Topology Proceedings

Web: http://topology.auburn.edu/tp/
Mail: Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA
E-mail: topolog@auburn.edu
ISSN: 0146-4124

Topology Proceedings
Vol. 20, 1995

A THREE DIMENSIONAL PRIME END THEORY

BEVERLY L. BRECHNER AND JOO S. LEE

ABSTRACT. *Prime end theory* is essentially a compactification theory for simply connected, bounded domains, U, in E^2, or simply connected domains in S^2 with nondegenerate complement. The planar case was originally due to Caratheodory and was later generalized to the sphere by Ursell and Young, and to arbitrary two manifolds by Mather. There are many applications of the two dimensional theory, including applications to fixed point problems, embedding problems, and homeomorphism (group) actions.

Several constructions of a three dimensional topological prime end theory appear in the literature, including work by Kaufmann, Mazurkiewicz, and Epstein.

In this paper, the authors develop a *simple* three dimensional prime end theory for certain open subsets of Euclidean three space. It includes conditions focusing on an "Induced Homeomorphism Theorem", which, the authors believe, provides the necessary ingredient for applications.

1. Introduction

Prime end theory is essentially a compactification theory for

This paper is a revision of an appendix to Joo S. Lee's dissertation at the University of Florida, August 1993. It is joint work with his thesis adviser, Beverly L. Brechner.

Many thanks to the referee for a careful reading of this paper and for valuable comments.

simply connected, bounded domains, U, in E^2, or simply connected domains in S^2 with nondegenerate complement. The planar case was originally due to Caratheodory [C], and was later generalized to the sphere by Ursell and Young [U-Y], and to arbitrary two manifolds by Mather [Mat]. For each such domain, U, there is given an associated structure of crosscuts, chains of crosscuts, prime ends, and impressions of prime ends. Caratheodory [C] and Ursell and Young [U-Y] proved the following:

Theorem 1.1. [C] *The prime ends of U are in 1-1 correspondence with the boundary points of the unit disk. That is, the compactification is by a manifold.*

Theorem 1.2. [C,U-Y] *There is a C-transformation $\phi : U \to \mathrm{Int}(D)$ such that ϕ is uniformly continuous on the collection of crosscuts of U, although not necessarily on U.*

Remark 1.1. The uniform continuity on the collection of crosscuts follows from a theorem in the appendix of [U-Y], and was observed and used earlier by the first author.

Theorem 1.3. [U-Y] (**The Induced Homeomorphism Theorem**) *Let U be a simply connected domain in the plane, and let $h : \mathrm{Cl}(U) \to \mathrm{Cl}(U)$ be a homeomorphism. Let $\phi : U \to \mathrm{Int}(D)$ be a C-transformation. Then $\phi h \phi^{-1} : \mathrm{Int}(D) \to \mathrm{Int}(D)$ can be extended to a homeomorphism of D onto itself.*

There are many applications of the two dimensional theory, including applications to fixed point problems, embedding problems, periodic points of homeomorphisms, and homeomorphism (group) action and extension problems. See, for example, [C-L], [Mas], [Ep2], [Br1,2,3], [Br-Mau], [Br-May], [May1,2], [Lew], and [Mat], among others.

Several constructions of a three dimensional topological prime end theory appear in the literature, including work by Kaufmann [Kau], Mazurkiewicz [Maz], and Epstein [Ep1]. These papers have not yet had any applications of which we are aware.

In this paper, we develop a *simple* three dimensional prime end theory for certain open subsets of Euclidean three space. It includes conditions not addressed by any of the above three authors. Our additional conditions focus on an "Induced Homeomorphism Theorem", which we believe provides the necessary ingredient for applications.

The first author would like to thank John Mayer for interesting discussions on the topic of this paper many years ago, and for the invitation to present these results at his AMS Special Session in Knoxville, Tennessee, in March, 1993. She would also like to express her deep gratitude to both Richard Wiegandt for translating significant portions of Kaufmann's work and to John Mayer for his translation of Caratheodory's original paper.

2. Definition of a Prime End Theory on E^3

The essential ingredients of the planar prime end theory are included in Theorems 1.1, 1.2, and 1.3 quoted above. A satisfactory three dimensional theory should certainly include these. Thus, we *define* a prime end theory for open subsets U of E^3 to be a theory which satisfies these conditions.

A PRIME END THEORY FOR OPEN SUBSETS OF E^3 MUST INCLUDE THE FOLLOWING:

(1) There exists a *prime end structure* on U including crosscuts, chains of crosscuts, prime ends, and impressions of prime ends, for suitable domains, U, such that the prime ends determine a *prime end compactification*, U^* of U, consisting of U plus its prime ends.

(2) There is defined a homeomorphism ϕ from U onto the the interior of some compact three manifold M^3 with nonempty boundary such that *ϕ is uniformly continuous on the collection of crosscuts of U.* (Note that, since $U \subset E^3$ and $Bd(M^3)$ is collared in M^3 [Bro], it follows that M^3 is embeddable in E^3.)

(3) The *prime end compactification, U^* of U, is homeomorphic to M^3.*
(4) The *Induced Homeomorphism Theorem* holds on M^3. That is, if $h : Cl(U) \to Cl(U)$ is an onto homeomorphism, then $\phi h \phi^{-1} : Int(M^3) \to Int(M^3)$ can be extended to an (induced) homeomorphism of M^3 onto itself.

In §3, we adopt a standing hypothesis for the domain U. We also present our definitions of "U has a prime end structure", of the space U^*, and of "C-transformation". We show that U^* is compact, so that we can indeed call it the "prime end compactification" of U. Our C-transformation is similar to Ursell and Young's [U-Y] C-transformation, and plays the role of the homeomorphism, ϕ, in our definition above. Of course, there are then two major problems: (1) to characterize those domains U which have prime end structures, and (2) to characterize those open subsets of E^3 which admit a C-transformation onto the interior of some compact 3-manifold. These remain open problems at present.

The Whitehead Example. This example illustrates the typical problem that we must avoid in our open sets. It is an example of a connected, simply connected, contractible, proper open subset of S^3 which is not homeomorphic to E^3. It can be constructed as the complement of the intersection in S^3, of "half-twisted", folded tori. Note that the open set is not 1-connected at infinity, since the fundamental group at infinity is infinitely generated. In particular, it is not $1 - ULC$ at infinity. Further, this example does not have a manifold compactification. A proof can be found in Husch [Hu1].

Below, we state some well known theorems which provide sufficient conditions for an open 3-manifold to be homeomorphic to the interior of some compact 3-manifold M^3 with non-empty boundary.

Theorem 2.1. [Ed] *Let U be a contractible open 3-manifold, each of whose compact subsets can be embedded in E^3. If U is*

1-connected at infinity, then U is homeomorphic to E^3.

C. T. C. Wall has a related theorem, for which a corollary is:

Theorem 2.2. [Wa] *If M is an open 3-manifold* in E^3 *which is 1-connected at infinity, then M is homeomorphic to E^3.*

Theorem 2.3. [Hu2] *Let M be a connected, orientable 3-manifold with compact boundary, and one end. The interior of M is homeomorphic to the interior of a compact 3-manifold iff there exists a positive integer n such that every compact subset of M is contained in the interior of a compact 3-manifold M' with connected boundary such that*

1. *$\pi_1(M - M')$ is finitely generated,*
2. *$genus(Bd(M')) \leq n$, and*
3. *every contractible 2-sphere in $M - M'$ bounds a 3-cell.*

3. A Three Dimensional Prime End Theory for E^3

In this section, we develop a three dimensional prime end theory for a class of domains in E^3. In §3.1, we make the necessary definitions to set up its structure, including the definition of an *admissible domain*; in §3.2, we prove the existence of such a prime end theory for admissible domains; and in §3.3, we define the *bubble domains* and prove that the bubble domains admit C-transformations and therefore are admissible. The proof requires the use of the Topological Dehn's Lemma, due to Repovs [Re]. We also give some examples of bubble domains to show that they form a large class of interesting domains in E^3. However, Figure 3.2(c) shows that there are admissible domains which are not bubble domains.

STANDING HYPOTHESIS FOR §3: U is a bounded *domain* (i.e., connected open set) in E^3, with finitely generated homology and finitely generated fundamental group.

3.1 DEFINITIONS

1. A *crosscut* is an open 2-cell D in U such that

(1) D separates U into exactly two complementary domains,
(2) $Cl(D)$ is a 2-cell, and
(3) $Cl(D) \cap Bd(U) = Bd(D)$.

2. A *chain of crosscuts* in U is a sequence $\{D_i\}_{i=1}^{\infty}$ of cross cuts such that

(1) D_{i+1} separates D_i from $\{D_{i+j}\}_{j=2}^{\infty}$,
(2) $Cl(D_i) \cap Cl(D_j) = \phi$ for $i \neq j$, and
(3) $lim_{i\to\infty}(diam(D_i)) = 0$.

3. Two chains of crosscuts, $\{Q_i\}$ and $\{R_i\}$, are *equivalent* iff

(1) For each Q_i, there exists $j > i$ such that Q_{i+1} separates Q_i from $Q_j \cup R_j$, and
(2) For each R_i, there exists $j > i$ such that R_{i+1} separates R_i from $R_j \cup Q_j$.

