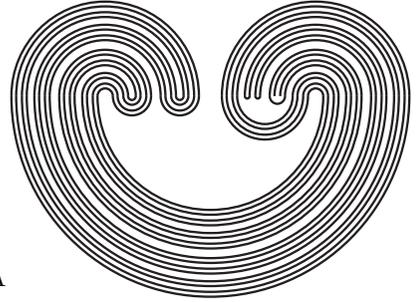


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## DOWKER SPACES, ANTI-DOWKER SPACES, PRODUCTS AND MANIFOLDS

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ABSTRACT. Assuming  $\diamond^*$ , we construct first countable, locally compact examples of a Dowker space, an anti-Dowker space containing a Dowker space, and a countably paracompact space with Dowker square. We embed each of these into manifolds, which again satisfy the above properties.

### INTRODUCTION

All spaces are Hausdorff. A space is normal if every pair of disjoint closed sets can be separated, binormal if its product with the closed unit interval  $I$  is normal, and countably paracompact if every countable open cover has a locally finite open refinement. Dowker proves that a normal space is binormal if and only if it is countably paracompact [Do]. A Dowker space is a normal space that is not countably paracompact.

There are essentially two Dowker spaces that do not require extra set-theoretic assumptions ([Ru1], [Bg2]). Neither of these is first countable or locally compact. Of course, given set-theoretic assumptions beyond ZFC, there are also small Dowker spaces—see [Ru2]. Here we construct a simple (and typical) small Dowker space assuming  $\diamond^*$ .

An anti-Dowker space is a countably paracompact, (regular) space that is not normal. Unlike Dowker spaces, there are many examples of such spaces that require no special set-theoretic assumptions—again, see [Ru2]. The (lighthearted) anti-Dowker space constructed here uses  $\diamond^*$ , since it contains

a small Dowker subspace.

Rudin and Starbird [RS] have shown that, for normal, countably paracompact  $X$  and metrizable  $M$ ,  $X \times M$  is normal iff it is countably paracompact. They asked whether a product of two normal, countably paracompact spaces could be a Dowker space. (Any normal first countable space with Dowker square is countably paracompact.) Bešlagić [Bs1] constructs a countably paracompact space with Dowker square assuming  $\diamond$ . He constructs such a space assuming CH [Bs2] and a perfectly normal example, again assuming  $\diamond$  [Bs3]. We construct a slight modification of Bešlagić's space in [Bs1] assuming  $\diamond^*$ .

Our aim is to construct various Dowker manifolds. Unlike [N3], we are interested primarily in the Dowker pathology and therefore take a hands on approach to the constructions. Each of the topologies constructed in Section 2 refines the usual order topology on  $\omega_1$  (or two disjoint copies in the case of the Dowker product), this makes them particularly suitable for embedding into the product of the long line and the open unit interval. We use the Prüfer technique, rather than the tangent bundles of [N3], and, since the technique is well-known (see for example [N2]), our discussion is quite informal. The first construction of a Dowker manifold, assuming  $\diamond^*$ , was published by Nyikos (see [N3]) and in [N3] a construction is given using the weaker  $\diamond$ . We use  $\diamond^*$  and none of our constructions works if we assume  $MA + \neg CH$ . Rudin [Ru3] has described a Dowker manifold (with a countable point separating open cover) assuming CH.

## 1. NOTATION AND COMBINATORICS

Notation and terminology are standard (see [E], [Ku], or [KV]). We regard an ordinal as the set of its predecessors, use the term *club set* to denote a closed, unbounded subset of an ordinal, and, following [Bs1], we say that a subset  $A$  of  $\omega_1^2$  is *2-unbounded* if for no  $\alpha \in \omega_1$  is  $A$  a subset of  $(\alpha \times \omega_1) \cup (\omega_1 \times \alpha)$ . For a function  $f : A \rightarrow B$ , we denote the image of a subset  $C$  of  $A$  by  $f''C$ , and for a subset  $A$  of  $\alpha \times \beta$ , we denote the set of

first coordinates by  $\text{dom } A$ , and the set of second coordinates by  $\text{ran } A$ . Recall that a space  $X$  is countably metacompact (paracompact) if and only if for every decreasing sequence  $\{D_n\}_{n \in \omega}$  of closed subsets of  $X$  with empty intersection, there is a sequence  $\{U_n\}_{n \in \omega}$  of open sets,  $U_n$  containing  $D_n$ , which also has empty intersection (whose closures have empty intersection). The two notions coincide in the class of normal spaces. A manifold for our purposes is a locally Euclidean, connected, Hausdorff space.

We use the Ostaszewski technique [O] for constructing locally countable, locally compact spaces. In order to facilitate the construction of the manifolds, the spaces described in Section 2 will have point set  $\omega_1$ . To move between disjoint stationary sets, we use the club sets chosen by the axiom  $\clubsuit^*$ , which is derived from  $\diamond^*$ .  $\diamond^*$  is true if  $V = L$ . It follows from results in [Bgl] that these constructions do not work if we assume  $\text{MA} + \neg\text{CH}$ .

Recall that  $\diamond^*$  is the assertion that, for every  $\alpha \in \omega_1$ , there is a countable family  $\mathcal{S}_\alpha$  of subsets of  $\alpha$  such that  $\{\alpha \in \omega_1 : X \cap \alpha \in \mathcal{S}_\alpha\}$  contains a club set, whenever  $X$  is a subset of  $\omega_1$ . The collection  $\{\mathcal{S}_\alpha : \alpha \in \omega_1\}$  is called a  $\diamond^*$ -sequence.

We shall let  $\clubsuit^*$  be the assertion that, for every limit ordinal  $\alpha \in \omega_1$ , there is a sequence  $R_\alpha$ , cofinal in  $\alpha$ , such that  $\{\alpha \in \omega_1 : X \cap R_\alpha \text{ is cofinal in } \alpha\}$  contains a club set, whenever  $X$  is an uncountable subset of  $\omega_1$ . The collection  $\{R_\alpha : \alpha \in \omega_1 \text{ and } \lim(\alpha)\}$  is called a  $\clubsuit^*$ -sequence.

