Topology Proceedings

Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings Department of Mathematics & Statistics Auburn University, Alabama 36849, USA
E-mail:	topolog@auburn.edu
ISSN:	0146-4124

Topology Proceedings
Vol. 20, 1995

ON THE TRANSFINITE DIMENSION DIM AND ESSENTIAL MAPPINGS

V.A.CHATYRKO

0. Throughout this note we shall consider only metrizable spaces with a countable base. By C we denote the Cantor set and by I^n we denote the n-dimensional cube. The terminology and notations follow [1].

Let X be a space and dim be the Lebesgue covering dimension.

It is well-known that

a) $dimX = dimX \times C$;

b) $dimX \geq n$ iff X admits an essential map onto I^n.

From a) and b) one can easily note that

c) $dimX \geq n$ iff $X \times C$ admits an essential map onto I^n.

In [2,3] P.Borst considered a transfinite extension of the covering dimension dim, namely the transfinite dimension dim and extended statements a) and c) as follows (here we shall denote Borst's dimension by $trdim$).

Let X be a locally compact space and α be a countable ordinal number, then

$a)_{tr}$ $trdimX = trdimX \times C$;

$c)_{tr}$ $trdimX \geq \alpha$ iff $X \times C$ admits an essential map onto H^α

(where $H^\alpha, \alpha < \omega_1$, are Henderson's cubes).

Remind that for every metrizable space X we have either $trdim X = \infty$ or $trdim X < \omega_1$ (see [4] or [5]). By definition we have $\infty > \alpha$ for every ordinal number α.

Besides, P. Borst [3] constructed a compact space X with $trdim X = \omega + 1$, which admitted no essensial map onto $H^{\omega+1}$. Hence the extension of statement b) to infinite ordinal numbers is imposible.

In [3, 6] P.Borst asked

Can the condition "locally compact" for $a)_{tr}$ and $c)_{tr}$ be weakened?

We shall prove the following statements

Theorem 1. *Let X be a space and $trdim X \times C \geq \omega^2$. Then*

$trdim X = trdim X \times C$.

Theorem 2. *Let X be a space and α be a countable ordinal number $\geq \omega^2$. Then*

$trdim X \geq \alpha$ *iff* $X \times C$ *admits an essential map onto* H^α.

1. Recall some definitions and propositions.

A finite sequence $\{(A_i, B_i)_{i=1}^m\}$ of pairs of disjoint closed sets in space X is called inessential if we can find open sets $O_i, i = 1, ..., m$ such that

$A_i \subset O_i \subset \bar{O}_i \subset X \setminus B_i$ and $\cap_{i=1}^m Fr O_i = \emptyset$.

Otherwise it is called essential.

We have the following charakterization of the dimension dim [1]:

$dim X \leq n$ iff every sequence $\{(A_i, B_i)_{i=1}^{n+1}\}$ of pairs of disjoint closed sets in X is inessential.

Let L be an arbitrarary set. By $Fin L$ we shall denote the collection of all finite, non-empty subsets of L. Let M be a subset of $Fin L$. For $\sigma \in \{\emptyset\} \cup Fin L$ we put

$M^\sigma = \{\tau \in Fin L | \quad \sigma \cup \tau \in M \text{ and } \sigma \cap \tau = \emptyset\}$.

Let $M^a = M^{\{a\}}$.

Define [2] the ordinal number $OrdM$ inductively as follows

$OrdM = 0$ iff $M = \emptyset$,

$OrdM \leq \alpha$ iff for every $a \in L$ $OrdM^a < \alpha$,

$OrdM = \alpha$ iff $OrdM \leq \alpha$ and $OrdM < \alpha$ is not true, and

$OrdM = \infty$ iff $OrdM > \alpha$ for every ordinal number α.

Let X be a space. Put

$L(X) = \{(A, B)| \quad A, B \subset X$, closed, disjoint $\}$

and

$M_{L(X)} = \{\sigma \in FinL(X)| \quad \sigma$ is essential in X $\}$.

Define [2]

$trdimX = OrdM_{L(X)}$.

A space X is called S-weakly infinite-dimensional [1] if for every sequence $\{(A_i, B_i)_{i=1}^{\infty}\}$ of pairs of disjoint closed sets in space X we can find open sets $O_i, i = 1, ...,$ such that

$A_i \subset O_i \subset \bar{O}_i \subset X \setminus B_i$ and $\cap_{i=1}^{m} FrO_i = \emptyset$ for some m.

Otherwise, X is S-strongly infinite-dimensional. If X is compact and S-weakly (strongly) infinite-dimensional then X is said to be weakly (strongly) infinite-dimensional.

Let us recall [2] that

$trdimX \neq \infty$ iff X is S-weakly infinite-dimensional.

Henderson's cubes and the essential maps are defined [7] as follows.

Let $H^1 = I^1, \delta H^1 = \delta I = \{0,1\}, p_1 = \{0\}$, and assume that for every $\beta < \alpha$ the compacta H^β, their "boundaries" δH^β, and the points $p_\beta \in \delta H^\beta$ have already been defined. If $\alpha = \beta + 1$, then we set

$$H^{\beta+1} = H^\beta \times I, \quad \delta H^{\beta+1} = (\delta H^\beta \times I) \cup (H^\beta \times \delta I)$$

and $p_{\beta+1} = (p_\beta, p_1)$. If α is a limit ordinal number, then K_β is the union of H^β and a half-open arc A_β such that $A_\beta \cap H^\beta = p_\beta$ $= \{$ endpoint of the arc $A_\beta\}$, $\beta < \alpha$. Let us define H^α as the one-point compactification of the free sum $\oplus_{\beta<\alpha} K_\beta$,

$$\delta H^\alpha = H^\alpha \setminus \cup_{\beta<\alpha}(H^\beta \setminus \delta H^\beta),$$

and let p_α be the compactifying point.