That is, two subsequences can be alternated or "interspersed" to form a new, equivalent chain of crosscuts.

4. A *prime end* of U is an equivalence class of chains of crosscuts of U.

5. Let $\{Q_i\}$ be a chain of crosscuts representing the prime end E of the domain U, and let U_i be the *associated* or *corresponding complementary domain* of Q_i; that is, that complementary domain of Q_i in U which contains $\bigcup_{j>i} Q_j$. We call the set $Cl(U_i)$ the *corresponding continuum*, and the set $(Cl(U_i)) \cap Bd(U)$ the *corresponding boundary compactum.* The *impression of* E, denoted by $I(E)$, is defined to be the set $\bigcap_i Cl(U_i)$. Clearly, $I(E) \subset Bd(U)$. If $\{Q_i\}$ converges to a single point x in $Bd(U)$, then x is called a *principal point* of E. As in the two dimensional theory, the set of principal points of E may be nondegenerate. However, if E is a prime end of a compact manifold, then E has exactly one principal point.

6. An *onto* homeomorphism $\phi : U \to Int(M^3)$, where M^3 is a compact 3-manifold, is called a *C-transformation* or a *C-map* iff all of the following hold:

(1) The image of every chain of crosscuts of U is a chain of crosscuts of $Int(M^3)$. In particular, the image of each crosscut of U is a crosscut of $Int(M^3)$.

(2) On each crosscut Q, ϕ extends to a homeomorphism from $Cl(Q)$ onto $Cl(\phi(Q))$. (However, ϕ does not necessarily extend to a homeomorphism from the union of the closures of all the crosscuts of U to the union of the closures of their images in $\phi(Q)$.)

(3) For each crosscut Q_i of a prime end of U, let U_i be its corresponding domain. Let (Q_i', U_i') be the image of (Q_i, U_i) under ϕ. We consider the following open sets on $Bd(M^3)$: $Int[Cl(U_i') \cap Bd(M^3)]$. We require that the collection of all such open disks on $Bd(M^3)$ form a basis for the topology of $Bd(M^3)$.

Remark 3.1. Note that if B_i' denotes the corresponding boundary compactum of Q_i', and if S_i' denotes the boundary of Q_i', then condition (3) of Definition 6 states that $\mathrm{Int}[\mathrm{Cl}(U_i') \cap Bd(M^3)] = B_i' - S_i'$.

7. We say that a domain U *has a prime end structure* iff for every $\epsilon > 0$ there exist a finite number of prime ends, $\{E_i\}_{i=1}^n$, of U, and a finite number of crosscuts, $\{Q_i\}_{i=1}^n$, with Q_i a crosscut of some chain representing E_i, such that

(1) $diam(Q_i) < \epsilon$,

(2) If U_i denotes the corresponding domain for Q_i, then $Bd(U) \cup \bigcup_{i=1}^n U_i$ is an ϵ-neighborhood of $Bd(U)$ in $Cl(U)$, and

(3) If B_i denotes the corresponding boundary compactum of Q_i, then $\bigcup_{i=1}^n (B_i - S_i) = Bd(U)$, where $S_i = Bd(Q_i)$.

Remark 3.2. We do not know whether (2) implies (3) in general. In particular, it is conceivable that for some $\epsilon > 0$, and for a given collection of crosscuts for this ϵ, some point p lies on some S_k of the collection, but not in $\bigcup_{i=1}^{n}(B_i - S_i)$. For example, suppose there is given a finite collection of crosscuts, $\{Q_i\}_{i=1}^{n}$, to $Bd(U)$ such that for some pair, say Q_1 and Q_2, $Bd(Q_1) \cap Bd(Q_2)$ is an arc containing the point p, but p is not in $B_i - S_i$, for all i. Suppose also, that Q_1 and Q_2 overlap in such a way that $Bd(U) \cup \bigcup_{i=1}^{n} U_i$ is, nevertheless, a neighborhood of $Bd(U)$. In this case, note that there is no accessible arc A to p that lies in some U_k, with $Cl(A) \cap Bd(Q_k) = \emptyset$.

8. A bounded domain $U \subset E^3$ is an *admissible domain* iff there exists a C-transformation $\phi : U \to Int(M^3)$, for some compact 3-manifold, M^3, with nonempty boundary. The triple (U, ϕ, M^3) is called an *admissible triple.*

3.2. EXISTENCE OF A PRIME END THEORY ON ADMISSIBLE DOMAINS

This section is divided into four parts, establishing the properties corresponding, respectively, to the four parts of the definition of a prime end theory. In constructing the prime end compactification, U^*, we assume only that U is a bounded domain in E^3 and that it has a prime end structure. *For the remainder of this section, we also assume that U is an admissible domain*, which includes, in particular, the existence of a C-transformation, ϕ, taking U onto the interior of some compact 3-manifold, M^3, with nonempty boundary.

3.2.1. THE PRIME END COMPACTIFICATION, U^*

The *raison d'être* of prime end theory is to use the prime ends as a compactification of the domain in question. Thus, we first define the space U^*. Then, in Theorem 3.1 below, we show that if U is a bounded domain which has a prime end structure, then U^* is compact. Of course, the question arises as to the *existence* of domains in E^3 with a prime end structure. In Theorem 3.2, we show that admissible domains have such structures.

To this end, let U be a bounded domain in E^3 which has a prime end structure, and let U plus the prime ends of U be denoted by U^*. We topologize U^* by declaring the topology of U^* to be generated by basic neighborhoods of the form described below:

The basic neighborhoods of a point of U are the same as the basic neighborhoods of that point in the topology of E^3. Now let E be a prime end of U, and let $\{Q_i\}$ be any chain of crosscuts representing E. Then a basic neighborhood of the prime end E is the corresponding domain W_i of any one of these crosscuts Q_i, plus all the prime ends of U represented by chains of crosscuts which are eventually in W_i.

Proposition 3.1. *A point p of $Bd(U)$ that is accessible from U corresponds to at least one prime end of U.*

Proof: Let A be an arc of accessibility to p from U. Let $\{\epsilon_n\}$ be a sequence of positive numbers with limit 0. Let $\{\mathcal{E}_i\}_{i=1}^{\infty}$, denote an infinite sequence of finite collections of crosscuts to $Bd(U)$ satisfying Definition 7. A tail end of the arc A lies in the corresponding domain of some element Q_1 of the collection $\mathcal{E}_1$, in such a way that $p \notin Cl(Q_1)$, by (3) of Definition 7. Let U_1 denote the corresponding domain of Q_1, and let $\delta = d(p, Cl(Q_1))$. Let ϵ_{n_2} be the first element of the sequence $\{\epsilon_n\}$ such that $\epsilon_{n_2} < \delta/2$, and look at the finite collection $\mathcal{E}_{n_2}$ corresponding to ϵ_{n_2}. The union of their corresponding domains forms an ϵ_{n_2}-neighborhood of $Bd(U)$, so one of these corresponding domains, say U_2 corresponding to Q_2, must contain a (sub)tail of A, in such a way that $p \notin Cl(Q_2)$, by (3) of Definition 7. Since $d(p, Cl(Q_1)) < \delta$, $(Q_2 \cup U_2)$ must lie in U_1. Further, $Cl(Q_1) \cap Cl(Q_2) = \emptyset$. Thus, we have found n_2 and Q_2 so that a tail end of the arc A lies in the corresponding domain U_2 of the element Q_2 of the collection $\mathcal{E}_{n_2}$, $(Q_2 \cup U_2) \subset U_1$, and $Cl(Q_1) \cap Cl(Q_2) = \emptyset$.

Similarly, find n_3 and Q_3 so that a tail end of the arc A lies in the corresponding domain U_3 of the element Q_3 of the collection $\mathcal{E}_{n_3}$, $(Q_3 \cup U_3) \subset U_2$, and $Cl(Q_2) \cap Cl(Q_3) = \emptyset$. Continue the process inductively. Then the collection of crosscuts, $\{Q_i\}_{i=1}^{\infty}$, defines a chain of crosscuts corresponding to a prime end E of U. Further, this chain of crosscuts converges to the point p of $Bd(U)$, and we may say that p corresponds to (at least) the prime end E of U. □

Proposition 3.2. *U^* is a Hausdorff space.*

Proof: Let E and F be distinct prime ends of U, defined by the chains of crosscuts $\{Q_i\}$ and $\{R_i\}$, respectively, and let $\{U_i\}$ and $\{V_i\}$, respectively, be their sequences of corresponding domains. We must exhibit disjoint neighborhoods of E and F. We may assume that the sequence $\{Q_i\}$ converges to a point q in $Bd(U)$, and that the sequence $\{R_i\}$ converges to a point r in $Bd(U)$. (The reason is that, since the limit of the diameters of each sequence is 0, some subsequence of each converges to a point, and we may use those subsequences to represent the respective prime ends.)