Simple modification of the proof of  $\clubsuit$  from  $\diamond$  shows that  $\clubsuit^*$  follows from  $\diamond^*$ . (Pick  $R_\alpha$  so that  $R_\alpha \cap S$  is cofinal in  $\alpha$  whenever  $S \in \mathcal{S}_\alpha$  is cofinal in  $\alpha$ .)

We use the following two consequences of  $\clubsuit^*$  to construct the space  $Z$  of Example 2.7:

$\clubsuit_{\omega_1 \times \omega_1}^*$  is the assertion that, for every limit ordinal  $\alpha \in \omega_1$ , there is a sequence  $T_\alpha$ , cofinal in  $\alpha \times \alpha$ , such that  $\{\alpha \in \omega_1 : X \cap T_\alpha \text{ is cofinal in } \alpha \times \alpha\}$  contains a club set, whenever  $X$  is a 2-unbounded subset of  $\omega_1 \times \omega_1$ . Notice that  $\{\text{dom } T_\alpha \cup \text{ran } T_\alpha : \alpha \in \omega_1 \text{ and } \lim(\alpha)\}$  is a  $\clubsuit^*$ -sequence, if  $\{T_\alpha : \alpha \in \omega_1 \text{ and } \lim(\alpha)\}$

$\lim(\alpha)\}$  is a  $\clubsuit_{\omega_1 \times \omega_1}^*$ -sequence.

$\clubsuit^{*2}$  is the assertion that there are two  $\clubsuit^*$ -sequences  $\{R_{\alpha,0} : \alpha \in \omega_1 \text{ and } \lim(\alpha)\}$  and  $\{R_{\alpha,1} : \alpha \in \omega_1 \text{ and } \lim(\alpha)\}$  such that  $R_{\alpha,0}$  and  $R_{\alpha,1}$  are disjoint for each  $\alpha$ .

It is easy to prove that  $\diamond^*$  implies  $\clubsuit^{*2}$ . To see that  $\clubsuit^*$  implies  $\clubsuit_{\omega_1 \times \omega_1}^*$  let  $f : \omega_1 \rightarrow \omega_1 \times \omega_1$  be any bijection and let  $T_\alpha = f^{\smile} R_\alpha$ .

## 2. DOWKER SPACES, ANTI-DOWKER SPACES, AND DOWKER PRODUCTS

In this section the constructions follow the same pattern: we inductively define a local base at each point  $\alpha$  of  $\omega_1$ , which then generates a topology on the point set  $\omega_1$ . To standardize the discussion, we shall use the following terminology: Suppose that we have defined a local base  $\mathcal{B}_\gamma$  at each  $\gamma < \alpha$ , which refines the usual neighbourhood topology at  $\gamma \in \omega_1$ . Let  $\{\alpha_j\}$  and  $\{\beta_k\}$  be sequences cofinal in  $\alpha$ , with  $\{\alpha_j\}$  a subsequence of  $\{\beta_k\}$ . For each  $\alpha_j$  let  $\beta(\alpha_j)$  be the predecessor of  $\alpha_j$  in  $\{\beta_k\}$ . Choose a neighbourhood  $B(\alpha_j)$  from  $\mathcal{B}_{\alpha_j}$  such that  $B(\alpha_j)$  is a subset of the interval  $(\beta(\alpha_j), \alpha_j]$ . We shall then say that  $\alpha$  is a *rigid limit of  $\{\alpha_j\}$  with respect to  $\{\beta_k\}$*  if the neighbourhood base at  $\alpha$  is defined to be the collection

$$\mathcal{B}_\alpha = \left\{ \{\alpha\} \cup \bigcup_{j \geq n} B(\alpha_j) : n \in \omega \right\}.$$

It is not hard to see that any topology defined in this way on  $\omega_1$  will be Hausdorff, first countable, locally countable, locally compact, locally metrizable, and zero-dimensional. Moreover, since the topology refines the usual order topology on  $\omega_1$ , such a space will be pseudo-normal (two disjoint closed sets can be separated provided one is countable).

**2.1 The Dowker space  $X(\clubsuit^*)$**  *There is a first countable, strongly zero-dimensional, locally countable, locally compact,  $\omega_1$ -compact, strongly collectionwise normal Dowker space, which*

has scattered length  $\omega$  and is hence  $\sigma$ -discrete and weakly  $\theta$ -refinable.

*Proof:* The point set for the space  $X$  is  $\omega_1$ . Partition  $X$  into  $\omega$  disjoint stationary sets  $S_n$ ,  $n \in \omega$ . We think of  $S_n$  as the  $n^{\text{th}}$  level of  $X$ . Let  $\{R_\alpha\}_{\text{lim}(\alpha)}$  be a  $\clubsuit^*$ -sequence. For each  $\alpha$  in  $S_{n+1}$ , let  $T_\alpha = R_\alpha \cap S_n$  if this is cofinal in  $\alpha$ , otherwise let  $T_\alpha$  be undefined. If  $\alpha$  is in  $S_0$ , or if  $\alpha$  is a successor, or if  $T_\alpha$  is undefined, then declare  $\alpha$  to be isolated with neighbourhood base  $\mathcal{B}_\alpha = \{\{\alpha\}\}$ . If  $T_\alpha$  is defined and  $\alpha$  is in  $S_{n+1}$ , then we declare  $\alpha$  to be the rigid limit of the sequence  $T_\alpha$  with respect to  $T_\alpha$ . Let  $\mathcal{T}$  be the topology generated by the neighbourhood bases so defined.

As mentioned above,  $(X, \mathcal{T})$  is Hausdorff, locally compact, locally countable, first countable, regular, pseudo-normal, zero-dimensional space. Furthermore,  $X$  has scattered height  $\omega$  and so is  $\sigma$ -discrete and, hence, weakly  $\theta$ -refinable by a result of [N1]. ( $X$  fails to have stronger covering properties: if  $\omega_1$ , with a topology refining the usual order topology, is  $\theta$ -refinable, then it is perfect [G3].)

