For $a, b \in {}^\omega\langle 0, +\infty)$ is $\delta(\mathcal{F}_a) \cap \delta(\mathcal{F}_b) = \delta(\mathcal{F}_{\min(a,b)}) = \delta(\langle \mathcal{F}_a \cup \mathcal{F}_b \rangle)$.*

Proof: (1^{st} eq., $\subseteq$) Let $j \in \delta(\mathcal{F}_a) \cap \delta(\mathcal{F}_b)$ i.e. $\mathcal{F}_a \cup \mathcal{F}_b \subseteq j$. We try to prove $\mathcal{F}_{\min(a,b)} \subseteq j$. Suppose $Y \in \mathcal{F}_{\min(a,b)}$ i.e. for $X = \omega \setminus Y$ we have $\sum_{n \in X} \min(a, b)(n) < +\infty$. Denote $X_1 = \{n \in X : a(n) < b(n)\}$, $X_2 = \{n \in X : a(n) = b(n)\}$ and $X_3 = \{n \in X : a(n) > b(n)\}$. Then $X_1 \in \mathcal{I}_a$ because on X_1 is $a(n) = \min(a(n), b(n))$, similarly $X_2 \in \mathcal{I}_a \cap \mathcal{I}_b$, and $X_2 \in \mathcal{I}_b$. So $\omega \setminus X_1 \in \mathcal{F}_a \subseteq j$, $\omega \setminus X_2 \in \mathcal{F}_a \subseteq j$, $\omega \setminus X_3 \in \mathcal{F}_b \subseteq j$ hence $Y = \omega \setminus X = \bigcap_{i=1}^{3} (\omega \setminus X_i) \in j$.

(1^{st} eq., $\supseteq$) Easy.

(2^{nd} eq.) As ω^* is regular and all sets involved are closed, it is enough to notice that $\delta(\mathcal{F}_a) \cap \delta(\mathcal{F}_b)$ and $\delta(\langle \mathcal{F}_a \cup \mathcal{F}_b \rangle)$ have same neighborhoods.

Corollary. $\delta(\mathcal{F}_a) \cap \delta(\mathcal{F}_b) = \emptyset$ *iff* $\min(a, b) \in \ell^1$.

Lemma. $\delta(\mathcal{F}_a) \cup \delta(\mathcal{F}_b) = \delta(\mathcal{F}_{\max(a,b)})$.

Proof: ($\supseteq$) Take $j \in \omega^*$ such that $\mathcal{F}_{\max(a,b)} \subseteq j$ we are going to prove that either $\mathcal{F}_a \subseteq j$ or $\mathcal{F}_b \subseteq j$. Suppose not, i.e. we have $X \in \mathcal{F}_a \setminus j$ and $Y \in \mathcal{F}_b \setminus j$ (i.e. $\sum_{n \in \omega \setminus X} a(n) < \infty$ and

$\sum_{n\in\omega\setminus Y} b(n) < \infty$). But then for $Z = (\omega \setminus X) \cap (\omega \setminus Y) = (\omega \setminus (X \cup Y))$ we have

$$\sum_{n\in Z} \max(a,b)(n) \leq \sum_{n\in\omega\setminus X} a(n) + \sum_{n\in\omega\setminus Y} b(n) < +\infty$$

hence $X \cup Y \in \mathcal{F}_{\max(a,b)} \subseteq j$ but $X \cup Y \in j$ gives, as j is an ultrafilter, either $X \in j$ or $Y \in j$ contradiction.

($\subseteq$) is again easy.

Definition For $a \in c_0 \setminus \ell^1$ and $X \in [\omega]^\omega$ denote $a \restriction X$ series defined by

$$(a \restriction X)(n) = a(n) \text{ if } n \in X,$$
$$(a \restriction X)(n) = 0 \text{ if } n \notin X.$$

Lemma. *(i)* $a \restriction X \in c_0 \setminus \ell^1$ *iff* $X^* \cap \delta(\mathcal{F}_a) \neq \emptyset$ *iff* $X \in \mathcal{I}_a^+$.
(ii) $\delta(\mathcal{F}_{a\restriction X}) = \delta(\mathcal{F}_a) \cap X^* = \delta(\langle \mathcal{F}_a \cup \{X\}\rangle)$.