Observe that either for some Q_n, there is $m > 0$, so that no R_{m+j}, for all $j > 0$, lies in U_n, or for some R_n, there is $m > 0$, so that no Q_{m+j}, for all $j > 0$, lies in V_n. Otherwise we could construct an alternating sequence of crosscuts from $\{Q_i\}$ and $\{R_i\}$, and it would follow that E and F would represent the same prime end. Thus, without loss of generality, assume that for some Q_n, there is $m > 0$, so that, for all $j > 0$, no R_{m+j} lies in U_n, the corresponding domain for Q_n. Then Q_n separates, in U, Q_{n+1} from R_{m+j}, for all $j > 0$. Thus, the neighborhoods of E and F defined by U_{n+1} and V_{m+1}, the corresponding domains of Q_{n+1} and R_{m+1}, respectively, are disjoint. □

Theorem 3.1. *Let U be a bounded domain in E^3 which has a prime end structure. Then U^* is compact.*

Proof: Let $\mathcal{V}$ be a basic open cover of U^*. Since U has a prime end structure, for every $\epsilon > 0$, there is a finite set of crosscuts, $\{Q_i\}_{i=1}^{n}$, coming from a finite set of prime ends, $\{E_i\}$, respectively, such that $diam(Q_i) < \epsilon$ and $[\bigcup_{i=1}^{n} W_i \cup Bd(U)]$ forms an ϵ-neighborhood of $Bd(U)$ in $Cl(U)$. (Note that $diam(W_i)$

is not necessarily less than ϵ, where W_i is the corresponding domain of Q_i, but W_i does lie in an ϵ-neighborhood of $Bd(U)$.)

We claim that there is $\epsilon_* > 0$ such that for each Q_i of the collection $\{Q_i\}_{i=1}^n$ for that ϵ_*, there is a $V_i \in \mathcal{V}$ such that if W_i is the corresponding domain of Q_i, then $[Q_i \cup W_i \cup$ {prime ends in W_i}$] \subset V_i$. For suppose not. We shall construct a chain of crosscuts, defining a prime end F of U^*, such that F is not contained in any member of the cover $\mathcal{V}$, leading to a contradiction.

To this end, let $\epsilon_i \to 0$, and let $Q_{i_1}, \dots, Q_{i_{n_i}}$ be a finite set of crosscuts of U with corresponding domains $W_{i_1}, \dots, W_{i_{n_i}}$, respectively, such that each of these crosscuts has diameter $< \epsilon_i$ and has corresponding domain lying in an ϵ_i-neighborhood of $Bd(U)$. For ϵ_1, there is at least one of these crosscuts, call it Q_{1_*}, such that $[Q_{1_*} \cup W_{1_*} \cup$ {prime ends in W_{1_*}}$]$ is not covered by a finite subcollection of elements of $\mathcal{V}$. Now $S_{1_*} = Bd(Q_{1_*})$ is a simple closed curve on $Bd(U)$ such that it collars into U (for example, along the crosscut), and therefore each point of S_{1_*} is accessible from U. Thus, by (3) of Definition 7, for each point p of S_{1_*}, there is a crosscut Q with boundary S and corresponding domain W, such that W contains a tail end of an accessible arc to p and $p \in B - S$, where B is the corresponding boundary compactum of Q, and $S = Bd(Q)$. Further, since $\mathcal{V}$ is an open cover of U^*, we may choose Q so that $[Q \cup W \cup$ {prime ends in W}$] \subset$ some basic open set $\in \mathcal{V}$. Note that this implies that a small open arc of S_{1_*}, containing p, lies in B.

Since S_{1_*} is compact, a finite number of such crosscuts, $Q_1, Q_2, \dots, Q_k$ induces a covering of S_{1_*}. That is, if B_i denotes the corresponding boundary compactum of Q_i, then $\bigcup_{i=1}^k B_i \supset S_{1_*}$.

Let $\delta_2 = d(S_{1_*}, C[Q_{1_*} \cup (\bigcup_{i=1}^k Cl(W_i))])$, where C denotes complementation and the complement is taken in the closure of U. Then $\delta_2 > 0$. Let ϵ_{n_2} be the first member of $\{\epsilon_i\}$ such that $\epsilon_{n_2} < \frac{\delta_2}{2}$. Without loss of generality, let $\epsilon_2 = \epsilon_{n_2}$. For ϵ_2, there is a finite set of crosscuts $\{Q_{2_j}\}_{j=1}^{2_{n_2}}$ satisfying the definition of prime end structure. Since each member of $\{W_i\}_{i=1}^k$ is a subset of an element of $\mathcal{V}$, our assumption that $[Q_{1_*} \cup W_{1_*} \cup$ {prime ends in W_{1_*}}] is not covered by a finite subcollection of $\mathcal{V}$, implies that at least one of the crosscuts, call it Q_{2_*}, from that subcollection of $\{Q_{2_j}\}_{j=1}^{2n_2}$ which is contained in W_{1_*}, has the property that $[Q_{2_*} \cup W_{2_*} \cup$ {prime ends in W_{2_*}}] is not covered by a finite subcollection of elements of $\mathcal{V}$. Further, by our choice of ϵ_2, we may assume that $Cl(Q_{2_*}) \cap Cl(Q_{1_*}) = \emptyset$.

We have now constructed the first two members of our chain of crosscuts of U^*. Continue the above process inductively, obtaining a chain of crosscuts, $\{Q_{i_*}\}_{i=1}^{\infty}$, with corresponding domains $\{W_{i_*}\}_{i=1}^{\infty}$. This chain defines a prime end, say F, of U^*, and is such that, for all $i > 0$, $[Q_{i_*} \cup W_{i_*} \cup$ {prime ends in W_{i_*}}] cannot be covered by a finite set of elements of $\mathcal{V}$. But F is a point of U^*, and so lies in some element of $\mathcal{V}$. So it follows from the definition of the topology of U^*, that for some i, $[W_{i_*} \cup$ {prime ends in W_{i_*}}] lies in some element of $\mathcal{V}$. Thus, $[Q_{i+1_*} \cup W_{i+1_*} \cup$ {prime ends in W_{i+1_*}}] lies in an element of $\mathcal{V}$, and this contradiction establishes our claim in paragraph two.

Thus there is some $\epsilon_* > 0$ such that for each Q_i of the collection $\{Q_i\}_{i=1}^{n}$ for that ϵ_*, there is a $V_i \in \mathcal{V}$ such that if W_i is the corresponding domain of Q_i, then $[Q_i \cup W_i \cup$ {prime ends in W_i}] $\subset V_i$. Let $V^* = \bigcup_{i=1}^{n} V_i$. Then $U - V^*$ is a compact subset, X, of U, so some (other) finite subcollection, say $\{O_j\}_{j=1}^{m}$ of $\mathcal{V}$ covers X. It follows that the union of these two finite subcollections of $\mathcal{V}$, $\{V_j\}_{j=1}^{n} \cup \{O_j\}_{j=1}^{m}$, is a finite subcollection of $\mathcal{V}$ covering U^*. This completes the proof of the theorem. □

Remark 3.3. *If U is not $1-ULC$, it may or may not have a prime end structure. (See Figure 3.2.)*

Lemma 3.1. *Let U be a bounded domain in E^3, let $f : U \to Int(M^3)$ be an onto homeomorphism, and let N be the full $\epsilon-$neighborhood of $Bd(U)$. Then there exists $\gamma > 0$ such that, if $f^{-1}(P_\gamma)$ is a neighborhood of $Bd(U)$, where P_γ denotes the full γ-neighborhood of $Bd(M^3)$, then $f^{-1}(P_\gamma)$ is an $\epsilon-$neighborhood of $Bd(U)$. That is, $f^{-1}(P_\gamma) \subset N$.*

Proof: Let $K = U - N$. Then $d(K, Bd(U)) = \epsilon > 0$, and $d(Bd(M^3), f(K)) = \delta > 0$ since $f(K)$ is compact. We take $\gamma \leq \delta$ such that

(1) Every γ-crosscut, R, to $Bd(M^3)$ has a 3-cell as its corresponding compactum, C; (Recall that C is the closure, in M^3, of the (small) corresponding domain of R.)
(2) $Diam(C) < \delta$ (so that $C \subset P_\delta$), where P_δ is the full δ-neighborhood of $Bd(M^3)$; and
(3) $f^{-1}(R) \subset N$.

Since M^3 is a compact 3-manifold, (1) and (2) can be obtained. Since $R \subset C \subset P_\delta$, (3) holds as well. Let P_γ denote the full γ-neighborhood of $Bd(M^3)$. Then $f^{-1}(P_\gamma) \subset N$, since N is the full ϵ-neighborhood of $Bd(U)$. The lemma follows. □

Lemma 3.2. *Let U be a bounded domain in E^3, and let $h : U \rightarrow Int(M^3)$ be an onto homeomorphism. Let $\{(Q_i, U_i, B_i, S_i)\}_{i=1}^n$ be a finite collection of quadruples, where Q_i is a small crosscut to $Bd(U)$ in an ϵ-neighborhood of $Bd(U)$, U_i is a complementary domain of Q_i which lies in that ϵ-neighborhood of $Bd(U)$, B_i is its corresponding boundary compactum (that is, $B_i = [Bd(U) \cap Cl(U_i)]$ in E^3), and $S_i = Bd(Q_i)$. Let $\{(Q_i', U_i', B_i', S_i')\}_{i=1}^n$ denote the collection of quadruples $\{(h(Q_i), h(U_i), h(B_i), h(S_i))\}_{i=1}^n$. If $Bd(M^3) \subset \bigcup_{i=1}^n (B_i' - S_i')$, then it follows that if p is a point of S_k for some k, then there exists j such that $p \in B_j - S_j$. Thus $Bd(U) \subset \bigcup_{i=1}^n (B_i - S_i)$.*

Remark 3.4. (1) We may assume that each B_i' is a closed disk on $Bd(M^3)$ with simple closed curve boundary, S_i'. (2) We note that the statement of Lemma 3.2 is long, and the lemma is easy to prove. However, it is a subtle point which must be dealt with in our proofs of the existence of a prime end structure in Theorems 3.2 and 3.7. Thus it is useful to state it explicitly.