A map $f : X \to H^\alpha$ is called essential if every continuous extension to X of the restriction $f|_{f^{-1}\delta H^\alpha}$ maps X onto $H^\alpha, \alpha < \omega_1$.

2. We need some lemmas in order to prove theorems 1, 2.

Lemma 1. [2, 3]. *Let X be a space.*

1) $trdim X = \infty$ *iff* $trdim X \times C = \infty$.

2) let A be a closed subset of X, then

$$trdim A \leq trdim X.$$

Lemma 2. [8]. *Let X be a S-weakly infinite-dimensional space. Then there exists a weakly infinite-dimensional compact space Y such that* $Y \supset X$ *and*

$$trdim Y \leq trdim X.$$

Lemma 3. *Let* X, Y *be S-weakly infinite-dimensional spaces such that* $X \subset Y$ *and* $trdim X \geq \omega^2$. *Then*

$$trdim X \leq trdim Y.$$

Proof: In [4] Y.Hattori (see also [5]) proved the following statement

(*) Let X, Y be S-weakly infinite-dimensional spaces such that $X \subset Y$ then $trdim X \leq \omega + trdim Y$. Moreover, as D.Malyhin remarked in [5], if $trdim Y \geq \omega^2$, then $trdim X \leq trdim Y$.

Note that since $trdimX \geq \omega^2$, then by first inequality of (*) we have $trdimY \geq \omega^2$. The lemma follows the second inequality of (*).

Proof of theorem 1.

Due to lemma 1.1) we need to consider only case when $trdimX \neq \infty$. In this case the spaces X and $X \times C$ are S-weakly infinite-dimensional. By lemma 2 there exists weakly infinite-dimensional compact space Y such that $Y \supset X$ and $trdimY \leq trdimX$. Since $Y \times C \supset X \times C$ and $trdimX \times C \geq \omega^2$, then by lemma 3 we have $trdimY \times C \geq trdimX \times C \geq \omega^2$. It remains to put together the following chain of inequalities

$trdimX \times C \geq trdimX$ (lemma 1) $\geq trdimY = trdimY \times C$ (statement $a)_{tr}$) $\geq trdimX \times C$.

The theorem is proved.

Lemma 4. [3]. *Let X be a space and α be a countable ordinal number.*

1) If $trdimX \geq \alpha$, then $X \times C$ admits an essential map onto H^α.

2) If $f : X \to H^\alpha$ is an essential map of X onto H^α, then $trdimX \geq \alpha$.

Proof of theorem 2:

Lemma 4.1) contains the necessity. Let us prove the sufficiency. If there exists an essential map $f : X \times C \to H^\alpha$, then by lemma 4.2) we have $trdimX \times C \geq \alpha \geq \omega^2$. By theorem 1 we get

$trdimX \times C = trdimX$.

Hence $trdimX \geq \alpha$. The theorem is proved.

Remark 1. (see [6]). There exists a compactum Y with $trdimY = \omega_0$ containing a subspace X with $trdimX = \omega_0 + 1$.

Question 1. Can one drop the condition $trdim X \times C \geq \omega^2$ from theorem 1 ?

Note (see [3]) that if X is a space and α is a limit ordinal number $< \omega_1$ then

$trdim X \geq \alpha$ iff X admits an essential map onto H^α.

Question 2. Is it true that if X is a space and α is an ordinal number $\geq \omega^2$ then

$trdim X \geq \alpha$ iff X admits an essential map onto H^α.

Remark 2. In lemma 2 in the case $trdim X \geq \omega^2$ we have

$trdim X = trdim Y$.

Remark 3. In theorems 1, 2 the condition "with a countable base" can be omitted.

References

1. P.S. Aleksandrov and B.A. Pasynkov, Introduction to Dimension Theory (Russian), Moscow 1973.
2. P. Borst, *Classification of weakly infinite-dimensional spaces, Part I: A transfinite extension of the covering dimension*, Fund.Math., **130** (1988), 1-25.
3. P. Borst, *Classification of weakly infinite-dimensional spaces, Part II: Essential mappings*, Fund.Math., **130** (1988), 74-99.
4. Y. Hattori, *Remarks on weak large transfinite dimension $w - dim$*, Q&A in General Topology, **4** (1986), 59-66.
5. D. Malyhin, *On monotonicity and boundedness of transfinite dimension*, General Topology. Mappings, products and dimension (Russian) Moscow, 1995, 58-63.
6. P. Borst, *On weakly infinite-dimensional subspaces*, Fund.Math., **140**, (1992), 225-235.
7. D.W. Henderson, *A lower bound for transfinite dimension*, Fund. Math., **64** (1968), 167-173.
8. V.A. Chatyrko, *On the transfinite dimension dim*, Q&A in General Topology, **9** (1991), 177-194.

NIISI RAN, Pattern Recognition and Expert Systems Division
Avtozavodskaja 23, Moscow 109280, Russia
e-mail: chatyrko@systud.msk.su
or
e-mail: vitali@sm.luth.se