If  $A_0$  and  $A_1$  are any uncountable subsets of  $S_n$ , then for  $i \in 2$ ,  $\{\alpha : A_i \cap S_n \cap R_\alpha \text{ is cofinal in } \alpha\}$  contains a club by  $\clubsuit^*$ . Hence  $A_0$  and  $A_1$  have uncountably many common limit points in  $S_{n+1}$  and

**Fact 2.2.**  *$X$  is  $\omega_1$ -compact and has no two disjoint uncountable closed subsets.*

Pseudo-normality is now enough to give normality. Since  $X$  is  $\omega_1$ -compact, discrete collections of sets are countable and  $X$  is also strongly collectionwise normal (discrete collections of closed sets can be separated by discrete collections of open sets). Moreover, closed initial segments are clopen and countable, and, since zero-dimensional, Lindelöf spaces are strongly zero-dimensional (see [E 6.2.7]),  $X$  is strongly zero-dimensional (i.e. any two functionally separated sets can be separated by disjoint clopen sets).

To show that  $X$  is not countably metacompact, let  $D_n = \bigcup_{j \geq n} S_j$  and let  $U_n$  be any open set containing  $D_n$ .  $\{D_n\}_{n \in \omega}$  is a decreasing sequence of closed subsets with empty intersection,  $X - U_n$  and  $D_n$  are disjoint closed sets, and  $D_n$  is uncountable. By Fact 2.2, each  $X - U_n$  is countable and so  $\bigcap U_n$  non-empty.  $\square$

If  $C$  is any functionally closed and  $X - C$  is uncountable, then there is some  $f : X \rightarrow [0, 1]$  such that  $C = f^{-1}\{0\}$  and some  $n \in \omega$  for which  $D = f^{-1}[1/n, 1]$  is uncountable. As both  $C$  and  $D$  are closed, Fact 2.2 implies

**Fact 2.3.** *Every functionally closed subset of  $X$  is either countable or co-countable.*

A space is realcompact if it can be embedded as a closed subset of  $\mathbb{R}^\kappa$  for some  $\kappa$  and a Tychonoff space is compact iff it is pseudocompact and realcompact (see [E]). There are examples of realcompact Dowker spaces (see [R2]), however,  $X$  is not realcompact:

Let  $F_\lambda = \{x \in X : x > \lambda\}$ . Since  $X - F_\lambda$  is countable,  $F_\lambda$  is a  $G_\delta$ . Since  $X$  is normal,  $F_\lambda$  is functionally closed. Let  $\mathcal{F}$  be the set of all functionally closed sets, and let  $\mathcal{G}$  be the filter of  $\mathcal{F}$  generated by the collection of all  $F_\lambda$ s. By Fact 2.3, every element of  $\mathcal{G}$  is co-countable and  $\mathcal{G}$  is an ultrafilter of  $\mathcal{F}$  with the countable intersection property. A Tychonoff space is realcompact if and only if no ultrafilter of  $\mathcal{F}$  with the countable intersection property is free ([E 3.11.11]), however,  $\mathcal{G}$  is clearly free.

$X$  also fails to be hereditarily normal: Consider the subspace  $Y = S_0 \cup S_1$  and let  $H$  and  $K$  be uncountable disjoint subsets of  $S_1$ . If we assume  $\text{MA} + \neg\text{CH}$ , then the subspace  $Y$  is a normal (non-metrizable Moore) space ([DS] and [Bg1]), however, assuming  $V = L$ , no such Dowker space can be hereditarily normal (see [G2]). Balogh's Dowker space [Bg2] is hereditarily normal. We do not know whether there is an hereditarily normal Dowker manifold.

Pseudocompact spaces are never Dowker, however, the following lemma says that every continuous,  $\mathbb{R}$ -valued function on  $X$  is eventually constant.

**2.4 Lemma.** *If  $f : X \rightarrow \mathbb{R}$  is any continuous function, then there is some  $\gamma \in \omega_1$  such that  $f$  is constant on  $X - \gamma$ .*

*Proof:* For each  $n \in \omega$ , let  $A_n$  be the set  $f^{-1} "[n, n + 1]$ . Pick some  $n$  for which  $A_n \cap S_0$  has size  $\omega_1$ . By Fact 2.3,  $A_n$  is co-countable. Inductively define subsets  $B_k$  of  $A_n$  such that:  $B_0$  is  $A_n$ , if  $B_k$  is the set  $f^{-1} "[b_k, b_k + 1/2^k]$ , then  $B_{k+1}$  is either the set  $f^{-1} "[b_k, b_k + 1/2^{k+1}]$  or the interval  $f^{-1} "[b_k + 1/2^{k+1}, b_k + 1/2^k]$ , and  $B_k$  is co-countable. Let  $\bigcap [b_k, b_k + 1/2^k] = \{r\}$ . For each  $n \in \omega$ , let  $C_n$  be the closed set  $f^{-1} "[r - 1/n, r + 1/n]$ . Clearly, each  $C_n$  is co-countable, from which it follows that the pre-image of  $r$  is co-countable completing the proof. (I would like to thank the referee for suggesting this much simpler proof.)  $\square$

Incidentally, this provides us with an alternative proof that  $X$  is not countably paracompact: A space is both normal and countably paracompact if and only if, for every  $g$  lower and  $h$  upper semicontinuous  $\mathbb{R}$ -valued functions on  $X$  such that  $h < g$ , there is a continuous  $\mathbb{R}$ -valued function  $f$  such that  $h < f < g$ . The constant zero valued function  $\mathbb{O}$  on  $X$  is (upper semi)continuous and the function  $g : X \rightarrow \mathbb{R}$  defined by  $g(x) = 1/n$  iff  $x \in S_n$  is lower semicontinuous and greater than  $\mathbb{O}$ . Any continuous  $\mathbb{R}$ -valued function on  $X$  is eventually constant. If  $\mathbb{O} \leq f \leq g$ , then eventually  $f = \mathbb{O}$ .

The next example is a locally compact anti-Dowker space. We prevent normality using two stationary sets  $H$  and  $K$ , each containing a Dowker subspace (c.f. the argument that the space  $X$  above is not hereditarily normal). To achieve countable paracompactness, we use  $\clubsuit^*$  to cap the two Dowker subspaces in such a way that the intersection of countably many uncountable closed subsets is again an uncountable closed subset.