Theorem 3.2. *Let U be a domain in E^3 which admits a C-transformation, ϕ. Then U has a prime end structure induced by ϕ.*

Proof: Let $\phi : U \to M^3$ be a C-transformation from U onto the interior of a compact 3-manifold, and let $\epsilon > 0$. By Property (3) of the definition of C-transformation and Theorem 3.3 (which does not depend on this theorem), for each $0 < \gamma < \epsilon$ and each point y of $Bd(M^3)$, there is a crosscut Q'_y of $Int(M^3)$ such that (1) $diam(Q'_y) < \gamma$, (2) $Q'_y = \phi(Q_y)$, where $diam(Q_y) < \gamma$, and (3) Q_y belongs to some chain of crosscuts representing a prime end of U. Let U_i and U'_i denote the corresponding domains of Q_i and Q'_i, respectively. Since M^3 is a manifold, and its boundary is compact, there is a finite subcollection, $\{Q'_i\}_{i=1}^n$, of these crosscuts such that $Bd(M^3) \cup \bigcup_{i=1}^n U'_i$ forms a small neighborhood of $Bd(M^3)$ in M^3. By Lemma 3.1, we can choose γ to insure that $Bd(U) \cup \bigcup_{i=1}^n U_i$ is a subset of the full ϵ-neighborhood of $Bd(U)$, and thus it is an ϵ-neighborhood of $Bd(U)$. Our collection now satisfies conditions (1) and (2) of the definition of prime end structure.

To insure that the subcollection $\{Q_i\}_{i=1}^n$ also satisfies condition (3) of the definition of prime end structure, we ask that the collection $\{Q'_i\}_{i=1}^n$ also have the property that $Bd(M^3) \subset \bigcup_{i=1}^n Int(B'_i)$, where B'_i denotes the corresponding boundary compactum of Q'_i. Then Lemma 3.2 guarantees that $\{Q_i\}_{i=1}^n$ satisfies condition (3).

Thus, the collection $\{Q_i\}_{i=1}^n$ forms the required finite collection in U. □

Corollary 3.1. *Let (U, ϕ, M^3) be an admissible triple. Then U has a prime end structure induced by ϕ.*

Proof: An admissible domain admits a C-transformation, by definition. □

3.2.2. C-TRANSFORMATIONS AND UNIFORM CONTINUITY

In Theorem 3.3 below, we show that a C-transformation is uniformly continuous on the *collection of crosscuts* of U.

The reader should note that, in general, it is *not necessarily* uniformly continuous on all of U. A simple example shows why: Let the domain U be the open unit cube in E^3 minus a two dimensional disk with a portion of its boundary in the boundary of the cube. Then U^* is a 3-cell, but it splits apart the interior of the aforementioned disk into two disjoint open disks. Thus, if p is a point of the interior of that disk, it becomes two points in the prime end compactification of U, so that a small neighborhood of p in the original space, when intersected with U, becomes large in diameter when viewed in U^*. See Figure 3.1 below.

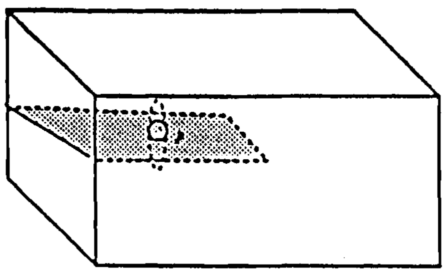


Figure 3.1

However, in Corollary 3.2, we show that a C-transformation *is* uniformly continuous on all of its domain, when that domain is the interior of a compact 3-manifold.

Lemma 3.3. *Let X be a nondegenerate continuum on a compact 2-manifold M. Suppose that $\{X \cap Int(D_i)\}$ is a basis of open sets for X, where each D_i is a disk in M. Then there exists D_{j*} such that $Bd(D_{j*})$ contains at least 2 points of X.*

Proof: Suppose that, for every D_i, $X \cap Bd(D_i)$ is empty or one point. For sufficiently small D_i, $X \cap Bd(D_i) \neq \emptyset$ since X is connected. So now we suppose $X \cap Bd(D_i)$ is one point. Let $p, q \in X$ and let D_j be a neighborhood of p, with $diam(D_j) < \frac{1}{2} d(p,q)$. If $X \cap Bd(D_j) = r$, then r separates p from q. Therefore each pair of points of X is separated by a third point of X, so that the continuum X is a dendrite [Wh, pp. 88-89].

Thus X contains an arc. This contradicts the assumption that for every D_i, $X \cap Bd(D_i)$ is one point, since a sufficiently small basic neighborhood of a point of the arc will intersect at least two points. □

Theorem 3.3. *Let $\phi : U \to Int(M^3)$ be a C-transformation. Then ϕ is uniformly continuous on the collection of crosscuts of U.*

Proof: Suppose that ϕ is not uniformly continuous on the collection of crosscuts of U. Then there exists $\epsilon > 0$ such that for every $\delta > 0$, there exists a crosscut Q_δ of diameter less than δ with $diam(\phi(Q_\delta)) \geq \epsilon$. In particular, for this $\epsilon > 0$, there exist a sequence of positive numbers $\{\delta_i\}$ with $\delta_i \to 0$ and a sequence of crosscuts $\{Q_i\}$ with $diam(Q_i) < \delta_i$ and such that $diam(\phi(Q_i)) \geq \epsilon$. By Whyburn [Wh, p. 11], there is a convergent subsequence $\{Q_i'\}$ of $\{Q_i\}$ such that $lim\,(Q_i') = m \in Bd(U)$; and there is also a subsequence $\{Q_i''\}$ of $\{Q_i'\}$ such that $\{\phi(Q_i'')\}$ converges to a limit continuum in M^3 (see [Wh, p. 14]).

Hence, without loss of generality, we may assume that $\{Q_i\}$ converges to a point m of $Bd(U)$ and that $\phi(\{Q_i\})$ converges to a limit continuum X in $Bd(M^3)$, with $diam(X) \geq \epsilon$.

For each $x \in Bd(M^3)$ and each α such that $x \in Int[(Cl(\phi(U_\alpha))) \cap Bd(M^3)]$, let $B^o_\alpha(x)$ denote $Int[(Cl(\phi(U_\alpha))) \cap Bd(M^3)]$. Now we take $x \in X$ and $B^o_\alpha(x)$ on $Bd(M^3)$ such that $X \cap Bd(B^o_\alpha(x))$ contains at least two points p, q. We can do this by Lemma 3.3 and Condition (3) of the definition of C-transformation. Let $\gamma = dist(p,q)$. We consider two sequences $\{p_1, p_2, ...\}$, $\{q_1, q_2, ...\}$, where $p_i, q_i \in \phi(Q_i)$ such that $p_i \to p$, $q_i \to q$. Then there exists an integer N such that for $n > N$, $dist(p_n, q_n) > \frac{\gamma}{2}$. Now the preimages of the sequences $\{p_i\}, \{q_i\}$ converge to the same point, $m \in Bd(U)$. Since M^3 is a manifold, it admits a prime end structure. Thus, let $R_1, \ldots, R_k$ be a finite collection of crosscuts to the boundary of $Int(M^3)$, with corresponding domains $V_1, \ldots, V_k$ respectively, with corresponding continua each of diameter $< \frac{\gamma}{2}$, and such that the interiors of the corresponding boundary compacta form an open cover of $Bd(M^3)$. (In particular, note that $diam\,[Cl(R_i \cup V_i)] < \frac{\gamma}{2}$.) Then since the collection $\{\phi^{-1}(V_i)\}_{i=1}^{k}$ forms (together with $Bd(U)$) a neighborhood of $Bd(U)$, it follows that infinitely many members of the collection $\{Q_i\}$ lie in one of the corresponding domains, say $\phi^{-1}(V_s)$, where $m \in \phi^{-1}(V_s)$. Now the image of this corresponding domain, $\phi(\phi^{-1}(V_s))$ has diameter $< \frac{\gamma}{2}$, but infinitely many pairs of points of this image are separated by a distance of at least $\frac{\gamma}{2}$. This contradiction completes the proof. □

Corollary 3.2. *If the map ϕ of the interior of the compact 3-manifold M^3 onto itself is a C-transformation, then each of the following three statements holds:*

(1) *ϕ is uniformly continuous on the collection of all crosscuts of $Int(M^3)$,*
(2) *ϕ can be extended to a homeomorphism $\overline{\phi}$ of M^3 onto itself, and*
(3) *$\overline{\phi}$ is uniformly continuous on all of M^3.*

Proof: Assume that ϕ is a C-transformation. Then (1) is a corollary of Theorem 3.3.