Proof: (i) is easy.

(ii, $\supseteq$) Let $j \in \delta(\mathcal{F}_a) \cap X^*$ i.e. $\mathcal{F}_a \subseteq j$, $X \in j$ and take $Y \in \mathcal{F}_{a\restriction X}$. We show that $Y \in j$. Let us calculate how the sum $\sum_{n\in\omega\setminus Y}(a \restriction X)(n) < \infty$ is brought up. For $n \in \omega \setminus (X \cup Y)$ is $(a \restriction X)(n) = 0$ so remaining part $\sum_{n\in X\setminus Y}(a \restriction X)(n) < \infty$ i.e. $X \setminus Y \in \mathcal{I}_a$ so

$\omega \setminus (X \setminus Y) \in \mathcal{F}_a \subseteq j$ and this together with $X \in j$ gives $\omega \setminus (X \setminus Y) \cap X = X \cap Y \in j$ hence $Y \in j$.

($\subseteq$) as $a \restriction X \leq a$ and $X \in \mathcal{F}_{a\restriction X}$ (if at all) and we are done.

Corollary. $(\forall a \in c_0 \setminus \ell^1)(\delta(\mathcal{F}_a)$ *is dense in itself*).

Proof: If not, then for some $X \subseteq \omega$ is $\delta(\mathcal{F}_a) \cap X^*$ an ultrafilter, but as we can easily see, for all $a \in c_0 \setminus \ell^1$, $\mathcal{F}_a$ is not an ultrafilter. Indeed, any $Y \in \mathcal{F}_a$ can be split to Y_1, Y_2 in such a way that $\sum_{n\in Y_1} a(n) = \sum_{n\in Y_2} a(n) = +\infty$, i.e. both $Y_1, Y_2 \in \mathcal{I}_a^+$.

Of course not all dense in itself nwd subsets of ω^* are of form $\delta(\mathcal{F}_a)$, but we do not deal with this characterization problem here.

We finished this introductory part and we recall now (a reformulated version of) results from [V1], [V2] and [CV]. We show how they are interrelated and what does this say about the structure of nowhere dense subsets of ω^* generated by series.

Rapid ulterafilters and their noncentered version.

Definition We define binary relation CONV $\subseteq c_0 \times [\omega]^\omega$ as follows

$$(a, X) \in \text{CONV iff } \sum_{n \in X} a(n) < \infty.$$

Theorem 1. (see [V1]). *Relation CONV and* $(<^* \cap ({}^\omega\omega)^2)$ *are Galois-Tukey equivalent (i.e. there are Galois-Tukey connections in both directions).*

Proof: (i) Galois-Tukey connection from $({}^\omega\omega, <^*)$ (exactly speaking we consider only increasing functions) to CONV. For $f \in {}^\omega\omega$ define $E(f)(i) = \frac{\log(n+1)}{n+1}$ if $i \in (f(n-1), f(n)\rangle$ for $n > 0$, else arbitrary and for $X \in [\omega]^\omega$ put $F(X) = e_X$ where e_X is the unique increasing enumeration of X. We have to prove that $(E(f), X) \in$ CONV implies $f <^* e_X$. Indeed, if there are infinitely many n's with $e_X(n) \leq f(n)$ then

$$\sum_{n \in X} a(n) = \sum_{n=0}^{\infty} a(e_X(n)) \geq \lim_{n \to \infty} (n+1) \frac{\log(n+1)}{n+1} = +\infty.$$

Let us note, that the proof of this part of the theorem owes much to a result of E. Copláková, which was a part of preliminary version of [CV] but did not appear in the final one.

(ii) conversely for $a \in c_0$ put $H(a)(k) = \min\{i : (\forall j \geq i)(a(j) < \frac{1}{2^k})\}$ and $K(g) = \text{rng}(g)$. Again easily $H(a) <^* g$ implies $(a, \text{rng}(g)) \in$ CONV.