**2.5 The anti-Dowker space  $Y$ .** ( $\clubsuit^*$ ) *There is a first countable, locally compact,  $\omega_1$ -compact, strongly zero-dimensional, pseudo-normal,  $\delta$ -normal, strongly collectionwise Hausdorff anti-Dowker space, containing a Dowker subspace, which satisfies all of the properties of the space  $X$ .*

*Proof:* The point set for the space  $Y$  is  $\omega_1$ . Let  $D$  be the set of all successors in  $\omega_1$ . Divide  $\omega_1 - D$  into two disjoint stationary subsets,  $H$  and  $K$ , and partition each into  $\omega$  many disjoint stationary sets,  $H_n$  and  $K_n$ ,  $1 \leq n \leq \omega$ . To simplify notation let  $D = H_0 = K_0$ , and denote  $x$  in  $Y$  by a pair  $x = (\alpha, \ell_n)$ , where  $\alpha$  is the actual element of  $Y = \omega_1$  that is  $x$ , and  $\ell$  is either  $H$  (if  $\alpha \in H_n$ ) or  $K$  (if  $\alpha \in K_n$ ), and  $\ell_0 = D$  (if  $\alpha$  is a successor).

For each limit ordinal  $\alpha$ , let  $R_\alpha$  be the cofinal sequence in  $\alpha$  furnished by  $\clubsuit^*$ . Let  $x = (\alpha, \ell_n)$ . Suppose that  $1 \leq n = m + 1 < \omega$ . If  $R_\alpha \cap \ell_m$  is cofinal in  $\alpha$ , then let  $T_\alpha = R_\alpha \cap \ell_m$ , otherwise let  $T_\alpha$  be undefined. Suppose that  $n = \omega$ . If  $R_\alpha \cap \bigcup_{m < \omega} \ell_m$  is cofinal in  $\alpha$ , then let  $T_\alpha = R_\alpha \cap \bigcup_{m < \omega} \ell_m$ , otherwise let  $T_\alpha$  be undefined. Let  $T_x$  denote the sequence  $T_\alpha$ . If  $x = (\alpha, \ell_n)$  and either  $\alpha$  is a successor (i.e.  $n = 0$ ), or  $T_x$  is undefined, then let  $x$  be isolated. If otherwise, let  $x$  be a rigid limit of  $T_x$  with respect to  $T_x$ . Let  $\mathcal{T}$  be the topology generated by  $\bigcup_{x \in Y} \mathcal{B}_x$ . As above,  $(Y, \mathcal{T})$  is Hausdorff, locally compact, regular, pseudo-normal, first countable, locally countable, zero-dimensional and locally metrizable. Furthermore  $Y$  has scattered height  $\omega_1$ .

The subspace  $X' = \bigcup_{n \in \omega} H_n$  is a Dowker space, which shares all the same properties as  $X$ , for the same reasons, and  $Y$  is not normal for exactly the same reasons that the Dowker space above is not hereditarily normal;  $H$  and  $K$  are two disjoint closed subsets which cannot be separated by disjoint open sets.

To show that  $Y$  is countably paracompact it is sufficient to show that each of the subspaces  $H \cup D$  and  $K \cup D$  is countably paracompact. Let us consider  $H \cup D$ . Let  $\{D_j\}_{j \in \omega} \subseteq H \cup D$  be a decreasing sequence of closed subsets with empty intersection. We need to find open sets  $U_j \supseteq D_j$  such that  $\bigcap_{j \in \omega} \overline{U_j} = \emptyset$ .

Suppose that for all  $j$  there is an  $n_j$  such that  $D_j \cap H_{n_j}$  is uncountable. Then by  $\clubsuit^*$  and the definition of the  $T_x$ , there is a club set  $C_j$ , for all  $j$ , such that every  $x$  in  $C_j \cap H_\omega$  is a limit point of  $D_j$ . Since  $D_j$  is closed, we have in particular  $C_j \cap H_\omega$  is a subset of  $D_j$  for all  $j \in \omega$ . But  $\bigcap_{j \in \omega} C_j$  is a club set so  $\bigcap_{j \in \omega} D_j$  contains  $\bigcap_{j \in \omega} (H_\omega \cap C_j) = H_\omega \cap \bigcap_{j \in \omega} C_j$  which is non-empty. Hence there must be some  $j_0 \in \omega$  such that  $D_k \cap H_n$  is countable for all  $k \geq j_0$  and all  $n \geq 0$ . Let  $\alpha = \sup\{\beta \in D_k : k \geq j_0\}$ , then  $\{x \in Y : x = (\beta, H_n), 0 \leq n \leq \omega, \beta \leq \alpha\}$  is a clopen, regular, countable and hence metrizable subspace of  $Y$ , containing  $D_k$  for all  $k \geq J_0$ . Hence  $Y$  is countable paracompact.  $\square$

$Y$  does have some separation: Reasoning as for  $X$  we see that  $Y$  is  $\omega_1$ -compact and therefore strongly collectionwise Hausdorff.  $Y$  is  $\delta$ -normal since it is countably paracompact. (A space is  $\delta$ -normal if two closed sets can be separated whenever one is a regular  $G_\delta$ . Mack has shown that a countably paracompact space has countably paracompact product with the closed unit interval if and only if it is  $\delta$ -normal [M].) To see that  $Y$  is strongly zero-dimensional, notice that no two uncountable closed sets can be (functionally) separated so the argument used for  $X$  suffices.

Having constructed the space  $Y$ , we can go one step further, by building the space  $Z$ , a Dowker space containing an anti-Dowker space which in turn contains a Dowker space ...