(2) Let $x \in Bd(M^3)$, and let $\{Q_i\}$ be a chain of crosscuts with $lim\ Q_i = x$. Define $\phi(x) = lim\ \phi(Q_i)$. Since a C-transformation takes chains of crosscuts to chains of crosscuts, ϕ is well defined. Now let $\{x_i\}$ be a sequence in $Bd(M^3)$ such that $x_i \to x$, and let $\{Q_{i_j}\}_j$ be a chain of crosscuts of mesh less than $1/2^i$ which converges to x_i. Since $\{Q_{i_j}\}_j$ determines a basis at x_i on $Bd(M^3)$, and $\{Q_i\}$ determines a basis at x on $Bd(M^3)$, and $x_i \to x$, it follows that $\phi(x_i) \to \phi(x)$. Thus ϕ has a (unique) extension to a continuous map, $\overline{\phi}$, of M^3 onto itself. That $\overline{\phi}$ is a homeomorphism follows from Theorem 3.4, which does not depend on this corollary.

(3) This is a corollary of (2) above. □

3.2.3. MANIFOLD COMPACTIFICATIONS

It is clear that the boundary points of a compact manifold with nonempty boundary are naturally in one-to-one correspondence with the prime ends of the interior of that manifold. In the following, we first show that there is a one-to-one correspondence between the prime ends of an admissible domain, U, and the prime ends of the interior of a compact 3-manifold, M^3, with nonempty boundary. Our main theorem (Theorem 3.5) of this part then follows easily. Thus, for admissible domains, the prime end compactification of U is indeed a manifold compactification, and we can think of the C-transformation as a "manifold compactification map".

Theorem 3.4. *Let ϕ be a C-transformation, $\phi : U \to Int(M^3)$. Then ϕ determines a one-to-one correspondence between the prime ends of the domain U and the prime ends of $Int(M^3)$.*

Proof: Let E be a prime end of U. Then $\phi(E)$ is a prime end of $Int\,(M^3)$, by the definition of C-transformation. We suppose that E_x and E_y are prime ends of U with $\phi(E_x) = \phi(E_y)$ and we take chains of crosscuts $\{E'_{x,i}\}$ and $\{E'_{y,i}\}$ as representatives of $\phi(E_x)$, $\phi(E_y)$, respectively. Let $E'_{x,i} = \phi(E_{x,i})$ and $E'_{y,i} = \phi(E_{y,i})$, so that $\{E_{x,i}\}$ and $\{E_{y,i}\}$ are representatives of E_x and E_y respectively. Now we construct a new chain of crosscuts of $Int\,(M^3)$, by forming an alternating sequence from the chains $\{E'_{x,i}\}$ and $\{E'_{y,i}\}$:

$$\{E'_{x,i_1}, E'_{y,i_1}, E'_{x,i_2}, E'_{y,i_2}, \ldots\}$$

which is equivalent to each of $\{E'_{x,i}\}$ and $\{E'_{y,i}\}$. Then

$$\{E_{x,i_1}, E_{y,i_1}, E_{x,i_2}, E_{y,i_2}, \ldots\}$$

forms a chain of crosscuts of U and is clearly equivalent to each of $\{E_{x,i}\}$ and $\{E_{y,i}\}$. Thus, $E_x = E_y$ and ϕ is one-to-one.

To show that this correspondence is onto, let F be a prime end of $Int(M^3)$ and let p be the principal point of F. We take a 3-cell neighborhood of p, $B_1(p)$. By Theorem 3.3, there exists δ_1 such that if Q is a crosscut of U with $diam(Q) < \delta_1$, then $diam(\phi(Q)) < 1/2$. We notice that the set of images of all crosscuts with diameter less than δ_1, determines a basis for $Bd(M^3)$ in the sense of Condition (3) of the definition of C-transformation. Therefore, by the definition of C-transformation and Theorem 3.3, we can find Q_1 such that:

(i) the "small" complementary domain of $\phi(Q_1)$ is contained in $B_1(p)$, and

(ii) $Int[Cl(\phi(U_1)) \cap Bd(M^3)]$ is a neighborhood of p on $Bd(M^3)$, where U_1 is the corresponding domain of Q_1.

Now we take ϵ'_2 so that $B_{\epsilon'_2}(p) \subset Int(Cl(\phi(U_1)))$. Let $\epsilon_2 = min\{1/4, \epsilon'_2\}$ and consider $B_{\epsilon_2}(p)$. Again by Theorem 3.3, for $\epsilon_2/2$, there exists δ'_2 such that if Q is a crosscut of U with $diam(Q) < \delta'_2$, then $diam(\phi(Q)) < \epsilon_2/2$. Let $\delta_2 = min\{\delta'_2, 1/4\}$. As above, we notice that the set of images of all crosscuts with diameter less than δ_2, determines a basis for $Bd(M^3)$ in the sense of Condition (3) of the definition of C-transformation. Therefore we can find Q_2 such that:

(i) the "small" complementary domain of $\phi(Q_2)$ is contained in $B_{\epsilon_2}(p)$, and

(ii) $Int[Cl(\phi(U_2)) \cap Bd(M^3)]$ is a neighborhood of p on $Bd(M^3)$.

By induction, we get a sequence of crosscuts $\{Q_i\}$ such that $Q_{i+1} \subset U_i$ (the corresponding domain of Q_i), $diam(Q_i) \to 0$ and $\{\phi(Q_i)\}$ is a chain of crosscuts defining the prime end F.

Note that we may assume that $Bd(Q_i) \cap Bd(Q_{i+1}) = \emptyset$ (i.e. $\{Q_i\}$ is a *chain* of crosscuts). In fact, suppose that there exists k such that $Bd(Q_k) \cap Bd(Q_j) \neq \emptyset$ for all $j \geq k$ and let $q \in \cap_{j=k}^{\infty} Bd(Q_j)$. Then since q is accessible from U (for example, along the crosscut Q_k), and since U has a prime end structure by Theorem 3.2, we can find a chain of crosscuts $\{R_i\}$ which converges to q by Proposition 3.1. The image of this chain of crosscuts, $\{\phi(R_i)\}$ is a chain of crosscuts since ϕ is a $C-$map. Further, $\{\phi(R_i)\}$ has the principal point p; i.e., $\{\phi(R_i)\}$ defines the prime end F. Then for sufficiently small $R_{k'}$, $\phi(R_{k'})$ must be contained in $\phi(U_k)$ where U_k is the corresponding domain of Q_k, since $\{\phi(R_i)\}$ and $\{\phi(Q_i)\}$ define the same prime end, F, with the principal point p.

We now take a point $r' \in R_{k'} - Cl(U_k)$. Then $\phi(r') \notin \phi(U_k)$ since Q_k separates U. This contradicts the fact that $\phi(R_{k'})$ must be contained in $\phi(U_k)$. Thus, we can find $j > k$, such that $Bd(Q_k) \cap Bd(Q_j) = \emptyset$. Continuing inductively, we can find a chain of crosscuts from $\{Q_i\}$. Therefore we may assume that $\{Q_i\}$ is a *chain* of crosscuts such that $\{\phi(Q_i)\}$ defines the prime end F. It follows that ϕ is onto. □

Theorem 3.5. *Let U be an admissible domain. Then U^* is homeomorphic to a compact 3-manifold-with-nonempty-boundary, M^3, in such a way that U is identified with $Int(M^3)$ and the prime ends of U are identified with $Bd(M^3)$. Thus, the prime end compactification of U is a manifold compactification.*

Proof: Since U is admissible, there exists a C-transformation, ϕ, taking U onto the interior of some compact 3-manifold, M^3. Then ϕ is one-to-one on U by definition, and by Theorem 3.4, ϕ determines a one-to-one correspondence between the prime ends of the respective domains. We need only show that ϕ is continuous at the prime ends of U.

Let W be a neighborhood of some prime end of M^3 (that is, of some point p in $Bd(M^3)$). By Property (3) of the definition of C-transformation and Theorem 3.3, there exists a crosscut Q_α of U, with corresponding domain U_α, such that $\phi(Q_\alpha \cup U_\alpha) \subset W$. By the definition of the topology of U^*, U_α plus the prime ends of U_α forms an open set of U^*, and since ϕ takes this set into W, ϕ is continuous on U^*.

Thus, the C-transformation ϕ induces a homeomorphism between U^* and all of M^3. That is, our prime end compactification of the admissible domain U is a manifold compactification. □

3.2.4. THE INDUCED HOMEOMORPHISM THEOREM

In our view, this is the most important theorem of our three dimensional prime end theory, since it is this theorem which gives rise to our anticipated applications.

