Topology Proceedings

Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings
	Department of Mathematics & Statistics
	Auburn University, Alabama 36849, USA
E-mail:	topolog@auburn.edu
ISSN:	0146-4124

Topology Proceedings
Vol. 19, 1994

ON PROBLEMS OF A. V. ARHANGEL'SKIĬ AND I. YU. GORDIENKO

YOSHIKAZU YASUI

ABSTRACT. A topological space X is said to be relatively locally finite, if each point of X has a neighborhood U such that every closed (in X) subset of U is finite.

We shall construct an infinite countably compact Hausdorff space X which is relatively locally finite. This gives the negative answers of A. V. Arhangel'skiĭ and I. Yu. Gordienko. Furthermore we shall give a characteriza ion of relatively local finiteness and the corollaries.

0. INTRODUCTION

Recently A. V. Arhangel'skiĭ and I. Yu. Gordienko [1] introduced a relatively locally finite space. Its property is very near to discreteness.

It is able to assert that discreteness is dual to compactness. Many generalizations and surrounding of compactness type properties (for example, countable compactness, pseudocompactness etc.) have been studied and distinguished themselves in the branch of general topology. Furthermore the necessary and sufficient conditions that imply compactness, are known.

So it is natural to consider that generalizations of discreteness will be in an important position in general topology. Indeed, the notion of scattered space has shown that it has played an important role in general topology and its applications can be clearly classified as a discreteness type notion.

A relatively locally finite property is equivalent to discreteness in the class of regular T_1 spaces. But it is known that there are non-discrete Hausdorff relatively locally finite spaces (see [1]).

In [1], they proved some sufficient conditions for a Hausdorff relatively locally finite space to be discrete.

In connection with this view, they posed the following interesting problems in [1]:

Problem A. *Is every Hausdorff countably compact relatively locally finite space finite?*

Problem B. *Is every Hausdorff relatively locally finite space of countable extent discrete?*

Recall that a space X is said to have *countable extent*, if every closed discrete subspace of X is at most countable. Hence countably compact space has countable extent. So, if an answer of problem B is positive, the answer of A is positive.

One of our purposes is to answer negatively for problem A (hence problem B). Really we shall have the following example:

There is a Hausdorff infinite space which is countably compact and relatively locally finite.

It is clear that this space is one of the counter examples for the above problems.

Another purpose of this paper is to give the necessary conditions of relatively local finiteness and to give their corollaries.

We follow the definitions and notations which will be used in this paper.

A subset A of a space ($=$ a topological space) X is *finitely located in* X if every subset of A that is closed in X is finite. A space X is said to be *relatively locally finite at a point* $x \in X$, if there is a nbd (=neighborhood) U of x in X which is finitely located in X.

If a space X is relatively locally finite at every point of X, a space X is said to be *relatively locally finite*.

For a set A, *card*($\mathcal{A}$) denotes the cardinality of A and $\mathcal{C}(A)$ denotes the collection of all the countably infinite subsets of A.

1. Negative answers of problems A and B

Theorem 1. *There is an infinite, countably compact Hausdorff space that is not relatively locally finite.*

Proof: Let N be a space of all natural numbers with discrete topology, and βN the Stone-Čech compactification of N. After this, we shall define the collection $\{X_\alpha \mid \alpha < \omega_1\}$ of subsets of βN by transfinite induction on ω_1, where ω_1 denotes the first uncountable ordinal number.

At first we shall let $X_0 = N$.

For each $M \in \mathcal{C}(N)$, we can select two distinct points $x_{(M,1)}$ and $y_{(M,1)}$ in $\mathrm{cl}_{\beta N}M - X_0$, because $\mathrm{card}(\mathrm{cl}_{\beta n}M) = 2^C$ (where $\mathrm{cl}_{\beta N}M$ denotes the closure of M in βN) by [2] or [3].

If we let

$$X_1 = X_0 \cup \cup\{\{x_{(M,1)}, y_{(M,1)}\} \mid M \in (N)\},$$

then it is clear that $\mathrm{card}(X_1) \leq c$.

Assume that for some α with $0 < \alpha < \omega_1$ we can define a collection $\{X_\beta \mid \beta < \alpha\}$ of subsets of βN with the following properties:

1) $\{ X_\beta \mid \beta < \alpha\}$ is monotone increasing, that is, $X_\beta \subset X_\gamma$ for any $\beta, \gamma < \alpha$ with $\beta < \gamma$,

2) $\mathrm{card}(X_\beta) \leq c$ for any $\beta < \alpha$

and

3) if $\beta < \alpha$ with $\beta + 1 < \alpha$, where $\beta + 1$ is the successor of β, then $x_{(M,\beta+1)}$ and $y_{(M,\beta+1)}$ are arbitrarily fixed and distinguished points in $X_{\beta+1} \cap (\mathrm{cl}_{\beta N}M - X_\beta)$.

For each $M \in \mathcal{C}(\cup_{\beta<\alpha}X_\beta)$, we can select two distinct points $x_{(M,\alpha)}$ and $y_{(M,\alpha)}$ in $\mathrm{cl}_{\beta N}M - (\cup_{\beta<\alpha}X_\beta)$, since

card $(\cup_{\beta<\alpha}X_\beta) \leq c$ from 2) and $\mathrm{card}(\mathrm{cl}_{\beta N}M) = 2^c$.

We let

$$X_\alpha = (\cup_{\beta<\alpha}X_\beta) \cup \cup\{\{x_{(M,\alpha)}, y_{(M,\alpha)}\} \mid M \in \mathcal{C}(\cup_{\beta<\alpha}X_\beta)\},$$

then it is seen that card $(X_\alpha) \leq c$.