**2.6 The space  $X'$ .** ( $\clubsuit^*$ ) *There is a first countable,  $\omega_1$ -compact, locally compact, strongly zero-dimensional, strongly collectionwise normal Dowker space in  $\mathcal{W}$  containing an anti-Dowker subspace which satisfies all of the properties that  $Y$  does.*

*Proof:* The point set for  $X'$  is, as always,  $\omega_1$ . Let  $D$  be the isolated points of  $\omega_1$  and partition  $X' - D$  into disjoint stationary sets  $H_r, K_r$ , for  $1 \leq r \leq \omega$ , and  $S_s$  for  $s < \omega$ . Again, let  $H_0 = K_0 = D$ . Write  $H$  for  $\bigcup\{H_r : 1 \leq r \leq \omega\}$ ,  $K$  for  $\bigcup\{K_r : 1 \leq r \leq \omega\}$ , and denote  $x$  in  $X'$  by the pair  $x = (\alpha, \ell)$

where  $\alpha$  is the actual element of  $X' = \omega_1$ , and  $\ell$  is the level ( $D$ ,  $H_r$ ,  $K_r$  or  $S_s$ ) containing  $x$ . For each limit ordinal  $\alpha$ , let  $R_\alpha$  be the cofinal sequence in  $\alpha$  given by  $\clubsuit^*$ . Let  $x = (\alpha, \ell_n)$  where  $\ell$  is either the letter  $H$  or  $K$  and  $0 < n \leq \omega$ , or the letter  $S$  and  $0 \leq n < \omega$ . If

- (1)  $n = m + 1$ , then let  $L = \ell_m$ ;
- (2)  $n = \omega$  (so  $\ell$  is either  $H$  or  $K$ ), then let  $L = \bigcup_{j \leq \omega} \ell_j$ ;
- (3)  $n = 0$  (so  $\ell$  is  $S$ ),  $\alpha$  is a limit of  $H_\omega \cup K_\omega$ , then let  $L = H_\omega \cup K_\omega$ .

In each case, if  $R_\alpha \cap L$  is cofinal in  $\alpha$ , then let  $T_\alpha = R_\alpha \cap L$ , otherwise let  $T_\alpha$  be undefined. Let  $T_x$  denote the sequence  $T_\alpha$ .

Topologize  $D \cup H \cup K \subseteq X'$  as for  $Y$  above. Let  $x = (\alpha, S_s)$ . If  $T_\alpha$  is not defined, then we declare  $x$  to be isolated. If  $s \geq 0$  and  $T_\alpha$  is defined, then let  $x$  be the rigid limit of  $T_x$  with respect to  $T_x$ . So level  $S_0$  provides the common limit points for  $H$  and  $K$  and above  $S_0$  the topology is similar to that of the space  $X$ . That  $X'$  is a Dowker space with all the required properties now follows by arguments similar to those used for  $X$  and  $Y$ .  $\square$

(If we let  $\Omega = \omega_1$  be the union of disjoint stationary sets  $\{S_\alpha, T_\alpha\}_{0 < \alpha < \omega_1}$ ,  $S_0 = T_0$  and  $\{S_\alpha\}_{\omega \leq \alpha \leq \omega_1}$ , and construct a topology in exactly the same way as above, except that a point  $\alpha$  in  $S_{\omega_1}$  is the limit of  $T_\alpha \cap (\bigcup_{\beta \leq \omega_1} S_\beta \cup \bigcup_{\beta < \omega_1} T_\beta)$ , then the resulting space is both normal and countably paracompact.)

**2.7 The space  $Z$ .** ( $\clubsuit^*$ ) *There is a first countable, locally countable, locally compact,  $\omega_1$ -compact, strongly collectionwise normal, strongly zero-dimensional, countably paracompact space  $Z$  whose Tychonoff square is a Dowker space. (In fact  $Z^2$  satisfies all of the listed properties that  $Z$  satisfies excepting, of course, that  $Z^2$  is not countably paracompact.)*

*Proof:* Our construction is similar to that used by Bešlagić in [Bs1]. We define three normal topologies  $\mathcal{T}_i$ ,  $i \in 3$ , on the point set  $W = \omega_1$ . The topologies  $\mathcal{T}_0$  and  $\mathcal{T}_1$  both refine  $\mathcal{T}_2$ , which is a Hausdorff topology, hence the diagonal  $\Delta$  of  $(W, \mathcal{T}_0) \times (W, \mathcal{T}_1)$  is a closed subspace of  $Z^2$ , where  $Z$  is the disjoint topological

sum of  $(W, \mathcal{T}_0)$  and  $(W, \mathcal{T}_1)$ .  $\clubsuit_{\omega_1 \times \omega_1}^*$  helps to ensure that the product  $Z^2$  is normal, and that  $\Delta$  is a Dowker space. Since  $\Delta$  is closed in  $Z^2$ ,  $Z^2$  is also a Dowker space. We use  $\clubsuit^{*2}$  to ensure that  $(W, \mathcal{T}_i)$ ,  $i \in 3$  is countably paracompact.

As for the space  $X$ , partition  $W$  into  $\omega$  disjoint stationary sets  $S_n$ ,  $n \in \omega$ . Let  $\{R_{\alpha,i} : \alpha \in \omega_1 \text{ and } \text{lim}(\alpha)\}$ ,  $i \in 2$ , be  $\clubsuit^{*2}$ -sequences, and let  $\{T_\alpha : \alpha \in \omega_1 \text{ and } \text{lim}(\alpha)\}$  be a  $\clubsuit_{\omega_1 \times \omega_1}^*$ -sequence. Let  $F'_\alpha = \text{dom } T_\alpha \cup \text{ran } T_\alpha$ . Suppose that  $\alpha \in S_n$ . If  $n = m + 1$  and  $F'_\alpha \cap S_n$  is cofinal in  $\alpha$ , then let  $F_\alpha = F'_\alpha \cap S_n$ , otherwise let  $F_\alpha$  be undefined. If  $n = 0$  and  $R_{\alpha,i} \cap \bigcup_{n>0} S_n$  is cofinal in  $\alpha$  for both  $i \in 2$ , then let  $G_{\alpha,i} = R_{\alpha,i} \cap \bigcup_{n>0} S_n$ , otherwise let  $G_{\alpha,0}$  and  $G_{\alpha,1}$  be undefined. If both are defined, then let  $G_{\alpha,2}$  be the sequence  $G_{\alpha,0} \cup G_{\alpha,1}$ , otherwise let  $G_{\alpha,2}$  be undefined.