Theorem 3.6. *Let (U, ϕ, M^3) be an admissible triple (that is, U is an admissible domain and ϕ is its associated C-transformation to M^3). If h is a homeomorphism of $Cl(U)$ onto itself then the induced homeomorphism, $\phi h \phi^{-1}$, of $Int(M^3)$ onto itself can be extended to a homeomorphism $\overline{\phi h \phi^{-1}}$ of all of M^3 onto itself.*

Proof: First we show that for a prime end E of $Int(M^3)$, $\phi h \phi^{-1}(E)$ is also a prime end of $Int(M^3)$. Let E be a prime end in $Int(M^3)$ and let $\{E_i\}$ be a representative chain of crosscuts of E. Then there exists a chain of crosscuts $\{D_i\}$ in U such that $\{\phi(D_i)\}$ is equivalent to $\{E_i\}$, by the definition of C-transformation and Theorem 3.3. Then, even though $\{\phi^{-1}(E_i)\}$ may not be a *chain* of crosscuts in U (that is, their diameters may not tend to 0), $\{D_i\}$ is equivalent to $\{\phi^{-1}(E_i)\}$ in the sense that there are subsequences of the crosscuts which alternate. Since h is uniformly continuous, $\{h(D_i)\}$ is also a chain of crosscuts of U; and $\{\phi h(D_i)\}$ is a chain of crosscuts in $Int(M^3)$ since ϕ is a C-map. But then $\{\phi h(D_i)\}$ is equivalent to $\{\phi h(\phi^{-1}(E_i))\}$ which is the same as $\{\phi h \phi^{-1}(E_i)\}$. Since $\{h(D_i)\}$ is a chain of crosscuts of U, $\{\phi(h(D_i))\}$ is a chain of crosscuts of $Int(M^3)$. Now small crosscuts of a *manifold* cut off small corresponding domains on that manifold. Thus, since $\{\phi h(D_i)\}$ is equivalent to $\{\phi h \phi^{-1}(E_i)\}$, the latter sequence has diameters tending to 0, so that it forms a *chain* of crosscuts of $Int(M^3)$. Thus, $\phi h \phi^{-1}(E)$ is a prime end of $Int(M^3)$.

We notice that the inverse of $\phi h \phi^{-1}$, namely $\phi h^{-1} \phi^{-1}$, behaves in the same way. Therefore $\phi h \phi^{-1}$ acts in a one to one and onto manner, taking prime ends to prime ends. But the prime ends of M correspond precisely to the points of $Bd(M)$.

To show that $\phi h \phi^{-1}$ is extendable to a continuous function, let $x \in Bd(M^3)$ and $\{E_i\}$ be a chain of crosscuts with $lim\, E_i = x$. Define $\overline{\phi h \phi^{-1}}(x) = lim\,(\phi h \phi^{-1}(E_i))$. Then $\overline{\phi h \phi^{-1}}$ is well defined, since for a prime end E of $Int(M^3)$, $\phi h \phi^{-1}(E)$ is also a prime end of $Int(M^3)$, and each chain of crosscuts representing a prime end of M^3 converges to a point of $Bd(M^3)$. Now if $\{x_i\}$ is a sequence in $Bd(M^3)$ such that $x_i \to x$, then Property (3) of the definition of a C-transformation guarantees that $\overline{\phi h \phi^{-1}}(x_i) \to \overline{\phi h \phi^{-1}}(x)$. It follows that $\phi h \phi^{-1}$ can be extended to the homeomorphism $\overline{\phi h \phi^{-1}}$ of M^3 onto itself. □

3.3. BUBBLE DOMAINS

In this part, we establish the existence of a nontrivial, interesting class of admissible domains. We define, and give examples of, the "bubble domains", and we prove that these domains are admissible. First we recall Bing's [Bi1] definition of $1-ULC$: A metric space M is $1-ULC$ iff for each $\epsilon > 0$, there is $\delta > 0$ so that each map of $Bd(D)$ into a δ-set of M can be shrunk to a point in an ϵ-set of M. We do not assume $0-ULC$.

A *bubble domain* in E^3 is a bounded, connected, $1-ULC$, open subset U, whose boundary contains a dense subset S, such that the following conditions are satisfied:

(1) There is a monotone map $f : Cl(U) \to M^3$, such that

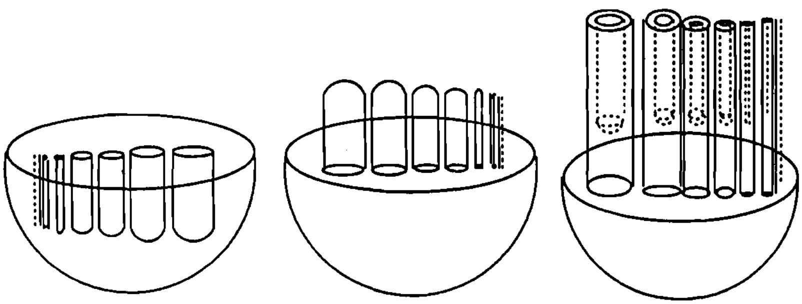
(a) $f|U$ is a homeomorphism onto $Int\,(M^3)$, and
(b) for each $x \in S$, $f^{-1}f(x) = x$.

(2) $Bd(M^3)$ admits a decreasing sequence of triangulations $\{T_i\}$, with $mesh\,(T_i) \to 0$, such that

(a) the one-skeleton of T_i lies in f(S), and
(b) the boundary of each two-simplex of T_i in $Bd(M^3)$ has inverse of diameter less than $\frac{1}{2^i}$. We call f^{-1}(1-skeleton of T_i) a *1-dimensional ϵ-triangulation of* $Bd(U)$, if $\frac{1}{2^i} < \epsilon$.

(3) S is collared into U; that is, there is a homeomorphism $g : S \times [0,1) \to U$ such that
  (a) $g(s,0) = s$, for all $s \in S$, and
  (b) $g(s,t) \in U$, whenever $t > 0$.

In this situation, U is called a *bubble domain* and $S^3 - U$ is called a *bubble continuum.* The triple (U, f, M^3) is called a *bubble triple.* (We use the word "bubble" because the crosscuts to the boundary look like bubbles on the boundary.)



(a) Bowling Ball (b) Bowling Glove (c) Modified Bowling Glove

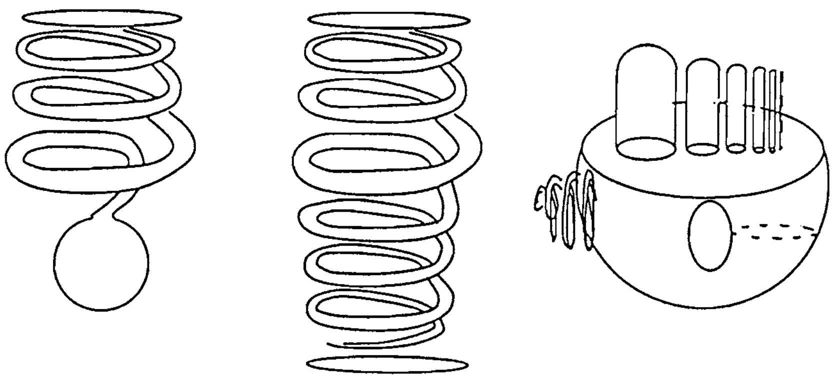
Figure 3.2

Two interesting examples are the "bowling ball" and "bowling glove" examples (Figure 3.2(a),(b)), which were constructed by John Mayer. Note that the bowling ball is not an admissible space since it does not admit a C-transformation, by Theorem 3.2. A third interesting example is the "modified bowling glove" of Figure 3.2(c), constructed by Bob Daverman, who pointed out to us that this latter example is not $1-ULC$, but does admit a C-transformation. It therefore also has a prime end structure, by our Theorem 3.2. Note that this example also shows that there are admissible domains which are not bubble domains.

These examples are basically three dimensional versions of the Warsaw circle, again with a limit segment. The bowling ball (with infinitely many fingers) is a bubble continuum, since its complement is a bubble domain. However, the interior of the bowling ball is not a bubble domain (so its complement, the bowling glove, is not a bubble continuum).

We note that the interior of the bowling ball does not have a prime end structure, and thus by Theorem 3.2, there cannot be a C-transformation from the interior of the bowling ball onto the interior of B^3.

We construct below a number of other exotic examples (Figure 3.3). Thus, we see that even the very restrictive definition of *bubble domain* gives rise to many examples to which our theory applies.



Spiral Double Spiral Distorted Torus

Figure 3.3.

Before proving that a bubble triple is an admissible triple, we will briefly review Dehn's Lemma which will play an important role in our three dimensional prime end theory.

In 1910, Max Dehn [De] first presented the lemma with a "proof"; however, in 1929 in [Kn], H. Kneser discovered a serious gap in the proof given by Dehn.

The following is the statement of Dehn's Lemma, as given by Bing [Bi2, p. 198]:

Dehn's Lemma. *If D is a PL singular disk in a PL 3-manifold-without-boundary M^3 such that $S(D) \cap Bd(D) = \emptyset$, then there exists a nonsingular PL disk, D_0, such that $D_0 \subset M^3$ and $Bd(D_0) = Bd(D)$. (Here $S(D)$ means the* singular set *of D.)*

In 1957, C. D. Papakyriakopoulos [Pa] proved the lemma. Since Papakyriakopoulos' proof of the lemma, there have been several generalizations and simplifications, including a paper by A. Shapiro and J. H. C. Whitehead [S-W] and a paper by D.W. Henderson [He].