Definition (G. Choquet) An ultrafilter $j \in \omega^*$ is said to be rapid if the family of functions enumerating elements of j, $\{e_X : X \in j\}$ is a dominating family in $({}^\omega\omega, <^*)$.

Corollary. *TFAE.*

(i) j *is rapid.*
(ii) $(\forall a \in c_0)(\exists X \in j)(\sum_{n \in X} a(n) < +\infty)$ *i.e.* $(\forall a \in c_0)(\exists X \in j)(X \in \mathcal{I}_a)$.
(iii) $j \in \bigcap_{a \in c_0} \sigma(\mathcal{I}_a)$.
(iv) $j \in \omega^* \setminus \bigcup_{a \in c_0} \delta(\mathcal{F}_a)$.

Proof (i) $\to$ (ii) If $\{e_X : X \in j\}$ is dominating then $\{K(e_X) : X \in j\} = j$ is CONV-dominating. K being that of Theorem 1 (ii). Conversely (ii) $\to$ (i) is emphasized by the mapping F of Theorem 1 (i). (ii) $\leftrightarrow$ (iii) $\leftrightarrow$ (iv) is easy.

Let us notify (though we will emphasize it later) that this gives a new type of characterization of a class of ultrafilters.

Q-POINTS.

Similar idea (though not precisely formulated there) is behind the main result of [CV]. Recall that $j \in \omega^*$ is a Q-point if for every disjoint partition of ω into finite pieces $\mathcal{R} \subseteq [\omega]^{<\omega}$ there is an $X \in j$ such that for all $R \in \mathcal{R}$ is $|X \cap R| \leq 1$. In order to fit in the previous pattern we have to change it (equivalently).

Definition. Denote $\mathbb{R}$ the system of all $\mathcal{R} \subseteq [\omega]^{<\omega}$ disjoint partitions of ω, put $\mathcal{I}_\mathcal{R} = \{X \subseteq \omega : (\exists k)(\forall R \in \mathcal{R})|R \cap X| \leq k\}$. The dual filter we denote $\mathcal{F}_\mathcal{R}$. For $\mathcal{R}, \mathcal{S} \in \mathbb{R}$ define $\mathcal{R} \preccurlyeq \mathcal{S}$ iff $\mathcal{I}_\mathcal{R} \supseteq \mathcal{I}_\mathcal{S}$.

Observation. *TFAE.*

(i) j *is a Q-point.*
(ii) $(\forall \mathcal{R} \in \mathbb{R})(\exists X \in j)(\exists k \in \omega)(\forall R \in \mathcal{R})(|R \cap X| \leq k)$ *i.e.* $X \in \mathcal{I}_\mathcal{R}$.
(iii) $j \in \bigcap_{\mathcal{R} \in \mathbb{R}} \sigma(\mathcal{I}_\mathcal{R})$.

(iv) $j \in \omega^* \setminus \bigcup_{\mathcal{R} \in \mathbb{R}} \delta(\mathcal{F}_{\mathcal{R}})$.

Proof: Easy, just observe that if $X \in j$ is such that for all $R \in \mathcal{R}$ is $|X \cap R| \leq k$ we can split X into k disjoint pieces, each hitting R at most once and as j is ultrafilter, one of these k-many pieces is in j.

Theorem 2. (see [CV]). *There is a Galois-Tukey connection from* $(\mathbb{R}, \preccurlyeq)$ *to* $({}^{\omega}\omega, \leq^*)$.