In this place we let $X = \cup\{X_\alpha \mid \alpha < \omega_1\}$ and we shall induce the following topology on X:

Let x be any point of X. We shall define the nbd base $\mathcal{U}(x)$ at x in X as follows:

$$\mathcal{U}(x) = \{(U(x) \cap (\cup_{\beta<\alpha(x)} X_\beta)) \cup \{x\} \mid U(x) \text{ is an open nbd of } x \text{ in } \beta N\},$$

where $\alpha(x)$ denotes the first of $\{\alpha < \omega_1 \mid x \in X_\alpha\}$.

It is clear that the space X is infinite and the new topology of X with the above nbd basis is strictly stronger than the subspace topology of βN, and so X is Hausdorff.

Therefore it will be sufficient to prove that X is countably compact and relatively locally finite.

To show that X is countably compact, we need the following useful claim:

Claim. *Let B be a subset of X and $y \in X$. Then the following are equivalent:*

(1) $y \in cl_X B$,

(2) $y \in cl_X(B \cap X_{\alpha(y)})$

Furthermore, if $B \subset \cup_{\beta<\alpha(y)} X_\beta$,

(3) $y \in cl_{\beta N} B$.

We shall continue to prove the countable compactness of X. Let A be any countably infinite subset of X and $\alpha < \omega_1$ an ordinal with $A \subset X_\alpha$, (it is possible by (1)). Since A is in $\mathcal{C}(X_\alpha)$, $x_{(A,\alpha+1)}$ is in $X_{\alpha+1} \subset X$ and $x_{(A,\alpha+1)}$ is an accumulation point of A in X by the above claim. Therefore X is countably compact.

Lastly we shall show that X is relatively locally finite. Let y be any point of X and $\alpha(y) =$ the first of $\{\alpha < \omega_1 \mid y \in X_\alpha\}$. If we let $O = (\cup_{\beta<\alpha(y)} X_\beta) \cup \{y\}$, then O is clearly open nbd of y in X.

It is sufficient to show that O is finitely located in X. For this purpose, let A be any infinite set of O, where we shall show that A is not closed in X. If B is any countably infinite subset of $A - \{y\}$ (and so, $B \subset \cup_{\beta<\alpha(y)} X_\beta$), then $x_{(B,\alpha(y))}, y_{(B,\alpha(y)}$ are distinct points of $\mathrm{cl}_{\beta N} B - (\cup_{\beta<\alpha(y)} X_\beta)$. By the definition of $X_\alpha(y)$, these points $x_{(B,\alpha(y))}$ and $y_{(B,\alpha(y))}$ are in $X_{\alpha(y)}$ and in

$\mathrm{cl}_X B$ by the above claim. We may assume that $y \neq x_{(B,\alpha(y))}$. Hence $x_{(B,\alpha(y))} \in \mathrm{cl}_X B \subset cl_X A$.

On the other hand, t is seen that $x_{(B,\alpha(y))} \notin O$, which contains A, and hence $x_{(B,\alpha(y))} \notin A$. From the above facts, $x_{(B,\alpha(y))} \in \mathrm{cl}_X A - A$. This means that A is not closed in X.

This implies the proof of Theorem 1.

2. A CHARACTERIZATION OF RELATIVE LOCAL FINITENESS

In this section we shall give some characterization that a Hausdorff space is relatively locally finite, and we shall do its applications.

Theorem 2. *A Hausdorff space X is relatively locally finite if and only if each point of X has an open nbd U such that any infinite subset of U has infinitely many accumulation points in X, one of which is in $X - U$.*

Proof: only if part: Let each x, U_x be an open nbd of x which is finitely located in X.

We assume that some infinite set S of some U_x has at most finitely many accumulation points in X. So we denote it by $\{x_1, x_2, \dots, x_n\}$. If we assume $n = 1$, then $U_{x_1} \cap S$ is infinite closed in X and a subset of U_{x_1}, which is contradictory. We select a mutually disjoint open collection $\{O_i \mid i = 1, 2, \dots, n\}$ such that each O_i is a nbd of x_i and finitely located in X. Since $O_1 \cap U_{x_1}$ is a nbd of x_1, $O_1 \cap U_{x_1} \cap S$ is an infinite subset of U_{x_1} and so $O_1 \cap U_{x_1} \cap S$ has at least two accumulation points in X.

Hence we can let y be an accumulation point of $O_1 \cap U_{x_1} \cap S$ with $y \neq x_1$. So y is an accumulation point of S, which means $y \in \{x_2, \dots, x_n\}$.

On the other hand, $y \notin \cup\{O_i \mid i = 2, \dots, n\}$ because $y \in \mathrm{cl}O_1$. This is contradictory.

Furthermore, if some infinite set S of some U_x does not have an accumulation point in $X - U_x$, then $\mathrm{cl}S$ is an infinite closed subset of U_α, which is contradictory.

if part: Let U be an open set satisfying the property in Theorem 1. Then every infinite subset S of U has an accumulation point $p \in X - U$. Therefore $p \in \mathrm{cl}S - S$; this means that S is not closed in X.

We have propositions 8 and 11 in [1] as corollaries of Theorem 4.

Acknowledgement The author would like to thank the referee for many kind comments and also Prof. Yukinobu Yajima for helpful advice about Theorem 2.

REFERENCES

[1] A. V. Arhangel'skiĭ and I. Yu. Gordienko: *Relatively locally finite Hausdorff spaces*, Ques. and Ans. in General Top., **12** (1994), 15-25.

[2] R. Engelking, General Topology, PWN, Warsaw, 1989.

[3] R. C. Walker, *The Stone-Čech compactification*, Springer-Verlag, 1974.

Osaka Kyoiku University
Asahigaoka, Kashiwara
Osaka 582, Japan
e-mail: yasui@cc.osaka-kyoiku.ac.jp