In 1988, there was an improvement of Dehn's Lemma by D. Repovs [Re]. In his paper, he extended the classical Dehn's Lemma to a topological version. He only required the map to be continuous and therefore a Dehn disk is a continuous Dehn disk. That is, a continuous map $f : D^2 \to M^3$ of a two dimensional disk, D^2, into a 3-manifold-without-boundary, M^3, is a Dehn disk iff $S_f \cap Bd(D^2) = \emptyset$, where $S_f = Cl\{x \in D^2 | f^{-1}f(x) \neq x\}$ is the *singular set* of f.

Topological Dehn's Lemma. *Suppose $f : D^2 \to M^3$ is a Dehn disk in a 3-manifold with boundary M^3. Then, for every neighborhood $U \subset M^3$ of $f(S_f)$, there is an embedding $F : D^2 \to M^3$ such that $F|Bd(D^2) = f|Bd(D^2)$ and $F(D^2) - U = f(D^2) - U$.*

Lemma 3.4. *Let M be a compact 2-manifold without boundary. Then there exists $\epsilon > 0$ such that if T is a triangulation of M with mesh of each simplex less than ϵ, then $Cl(st(v, T))$ is a 2-cell for all $v \in T$.*

Lemma 3.5. *Let M^3 be an orientable 3-manifold, and let $Bd(M^3)$, T, and $\frac{\epsilon}{3}$ satisfy the hypotheses of Lemma 3.4 above. Since there are only a finite number of vertices v in T, the collection $\{Bd(Cl(st(v,T)))\}_{v \in K^0}$ forms a finite number of simple closed curves, say $\{C_i\}$, whose interiors form a finite open cover of $Bd(M^3)$. Let Q_i be a crosscut to $Bd(M^3)$ in $Int(M^3)$, such that $Bd(Q_i) = C_i$ and $diam(Q_i) < \epsilon$. Then the union of the corresponding domains of $\{Q_i\}$ plus $Bd(M^3)$ forms an ϵ-neighborhood of $Bd(M^3)$ in M^3.*

In the next two theorems, we prove that a bubble domain is admissible, by showing that it admits a C-transformation.

Theorem 3.7. *If (U, f, M^3) is a bubble triple, then f induces a prime end structure on U.*

Proof: The proof uses ideas from Bing [Bi1], and the Topological Dehn's Lemma [Re].

We notice that $Int(M^3)$ has the prime end structure induced by the triangulations $\{T_i\}$ of $Bd(M^3)$ from the definition of bubble triple. Let K_i^1 be the 1-skeleton of the triangulation T_i so that $K_i^1 \subset f(S)$, by the definition of bubble domain. Then $S \supseteq \bigcup_{i=1}^{\infty} f^{-1}(K_i^1)$.

Let $\epsilon > 0$. For this ϵ, let $\gamma > 0$ be the number given by Lemma 3.1. Let O be the full γ-neighborhood of $Bd(M^3)$. Then $f^{-1}(O)$ is a subset of the full ϵ-neighborhood of $Bd(U)$. Let $\beta > 0$ be a number such that the full β-neighborhood of $Bd(U)$ is a subset of $f^{-1}(O)$.

For $\frac{\beta}{3}$, there exists $\delta > 0$ such that every closed path in a δ-subset of U can be shrunk to a point in a $\frac{\beta}{3}$-subset of U, since U is $1 - ULC$. For $\frac{\beta}{3}$, let T_j be the first member of the collection $\{T_i\}$ such that the diameter of the inverse image of the boundary of the star of each $v \in T_j$ is less than δ. We notice that the closed star of $v \in T_j$ is 2-cell by Lemma 3.4, so it has a simple closed curve boundary, say $\partial\Delta_v$. Further, $Bd(M^3)$ is orientable [Mo, p. 170], so that it is a sphere with n handles.

Since S can be collared into U we can pull $f^{-1}(\partial\Delta_v)$ into U along the collar, by a very small move. Let g be the homeomorphism giving the collar. Then the simple closed curve $g[f^{-1}(\partial\Delta_v)\times 1/2]$ can be shrunk to a point in a $\frac{\beta}{3}$-subset of U.

Without loss of generality, we may assume that the singular disk D'_v obtained by shrinking the simple closed curve $g[f^{-1}(\partial\Delta_v)\times 1/2]$ to a point, has no singular points on its boundary. In fact, suppose that there are singular points on the boundary. We notice that the singular disk D'_v, which is produced by the definition of $1-ULC$, is entirely in U. Therefore the distance between $f^{-1}(\partial\Delta_v)$ and the singular disk D'_v is nonzero, and there is an annulus between them in U. We tack onto D'_v the collar beginning at $g[f^{-1}(\partial\Delta_v)\times 1/4]$, which is a simple closed curve lying in this annulus. Then the new singular disk

$$g[f^{-1}(\partial\Delta_v)\times[1/4,1/2)\,]\cup D'_v$$

has no singular points on its boundary. Consequently we can consider the singular disk D'_v as having no singular points on its boundary.

Hence, from the Topological Dehn's Lemma [Re], for every (small) neighborhood W of the singular set of D'_v, there exists a real topological disk D_v in $D'_v\cup W$, such that D_v differs from D'_v (setwise), only inside W. Thus, we may assume that $diam(D_v) < \frac{2\beta}{3}$, and that the collection $\{g[f^{-1}(\partial\Delta_v)\times [0,1/2)\,]\cup D_v\}_{v\in K_j^0}$, is a collection of crosscuts to the boundary of U of diameter $< \beta$, induced by the triangulation T_j. Let us call this collection $\{Q_v\}_{v\in K_j^0}$. Then the collection $\{f(Q_v)\}_{v\in K_j^0}$ is a finite collection of crosscuts to $Bd(M^3)$, obtained from the stars of the vertices of T_j. So the union of their corresponding domains and $Bd(M^3)$ forms a neighborhood of $Bd(M^3)$ in M^3, by Lemma 3.5. It follows that the union of the corresponding domains of the crosscuts in the collection $\{Q_v\}_{v\in K_j^0}$ plus $Bd(U)$ is an ϵ-neighborhood, V, of $Bd(U)$ in $Cl(U)$. We have now satisfied conditions (1) and (2) of the definition of prime end structure.

To see that condition (3) of the definition of prime end structure is also satisfied, we note that our crosscuts to $Bd(U)$ are the inverses of stars of vertices on $Bd(M^3)$. But, on $Bd(M^3)$, the union of the *interiors* of these stars contains $Bd(M^3)$. Thus, by Lemma 3.2, condition (3) of Definition 7 is also satisfied.

Since ϵ can be chosen to be arbitrarily small, all the conditions of Definition (7) are satisfied. Consequently, we have a prime end structure on U induced by f.

Theorem 3.8. *If (U, f, M^3) is a bubble triple, then the map f is a C-transformation.*

Proof: We first show that the image of a crosscut is a crosscut. Since the map f is monotone on $Cl(U)$ and the closure of a crosscut is a disk, it follows that, for every crosscut Q, $Bd(f(Q))$ is a simple closed curve or a single point. It suffices to show that it cannot be degenerate. To this end, suppose that Q is a crosscut of U, with corresponding domain W, and such that $Bd(f(Q))$ is degenerate. Then any sufficiently small crosscut to the boundary of U, lying in W, maps to an open disk with degenerate boundary (tangent to $Bd(M^3)$). But, since U has a prime end structure induced by f (Theorem 3.7), we can find a crosscut $Q^* \subset W$ such that $f(Q^*)$ is a (small) crosscut of $Int(M^3)$ with $Bd(f(Q^*))$ being the boundary of a two-simplex of T_i, for some i. This contradicts our assumption that $Bd(f(Q))$ is degenerate. It follows that $f(Q)$ is a crosscut, for every crosscut Q in U.

We next show that f maps $Cl(Q)$ *homeomorphically* onto $Cl(f(Q))$. Suppose that f has singular points on $Bd(Q)$ for some Q in U, and let $p \in Bd(f(Q))$ such that $f^{-1}(p)$ is nondegenerate, necessarily an arc. Let S be the dense set of the definition of bubble domain. Then $p \notin f(S)$ since each point of S is an inverse set. Thus, we can take a chain of crosscuts $\{R_i\}$ in $Int(M^3)$, where $Bd(R_i)$ is the boundary of the star of a vertex from the triangulation $\{T_i\}$ in the definition of bubble domain, and p lies in the interior of each associated

2-cell on $Bd(M^3)$. We also notice that $f^{-1}(p)$ is an arc $[a, b]$, since f is a monotone map. Since $Bd(R_i) \cap Bd(f(Q))$ has two points a_i, b_i for sufficiently large i, we consider the two sequences $\{a_i\}$ and $\{b_i\}$ which approach p. Then the two sequences $\{f^{-1}(a_i)\}, \{f^{-1}(b_i)\}$ lie on $Bd\,(Q)$ and approach the opposite endpoints of the arc $f^{-1}(p)$, respectively. Thus, the points $f^{-1}(a_i)$ and $f^{-1}(b_i)$ are separated by some fixed positive number α. This contradicts the fact that $f^{-1}(T_i^1)$ forms a 1-dimensional ϵ_i-triangulation, where $\{\epsilon_i\}$ tends to 0, which fact comes from the definition of bubble domain. This contradiction shows that f maps $Cl(Q)$ homeomorphically onto $Cl(f(Q))$, and proves Condition (2) of the definition of C-transformation.