Proof: First for given partition $\mathcal{R}$ we construct a mapping $f_{\mathcal{R}}$ (the E-mapping of the very connection). We follow the proof of [CV]. By glueing together elements of $\mathcal{R}$ and "rounding" it we can obtain an interval partition $\mathcal{R}' \succcurlyeq \mathcal{R}$, defined by function $f_{\mathcal{R}}$. By induction, enumerate $\mathcal{R} = \{R_n : n \in \omega\}$,

$$f_{\mathcal{R}}(0) = \max\{\max(R) : R \in \mathcal{R} \& R \cap \langle 0, \max(R_0)\rangle \neq \emptyset\},$$
$$f_{\mathcal{R}}(n+1) = \max\{\max(R) : R \in \mathcal{R} \& R \cap \langle 0, \max(R_{n+1}) + f_{\mathcal{R}}(n) + 1\rangle \neq \emptyset\},$$

put $\mathcal{R}' = \{(f_{\mathcal{R}}(n), f_{\mathcal{R}}(n+1)\rangle : n \in \omega\} \cup \{\langle 0, f_{\mathcal{R}}(0)\rangle\}$. For $X \in \mathcal{I}_{\mathcal{R}'}$ take k such that $(\forall R \in \mathcal{R}')(|X \cap R| \leq k)$ then $(\forall R \in \mathcal{R})(|X \cap R| \leq 2k+1)$. For a monotone function $g \in {}^{\omega}\omega$ we would like to define $F(g)$ a partition such that the apropriate implication involved in this connection is valid. $F(g)$ will be an interval partition generated by function $\overline{g}$ defined by induction $\overline{g}(0) = g(0)+1$, $\overline{g}(n+1) = g(\overline{g}(n) + g(n) + 1) + 1$. Note that if for some monotone $f \in {}^{\omega}\omega$ dominated by g and $\xi \in \omega$ we have $f(\xi) < \overline{g}(n) \leq f(\xi+1)$ then $\overline{g}(n+1) > g(\overline{g}(n) + g(n) + 1) \geq f(\overline{g}(n) + g(n) + 1) \geq f(f(\xi)+1) \geq f(\xi+1)$. Now assume $E(\mathcal{R}) = f_{\mathcal{R}} \leq^* g$ and $F(g)$ is the partition given by $\overline{g}$. If $X \subseteq \omega$ and $k \in \omega$ are such that $(\forall R \in F(g))(|X \cap R| \leq k)$ then $(\forall R \in \mathcal{R}')(|X \cap R| \leq 2k+c)$ (the constant c depending on where from g dominates $f_{\mathcal{R}}$) and $(\forall R \in \mathcal{R})(|X \cap R| \leq 2(2k+c)+1)$ i.e. $E(\mathcal{R}) \leq^* g$ implies $\mathcal{R} \preccurlyeq F(g)$.

EXISTENCE THEOREMS.

Definition Let $\mathfrak{n}(\omega^*)$ be minimal size of a family of nowhere dense subsets of ω^* covering the whole ω^*.

Note first that each Q-point is rapid, but because of the later problem of considering properties of induced ordering of nwd subsets of ω^* also downwards we deal with both types of existence theorems. As the size of $c_0 \setminus \ell^1$ and of $\mathbb{R}$ is $\mathfrak{c}$ and ideals $\mathcal{I}_a$ and $\mathcal{I}_\mathcal{R}$ are tall it can be proved that $\mathfrak{t} = \mathfrak{c}$ implies there are Q-points (and rapids) just by induction building a tower of witnesses for each $a \in c_0 \setminus \ell^1$ (or $\mathcal{R} \in \mathbb{R}$). Using tallness of ideals parallelly we can by induction (under $\mathfrak{h} = \mathfrak{c}$) even build a MAD-families of such witnesses and every long chain in such a matrix (if there are) produces rapids, Q-points. Moreover, notice that if $\mathfrak{n}(\omega^*) > \mathfrak{c}$ then neither $\bigcup_{a\in c_0\setminus\ell^1} \delta(\mathcal{F}_a)$ nor $\bigcup_{\mathcal{R}\in\mathbb{R}} \delta(\mathcal{F}_\mathcal{R})$ can cover the whole ω^* and hence there are rapids and Q-points. (To compare this estimates with that of others see exhaustive references in [CV]). Moreover note that it is known to be consistent with set-theory that there are no P-points and that there are no rapids (see references in [CV]). We now give an existence theorem which is stronger (or at least not weaker) than those known from literature (see [CV]).