Again we define the topologies  $\mathcal{T}_i$  by induction along  $\omega_1$ , defining at each  $\alpha$  three neighbourhood bases  $\mathcal{B}_{\alpha,i} = \{B_{\alpha,i}(k)\}_{k \in \omega}$ ,  $i \in 3$ . If  $\alpha$  is a successor (or 0), or  $\alpha \in S_0$  and  $G_{\alpha,2}$  is undefined, or  $\alpha \in S_{m+1}$  and  $F_\alpha$  is undefined, then let  $\alpha$  be isolated. If  $\alpha$  is in  $S_0$  and  $G_{\alpha,2}$  is defined, then, for  $i \in 3$ , let  $\alpha$  be the  $\mathcal{T}_i$ -rigid limit of  $G_{\alpha,i}$  with respect to  $G_{\alpha,2}$ , ensuring that  $B_{\alpha,0}(k) \cup B_{\alpha,1}(k)$  is a subset of  $B_{\alpha,2}(k)$  for each  $k \in \omega$ . Note that this means that, for all  $\alpha$  in  $S_0$ ,  $B_{\alpha,0}(0) \cap B_{\alpha,1}(0) = \{\alpha\}$ . If  $\alpha$  is in  $S_{m+1}$  and  $F_\alpha$  is defined, then let  $\alpha$  be the  $\mathcal{T}_i$  rigid limit of  $F_\alpha$  with respect to  $F_\alpha$ , again ensuring that  $B_{\alpha,0}(k) \cup B_{\alpha,1}(k)$  is a subset of  $B_{\alpha,2}(k)$ .

Clearly both  $\mathcal{T}_0$  and  $\mathcal{T}_1$  refine  $\mathcal{T}_2$ , and each  $(W, \mathcal{T}_i)$  is regular, first countable, locally countable, locally compact, zero-dimensional and locally metrizable.

Since  $\{F_\alpha : \alpha \in \omega_1 \text{ and } \text{lim}(\alpha)\}$  is  $\clubsuit^*$ -sequence we have

**Claim 2.8.** *For each  $i \in 3$ ,  $(W, \mathcal{T}_i)$  is  $\omega_1$ -compact and has no two disjoint uncountable closed subsets.*

Since  $\alpha + 1$  is a clopen subset of  $(W, \mathcal{T}_i)$  for all  $\alpha \in \omega_1$ , strong collectionwise normality, and strong zero-dimensionality all follow as for  $X$ .

**Claim 2.9.**  *$(W, \mathcal{T}_i)$  is countably paracompact for each  $i \in 3$ .*

*Proof:* Fix  $i \in 3$ . Let  $\{D_n\}_{n \in \omega}$  be a decreasing sequence of closed subsets of  $(W, \mathcal{T}_i)$  that has empty intersection. Suppose that each  $D_n$  is uncountable. By Claim 2.8,  $D_n \cap (\omega_1 - S_0)$  is uncountable for every  $n \in \omega$  and therefore  $C_n = \{\alpha \in \omega_1 : G_{\alpha,i} \cap D_n \cap (\omega_1 - S_0) \text{ is cofinal in } \alpha\}$ , and hence  $C = \bigcap_{n \in \omega} C_n$ , contains a club. But then  $C \cap S_0$  is non-empty, and every  $\alpha$  in  $C \cap S_0$  is a limit of every  $D_n$ . Since the  $D_n$  are all closed we have a contradiction. Therefore there is some  $n_0$  such that  $D_n$  is countable whenever  $n > n_0$ .  $W$  is now easily seen to be countably paracompact.

**Claim 2.10.** *For  $i, j \in 2$ ,  $(W, \mathcal{T}_i) \times (W, \mathcal{T}_j)$  is normal*

*Proof:* Let  $C$  and  $D$  be disjoint closed subsets of  $(W, \mathcal{T}_i) \times (W, \mathcal{T}_j)$ . If  $C$  is 2-unbounded, then there is some  $n$  for which  $C_n = C \cap (S_n \times S_n)$  is 2-unbounded. As  $\{\alpha : T_\alpha \cap C_n \text{ is cofinal in } \alpha \times \alpha\}$  contains a club,  $C_{n+1}$  is 2-unbounded. If both  $C$  and  $D$  are 2-unbounded, then there is some  $n > 0$  for which both  $C_n$  and  $D_n$  are 2-unbounded. But then by  $\clubsuit_{\omega_1 \times \omega_1}^*$ ,  $E = \{\alpha : \text{both } T_\alpha \cap C_n \text{ and } T_\alpha \cap D_n \text{ are cofinal in } \alpha\}$  contains a club. Let  $\alpha$  be an element of  $E \cap S_{n+1}$ . By the definition of the topologies  $\mathcal{T}_i$  and  $\mathcal{T}_j$  both  $C_n \cap (F_\alpha \times F_\alpha)$  and  $D_n \cap (F_\alpha \times F_\alpha)$  are cofinal in  $\alpha$ , and  $\alpha$  is in  $C \cap D$ . Hence at least one of  $C$  and  $D$  is not 2-unbounded.

Let us suppose that  $C$  is a subset of  $A = (\alpha \times \omega_1) \cup (\omega_1 \times \alpha)$  and that  $\alpha$  is a successor so that  $A$  is clopen in  $(W, \mathcal{T}_i) \times (W, \mathcal{T}_j)$ . Lemma 2.8 of [Bs1] tells us that, if  $X$  is a normal, countably paracompact space and  $M$  is a countable metric space, then  $X \times M$  is normal. It is easy to see, then, that  $A$  is normal. Since  $A$  is clopen,  $(W, \mathcal{T}_i) \times (W, \mathcal{T}_j)$  is now, itself, seen to be normal—proving the claim.