We now show that, for every chain of crosscuts $\{Q_i\}$ of U, $\{f(Q_i)\}$ is a chain of crosscuts of M^3. Since f is uniformly continuous on $Cl(U)$, it suffices to show that $Bd(f(Q_i)) \cap Bd(f(Q_{i+1})) = \emptyset$. Suppose that there exists a chain of crosscuts $\{Q_i^*\}$ such that, for some i, $Bd(f(Q_i^*)) \cap Bd(f(Q_{i+1}^*)) \neq \emptyset$. Let $p \in Bd(f(Q_i^*)) \cap Bd(f(Q_{i+1}^*))$ and consider $f^{-1}(p)$. Then $f^{-1}(p)$ contains the points a_i and a_{i+1} in Q_i^* and Q_{i+1}^*, respectively, as well as a continuum containing both these points, since f is monotone. We again take a chain of crosscuts $\{R_i\}$ in $Int(M^3)$, using the definition of bubble domain, as in the above paragraph, such that the prime end induced by $\{R_i\}$ has p as its principal point. Let $a_{i,j}$ and $a_{i+1,j}$ be points in $Bd(f(Q_i^*)) \cap Bd(R_j)$ and $Bd(f(Q_{i+1}^*)) \cap Bd(R_j)$ respectively, for $j = 1, 2, \ldots$. Then $\{f^{-1}(a_{i,j})\}$ and $\{f^{-1}(a_{i+1,j})\}$ converge to a_i and a_{i+1}, respectively, which are a positive distance apart. Therefore we have a contradiction similar to that in the above paragraph. This completes the proof of Condition (1) of the definition of $C-$transformation.

Since the collection $\{f^{-1}(T_i)\}$ is a collection of 1-dimensional ϵ_i-triangulations of $Bd(U)$, the set of images of small crosscuts and their complementary domains which are induced by f and $\{T_i\}$, clearly satisfies Condition (3) of the definition of $C-$transformation. This completes the proof of Theorem 3.8. □

Corollary 3.3. (Induced Homeomorphism Theorem for Bubble Domains) *Let* (U, f, M^3) *be a bubble triple (that is, U is a bubble domain and f is its associated homeomorphism onto the interior of the compact 3-manifold-with-nonempty-boundary, M^3). If h is a homeomorphism of $Cl(U)$ onto itself, then the induced homeomorphism, fhf^{-1}, of $Int(M^3)$ onto itself can be extended to a homeomorphism $\overline{fhf^{-1}}$ of M^3 onto itself.*

Proof: By the definition of bubble triple, U is an admissible domain and by Theorem 3.8, f is a C-transformation of U onto the interior of the compact 3-manifold, M^3. Thus, by Theorem 3.6, the Induced Homeomorphism Theorem holds. □

4. Open Problems

The results of this paper lead to the following interesting and important open questions:

(1) Let U be a bounded, simply connected, $1-ULC$ domain in E^3. Does there necessarily exist a C-transformation $f : U \to Int(B^3)$? That is, is every bounded, simply connected, $1 - ULC$ domain in E^3 admissible?
By work of L. Husch [Hu2], C.H. Edwards [Ed], and C. T. C. Wall [Wa], we know that U is homeomorphic to the interior of the unit 3-ball, B^3. Also by work of L. Husch [Hu2], if we omit the simply connected hypothesis and add some other conditions, U is homeomorphic to the interior of a compact 3-manifold. Therefore we have the same question as above: Is such a domain admissible?

(2) In general, let U be a bounded, $1 - ULC$ domain in E^3. Must U be admissible? That is, is $1 - ULC$ sufficient? Note that there are domains which admit C-transformations and are not $1 - ULC$ (See Figure 3.2(c)), so that the $1 - ULC$ property is not necessary.

(3) Characterize the admissible domains in E^3.

(4) Characterize those domains which have a prime end structure.

References

[Bi1] R. H. Bing, *A surface is tame if its complement is* $1 - ULC$, Trans. Amer. Math. Soc., **101** (1961), 294-305.

[Bi2] R. H. Bing, *The Geometric Topology of 3-manifolds* (Amer. Math. Soc. Colloquium Publications, Providence RI, 1983).

[Br1] B. Brechner, *On stable homeomorphisms and imbeddings of the Pseudo Arc*, Illinois J. Math., **22** (1978), 630-661.

[Br2] ——, *Prime ends and group actions*, unpublished preprint (1984).

[Br3] ——, *Extendable periodic homeomorphisms on chainable continua*, Houston J. Math., **7** (1981), 327-344.

[Br-Mau] B. Brechner and R. D. Mauldin, *Positively regular homeomorphisms of Euclidean spaces*, Topology and Its Applications, **11** (1980), 231-242.

[Br-May] B. Brechner and J. C. Mayer, *The prime end structure of indecomposable continua and the fixed point property, in:* McAuley and Rao, ed., General Topology and Modern Analysis (Academic Press, New York, NY, 1981), 151-168.

[Bro] M. Brown, *Locally flat imbeddings of topological manifolds,* Annals of Math., **75 No. 2**, (1962), 331-341.

[C] C. Caratheodory, *Uber die Begrenzung einfach zusammenhangender Gebiete,* Math. Ann., **73** (1913), 323-370.

[C-L] M. L. Cartwright and J. E. Littlewood, *Some fixed point theorems,* Annals of Math., **54** (1951), 1-37.

[De] M. Dehn, *Uber die Topologie des dreidemensionalen Baumes,* Math. Ann., **69** (1910), 137-168.

[Ed] C. H. Edwards, Jr., *Open 3-manifolds which are simply connected at infinity,* Proc. Amer. Math. Soc., **14** (1963), 391-395.

[Ep1] D. B. A. Epstein, *Prime ends,* Proc. London Math. Soc., **42** (1981), 385-414.

[Ep2] D. B. A. Epstein, *Pointwise periodic homeomorphisms*, Proc. London Math. Soc., **42** (1981), 415-460.

[He] D. W. Henderson, *Extensions of Dehn's lemma and the loop theorem,* Trans. Amer. Math. Soc., **120** (1965), 448-469.

[Hu1] L. Husch, *The monotone union property of manifolds,* Trans. Amer. Math. Soc., **131** (1968), 345-355.

[Hu2] ——, *Finding a boundary for a 3-manifold,* Proc. Amer. Math. Soc., **21** (1969), 64-68.

[Kau] Boris Kaufmann, *Uber die Berandung ebener und raumlicher Gebiete,* Math. Ann., **103** (1930), 70-144.

[Kn] H. Kneser, *Geschlossene Flachen in dreidemensionalen Man-*

nigfaltigkeiten, Jber. Deutsch. Math.-Vereinig, **38** (1929), 248-260.

[L1] J. S. Lee, *The Hilbert-Smith conjecture and prime end theory on 3-manifolds,* Doctoral Dissertation, Univ. of Florida, Gainesville, FL (1993).

[L2] J. S. Lee, *Almost periodic homeomorphisms and p-adic transformation groups on compact 3-manifolds.* Proc. Amer. Math. Soc., **121** (1994), 267-273.

[Lew] Wayne Lewis, *Embeddings of the pseudo arc in* E^2, Pacific J. Math., **93** (1981), 115-120.

[Mas] W. K. Mason, *Fixed points of almost periodic homeomorphisms of the two-sphere,* Trans. Amer. Math. Soc., **202** (1975), 243-258.

[Mat] J. N. Mather, Topological proofs of some purely topological consequences of Caratheodory's theory of prime ends, in: Th.M. Rassias, G.M. Rassias, eds., Selected Studies (North-Holland Publ. Co., Amsterdam, 1982), 225-255.

[May1] John C. Mayer, *Embeddings of plane continua and the fixed point property,* Doctoral Dissertation, Univ. of Florida, Gainesville, FL (1982).

[May2] ——, *Embeddings and prime end structure of chainable continua,* Houston J. Math., **8** (1982), 221-253.

[Maz] S. Mazurkiewicz, *Recherches sur la theorie des bouts premiers,* Fund. Math., **33** (1945), 177-228.

[Mo] E. Moise, Geometric Topology in Dimension 2 and 3 (Springer-Verlag, Berlin and New York, 1977).

[Pa] C. D. Papakyriakopoulos, *On Dehn's lemma and the asphericity of knots,* Annals of Math., **66** (1957), 1-26.

[Re] D. Repovs, *A topological Dehn's lemma,* Mathematica Balkanica New Series, **2** (1988), 307-313.

[S-W] A. Shapiro and J. H. C. Whitehead, *A proof and extension of Dehn's lemma,* Bull. Amer. Math. Soc., **64** (1958), 174-178.

[U-Y] H. D. Ursell and L.C. Young, *Remarks on the theory of prime ends*, Memoirs Amer. Math. Soc., **3** (1951), 1-29.

[Wa] C. T. C. Wall, *Open 3-manifolds which are 1-connected at infinity,* Quart. J. Math., **16** (1965), 263-268.

[Wh] G. T. Whyburn, Analytic Topology, (Amer. Math. Soc. Colloq. Publications, Providence, R.I., 1955).

University of Florida
Gainesville, FL 32611
e-mail: brechner@math.ufl.edu

Dongguk University
Seoul 100-715, KOREA
e-mail: jsl@cakra.dongguk.ac.kr