Corollary. ([CV]). *If* $\mathfrak{n}(\omega^*) > \mathfrak{d} = \mathfrak{d}({}^\omega\omega, <^*)$ *then there are Q-points.*

Proof: Using the F mapping of previous theorem we can convert any dominating family $D = \{g_\alpha : \alpha \in \mathfrak{d}\}$ of $({}^\omega\omega, <^*)$ into a family of partitions $\{F(g_\alpha) : \alpha \in \mathfrak{d}\}$ such that as for every $\mathcal{R} \in \mathbb{R}$ there is an $\alpha \in \mathfrak{d}$ with $f_\mathcal{R} \leq^* g_\alpha$, and hence $\mathcal{R} \preccurlyeq F(g_\alpha)$ i.e. $\mathcal{I}_\mathcal{R} \supseteq \mathcal{I}_{F(g_\alpha)} = \mathcal{I}_\alpha$ (the dual filter denote $\mathcal{F}_\alpha$) i.e. $\delta(\mathcal{F}_\mathcal{R}) \subseteq \delta(\mathcal{F}_\alpha)$. So

$$\bigcup_{\mathcal{R}\in\mathbb{R}} \delta(\mathcal{F}_\mathcal{R}) = \bigcup_{\alpha\in\mathfrak{d}} \delta(\mathcal{F}_\alpha).$$

But as for any partition $\mathcal{R}$, $\mathcal{F}_\mathcal{R}$ is nowhere dense in ω^* and as

$\mathfrak{d} < \mathfrak{n}(\omega^*)$ these $\mathfrak{d}$-many filters cannot cover the whole ω^* so

$$\omega^* \setminus \bigcup_{\mathcal{R}\in\mathbb{R}} \delta(\mathcal{F}_{\mathcal{R}}) = \omega^* \setminus \bigcup_{\alpha\in\mathfrak{d}} \delta(\mathcal{F}_\alpha) \neq \emptyset$$

i.e. there are Q-points.

Similarly we need to estimate the number of series necessary to define rapid filters (via induced filters, nowhere dense in ω^*). Though every Q-point is rapid and existence follows, we deal with this for other reasons, as mentioned above.

Definition.

(i) $\mathbb{F}_q = \{\delta(\mathcal{F}_{\mathcal{R}}) : \mathcal{R} \in \mathbb{R}\}$.
(ii) $\mathbb{F}_r = \{\delta(\mathcal{F}_a) : a \in c_0 \setminus \ell^1\}$.

Note that we showed $(\mathbb{F}_q, \subseteq) \to (\mathbb{R}, \preccurlyeq) \to ({}^\omega\omega, \leq^*)$ (the first connection being easy as $\mathcal{I}_{\mathcal{R}} \supseteq \mathcal{I}_{\mathcal{S}}$ implies $\delta(\mathcal{F}_{\mathcal{R}}) \subseteq \delta(\mathcal{F}_{\mathcal{S}})$). A similar result holds for $\mathbb{F}_r$.

Theorem 3. $(\mathbb{F}_r, \subseteq) \to (c_0 \setminus \ell^1, \leq^*) \leftrightarrow ({}^\omega\omega, \leq^*)$.

Proof: The first connection is easy as $a <^* b$ implies $\mathcal{I}_a \supseteq \mathcal{I}_b$ and this gives $\delta(\mathcal{F}_a) \subseteq \delta(\mathcal{F}_b)$. To establish the last, first $\to$: define $E(a)(n) = \min\{i : (\forall j > i)\ a(j) < \frac{1}{n+1}\}$ and $F(f)(i) = \frac{1}{n+1}$ if $i \in \langle f(n), f(n+1))$ fulfills, that $E(a) <^* f$ implies $a <^* F(f)$ conversely for $\leftarrow$ same mappings fulfill $F(f) <^* b$ implies $f <^* E(b)$.

Corollary. *If* $\mathfrak{n}(\omega^*) > \mathfrak{d}$ *then there are rapid ultrafilters.*

Proof: Using F-mapping of connection from previous Theorem we convert arbitrary dominating family of $({}^\omega\omega, <^*)$ of minimal size into system of nowhere dense sets which covers the same portion of ω^* as all filters generated by series do. But as there are not enough of them to cover the whole ω^*, there are rapid ultrafilters.