Notice that for each  $\alpha$  in  $S_0$ ,  $(\alpha, \alpha)$  is isolated as a point of  $\Delta$ , so that  $\Delta$  is homeomorphic to a copy of the space  $X$  of 2.1, built using the  $\clubsuit^*$ -sequence  $\{F_\alpha : \alpha \in \omega_1 \cap \text{lim}\}$ , and is closed in  $(W, \mathcal{T}_0) \times (W, \mathcal{T}_1)$  since  $\mathcal{T}_0$  and  $\mathcal{T}_1$  both refine the Hausdorff topology  $\mathcal{T}_2$ . Hence the subspace  $\Delta = \{(\alpha, \alpha) : \alpha \in \omega_1\}$  of

$(W, \mathcal{T}_0) \times (W, \mathcal{T}_1)$  is closed and not countably metacompact and we are done.  $\square$

### 3. MANIFOLDS

Let us embed the spaces  $X, Y$  and  $Z$  into manifolds:

Let  $L^*$  be the set  $\omega_1 \times [0, 1)$  with the topology induced by the lexicographic order and let  $L$  be the subspace  $L^* - \{0, 0\}$  (the long line). Let  $M^*$  be the manifold  $L^* \times (0, 1)$  and  $M = L \times (0, 1)$ . Let  $I_\alpha$  be a copy of  $[0, 1)$  for each  $\alpha \in \omega_1$ . Let  $P^* = M^* \cup \bigcup_{0 < \alpha} I_\alpha$  and  $P = M \cup \bigcup_{0 < \alpha} I_\alpha$ .

We shall refer to elements of  $M$  as pairs  $(l, r)$  where  $l$  is in  $L$  and  $r$  is in  $(0, 1)$ . Let  $M_\alpha$  be the set  $\{(l, r) \in M : l < \alpha\}$  and  $M(\gamma, \varepsilon) = \{(l, r) \in M : \gamma < l \in L \text{ and } r \in (0, 1 - \varepsilon)\}$ . We shall say that a set  $A$  is *bounded in  $M$*  if  $A$  is a subset of some  $M_\alpha$ . Both  $M$  and hence  $M(\gamma, \varepsilon)$  are collectionwise normal, and  $M_\alpha$  is metrizable, indeed homeomorphic to  $R = (-1, 1) \times (0, 1)$ .

We will refer to a point of  $I_\alpha$  as  $x_\alpha$  where  $x$  is the corresponding point of  $[0, 1)$ . Let  $O_\alpha = M_\alpha \cup \bigcup_{\beta < \alpha} I_\beta$ ,  $P_\alpha = M_{\alpha+1} \cup \bigcup_{\beta \leq \alpha} I_\beta$  and  $Q_\alpha = P_\alpha - I_\alpha$ . A subset of  $P$  is said to be *bounded in  $P$*  if it is a subset of some  $P_\alpha$ . If  $x$  is in  $P$ , we shall let  $\alpha, s \preccurlyeq x$  mean that either  $x$  is an element of  $\bigcup_{\alpha \leq \beta} I_\beta$ , or  $x = (l, r)$  is an element of  $M \subseteq P$  and  $\alpha \leq l$  and  $s \leq r$ , we also let  $\alpha, s \succcurlyeq x$  mean that  $\alpha, s \not\preccurlyeq x$ . We shall say that a set  $A$  is *2-unbounded in  $M^2$*  if for no  $\alpha \in \omega_1$  is  $A$  a subset of  $(M_\alpha \times M) \cup (M \times M_\alpha)$ .

The following fact is essentially Lemma 3.4 of [N2].

**Fact 3.1.** *Every closed non-metrizable subspace of  $M$  contains a closed copy of  $\omega_1$ . For every copy  $K$  of either  $\omega_1$  or  $L$  in  $M$ , there is an  $\alpha \in \omega_1$  and an  $r$  in  $(0, 1)$  such that  $K - M_\alpha$  is a subset of  $L \times \{r\}$ .  $\square$*

Fact 3.2 has a similar proof, bearing in mind the comments of 3.5 [N2].

**Fact 3.2.** *If  $A$  is a closed 2-unbounded subset of  $M^2$ , then there are  $\alpha \in \omega_1$ , and  $r, s$  in  $(0, 1)$  such that  $\{((\gamma, r), (\gamma, s)) : \alpha \leq \gamma \in L\}$  is a subset of  $A$ .  $\square$*

To define a locally Euclidean topology on  $P$  we use the Prüfer construction, illustrated in Figure 1. In the diagram we use broken lines to enclose open sets and solid lines to enclose closed sets.

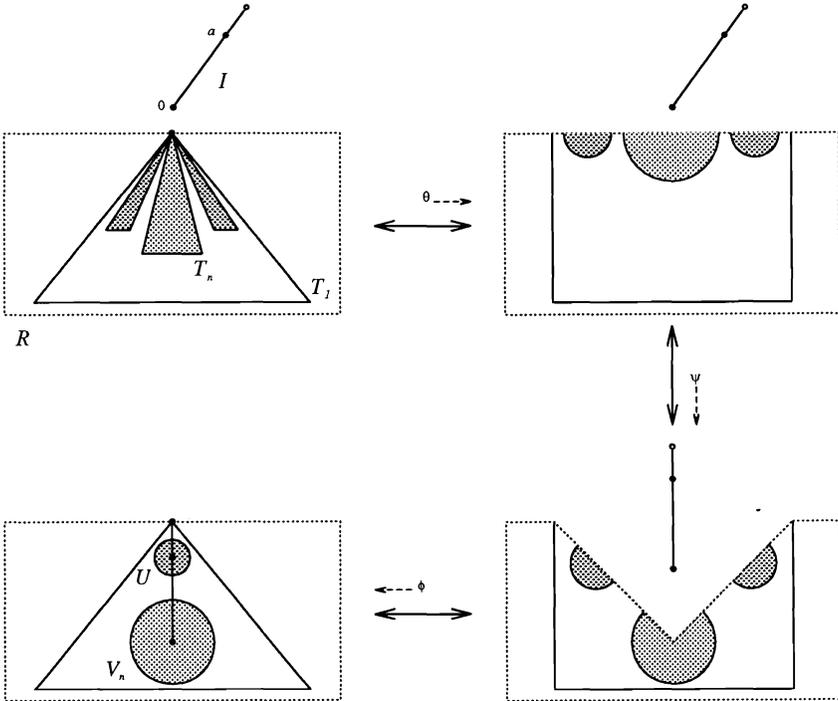


Figure 1

$R$  is a copy of the Euclidean space  $(-1, 1) \times (0, 1)$ ,  $I$  is a copy of  $[0, 1)$ .  $R$  is given the normal Euclidean topology, and the topology about points of  $I$  is chosen so that each of the maps  $\theta$ ,  $\psi$  and  $\phi$  are homeomorphisms onto another copy of the Euclidean space  $(-1, 1) \times (0, 1)$ . The collection  $\{T_n\}_{n \in \omega}$  is a decreasing, nested sequence of closed triangles in  $R$ , each with a