So looking to $(\mathbb{F}_r, \subseteq)$ and $(\mathbb{F}_q, \subseteq)$ upwards, both have the $\mathfrak{d}$-numbers smaller than or equal to $\mathfrak{d}({}^\omega\omega, <^*)$ and hence both under $\mathfrak{n}(\omega^*) > \mathfrak{d}$ define a nonempty class of ultrafilters.

There is yet another interesting feature of looking to these ordering downwards. For $c_0 \setminus \ell^1$ (or even ${}^\omega\langle 0, +\infty) \setminus \ell^1$) we can look to $<^*$ as an ordering from the comparison test for absolute divergence. The smaller series the more information about divergence it carries (in the forcing sense). Note that for absolute convergence the analogous problem dealt with upwards directed structure. Nevertheless here the problem is "Boolean like", as there are two divergent series (e.g. one with divergency concentrated to odd numbers (i.e. $\delta(\mathcal{F}_a) \subseteq (2\mathbb{N}+1)^*$) and one with divergency concentraced to even number (i.e. $\delta(\mathcal{F}_a) \subseteq (2\mathbb{N})^*$)) with no divergent series below both of them. So the problem of how efficient is the comparison ordering for the absolute divergence is no more a problem of characterizing some cardinal invariants (as $\mathfrak{b}(c_0 \setminus \ell^1, \geq^*) = 2$ and $\mathfrak{d}(c_0 \setminus \ell^1, \geq^*) = 2^\omega$) but more a problem of characterizing the Boolean structure generated by this partial ordering.

Partial orders and complete Boolean algebras.

Assume $(P, <)$ is a partial ordering without the smallest element. We say that $x, y \in P$ are compatible $(x \mid y)$ if there is a $z \in P$ with both $z \leq x$ and $z \leq y$. Elements $x, y \in P$ are incompatible $(x \perp y)$ if they are not compatible. For a partial ordering $(P, <)$ we can find a complete Boolean algebra B such that $(P, <)$ is order preserving mapped onto a dense subset of B in the following way (see [J]). On P define a topology generated by basic open sets of the form $\{x : x \leq p\}$ for $p \in P$ (called the cut-topology). The system of regular open sets in this topology forms a complete Boolean algebra and for every p there can be assigned the set $\mathrm{Int}(\mathrm{cl}(\{x : x \leq p\}))$. This is an order preserving mapping from $(P, <)$ into $(\mathrm{RO}(P, <), \subseteq)$. This mapping is one-to-one if P is separative (i.e. if for any $x \not\leq y$ there is a $z \leq x$ with $z \perp y$).

Theorem 4. ([V2]). *If* $\mathfrak{p} = cf(2^\omega)$ *then* $RO(\ell^\infty \setminus \ell^1, \leq^*) \cong RO(\mathcal{P}(\omega)/fin)$.

This theorem states that looking to $(\ell^\infty \setminus \ell^1, \leq^*)$ as a partial ordering downwards, it generates (uniquely) a complete Boolean algebra (which at least consistently is isomorphic to the complete Boolean algebra of regular open subsets of ω^*).

SEPARATIVE FACTORIZATION OF THE COMPARISON ORDERING OF DIVERGENT SERIES AND ω^*.

Note that the ordering $(\ell^\infty \setminus \ell^1, \leq^*)$ is not separative because e.g. $\{\frac{1}{n}\}_{n=0}^\infty \not\leq^* \{\frac{1}{2n}\}_{n=0}^\infty$ but for every $a \in \ell^\infty \setminus \ell^1$, $a \leq^* \{\frac{1}{n}\}_{n=0}^\infty$, we have also $\min(a(n), \frac{1}{2n}) \geq \frac{1}{2}\min(a(n), \frac{1}{n}) \notin \ell^1$. (For terminology needed see [J].) In [J] there is described a way, how a partial ordering which is not separative can be converted to a separative one: factorizing by a suitable equivalence, which topologically says, the generated cuts have the same interior of closure.

Lemma. *TFAE.*

(i) $a \mid b$.
(ii) $\min(a, b) \notin \ell^1$.
(iii) $\delta(\mathcal{F}_a) \cap \delta(\mathcal{F}_b) \neq \emptyset$.

Proof: (i) $\leftrightarrow$ (ii) by the definition of being compatible. (ii) $\leftrightarrow$ (iii) follows from Lemma stating $\delta(\mathcal{F}_a) \cap \delta(\mathcal{F}_b) = \delta(\mathcal{F}_{\min(a,b)})$.

Theorem. *In the partial ordering* $(\ell^\infty \setminus \ell^1, \leq^*)$ *with the cut-topology*
$Int(cl(\{c : c \leq^* a\})) = Int(cl(\{c : c \leq^* b\}))$ *iff* $\delta(\mathcal{F}_a) = \delta(\mathcal{F}_b)$.

Proof: By [J] we see, that it is enough to show that $(\forall c)(c \mid a \leftrightarrow c \mid b)$ iff $\delta(\mathcal{F}_a) = \delta(\mathcal{F}_b)$.

Sufficiency. Assume $\delta(\mathcal{F}_a) = \delta(\mathcal{F}_b)$ and we have a c with $c \mid a$ by previous Lemma $c \mid a$ iff $\delta(\mathcal{F}_c) \cap \delta(\mathcal{F}_a) \neq \emptyset$ iff $\delta(\mathcal{F}_c) \cap \delta(\mathcal{F}_b) \neq \emptyset$ iff $c \mid b$.

Necessity. Assume by contradiction that $(\forall c)(c \mid a \leftrightarrow c \mid b)$ but, say, there is a $j \in \delta(\mathcal{F}_a) \setminus \delta(\mathcal{F}_b)$. As ω^* is a regular topological space there is an $X \in j$ with $X^* \cap \delta(\mathcal{F}_b) = \emptyset$. As $j \in \delta(\mathcal{F}_a) \cap X^*$ the series $c = a \upharpoonright X \notin \ell^1$, clearly $c \mid a$ and $c \perp b$, contradiction.

Corollary. *The ordering* $(\mathbb{F}_r, \subseteq)$ *is the canonical separative factorization of the ordering* $(c_0 \setminus \ell^1, \leq^*)$.

We finish our paper by the following motivation of a problem. We showed that rapid ultrafilters and Q-points in ω^* are defined (besides topological and combinatorial definitions) as those points in ω^* which are not covered by a certain family of nowhere dense sets. (It is not our aim to show here that this is also true for other classes of ultrafilters in ω^*.) Up to this the rôle of a family $\mathbb{F}$ of nowhere dense subsets of ω^* is described by the number $\mathfrak{d}(\mathbb{F}, \subseteq)$ as the number of nwd sets necessary to cover everything what is possible to cover by whole $\mathbb{F}$, which leads to theorems of type $\mathfrak{n}(\omega^*) > \mathfrak{d}(\mathbb{F}, \subseteq)$ then there are "$\mathbb{F}$-points". Moreover we showed, that (at least for $\mathbb{F}_r$) this family $(\mathbb{F}, \subseteq)$ considered downwards as a partial ordering can be Boolean-isomorphic to some other Boolean algebra (we showed at least under $\mathfrak{p} = \mathrm{cf}(2^\omega)$ that $\mathrm{RO}(\mathbb{F}_r, \subseteq) \cong \mathrm{RO}(\omega^*)$). This can be also shown for other types of ultrafilters (see e.g. [KV]). But these are not problems we would like to point out.

We would like to emphasize the problem which we find topologically interesting: Characterize those families $\mathbb{F}$ of nowhere dense subsets of ω^* which are (at least consistently) Boolean isomorphic to $\mathrm{RO}(\omega^*)$ (i.e. $\mathrm{RO}(\mathbb{F}, \subseteq) \cong \mathrm{RO}(\omega^*)$). Moreover we have a feeling that there is a new form of duality hidden behind this phenomenon, though we are not able to formulate it precisely.

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