vertex at the point  $(0, 1)$ . We can define the homeomorphisms  $\phi$ ,  $\psi$  and  $\theta$  in such a way that we may take the  $n^{\text{th}}$  basic compact neighbourhood about the point  $0$  in  $I$  to be the set  $T_n \cup [0, 1/n]$ . (For details see [N2].)

Again, the three constructions are very similar.  $M$  is given its usual topology, and we use the Prüfer technique inductively to define the topology at points of  $I_\alpha$ : Let  $\mathcal{T}$  be a 0-dimensional, locally countable, locally compact topology on  $\omega_1$ , which refines the usual order topology. Suppose we have defined the topology on  $O_\alpha$  in such a way that  $O_\alpha$  is homeomorphic to  $(0, 1)^2$ , and so that the subspace  $\{0_\beta : \beta < \alpha\}$  is homeomorphic to the subspace  $\alpha$  of  $(\omega_1, \mathcal{T})$ . Clearly, there is a homeomorphism  $\mu_\alpha$  from  $Q_\alpha$  to  $R$  such that  $\mu_\alpha " O_\alpha$  is  $(-1, 0) \times (0, 1)$  and  $\mu_\alpha " (\{\alpha\} \times (0, 1))$  is  $\{0\} \times (0, 1)$ . Let  $\chi_\alpha : P_\alpha \rightarrow R \cup I$  be such that  $\chi_\alpha \upharpoonright Q_\alpha = \mu_\alpha$  and  $\chi_\alpha \upharpoonright I_\alpha : I_\alpha \rightarrow I$  is the identity. Let  $P_\alpha$  have the topology  $\mathcal{T}_\alpha$  defined so that  $\chi_\alpha$  is a homeomorphism when  $R \cup I$  is given the Prüfer topology.

For  $\alpha \in \omega_1$ , we shall say that  $I_\alpha$  is *inserted into  $M$  with respect to  $\mathcal{T}$*  if the topology of  $P_\alpha$  is defined as above with the map  $\mu_\alpha$  satisfying the following conditions:

- (1) for  $\beta \in \alpha$ ,  $\mu_\alpha " I_\beta$  meets  $T_j$  iff  $\beta$  is in the  $j^{\text{th}}$  neighbourhood  $B_\alpha(j)$  of  $\alpha$ ;
- (2) if  $\alpha$  is the rigid limit of the sequence  $\{\alpha_j\}_{j \in \omega}$ , then  $\mu_\alpha "[0, 1 - 1/j]_{\alpha_j}$  is a subset of  $T_j$ ;
- (3)  $T_j$  is a subset of  $\mu_\alpha " \{(l, r) : l < \alpha + 1 \in L \text{ and } 1/j \leq r\}$ .

This is possible since  $(\omega_1, \mathcal{T})$  is a 0-dimensional, locally countable, locally compact topology, the subspace  $\alpha$  is homeomorphic to a subset of  $\mathbb{Q}$ , and  $Q_\alpha$  is homeomorphic to  $(-1, 1) \times (0, 1)$ .

Since we have only redefined the topology at points of  $I_\alpha$ , the induction continues along  $\omega_1$ . Let  $P$  have the topology  $\mathcal{T}'$  generated by  $\bigcup_{\alpha \in \omega_1} \mathcal{T}_\alpha$ . Since an increasing  $\omega$ -sequence of spaces, each homeomorphic to  $\mathbb{R}^2$ , is again homeomorphic to  $\mathbb{R}^2$  (see [N2 p652]),  $(P, \mathcal{T}')$  is a manifold. The subspace  $\{0_\alpha : \alpha \in \omega_1 = X\}$  is homeomorphic to the space  $(\omega_1, \mathcal{T})$  and is a closed subset of  $(P, \mathcal{T}')$ .

**3.3 Example.** ( $\clubsuit^*$ ) *There is a Dowker manifold.*

*Proof:* Let  $(X, \mathcal{T})$  be the Dowker space constructed in Example 2.1. Give  $P$  the topology  $\mathcal{T}'$  generated as described above when the  $I_\alpha$  are inserted inductively along  $\omega_1$  with respect to  $\mathcal{T}$ .  $P$  is not countably metacompact since it contains  $X$  as a closed subspace. To see that  $P$  is normal we use the following claim.

**Claim 3.4.** *If  $C$  is an uncountable subset of  $X$  which meets uncountably many  $I_\beta$ , for  $\beta$  in some  $S_n$ , then  $C$  has uncountably many limit points  $0_\alpha$ , for  $\alpha$  in  $S_{n+1}$ . If  $C$  and  $D$  are closed sets such that both  $C$  and  $D$  meet uncountably many of the  $I_\alpha$ , then  $C$  and  $D$  are not disjoint.*

*Proof of Claim:* For some integers  $k$  and  $n$  in  $\omega$ , there is an uncountable subset  $A$  of  $S_k$ , the  $k^{\text{th}}$  level of  $X$ , such that  $C \cap [0, 1 - 1/n]_\beta$  is non-empty for each  $\beta$  in  $A$ .  $\clubsuit^*$  implies that for all but a non-stationary subset of  $S_{k+1}$ ,  $A \cap T_\alpha$  is cofinal in  $\alpha$  (where  $T_\alpha$  is as in 2.1). Hence, by 2) above, for all but a non-stationary subset of  $S_{k+1}$ ,  $0_\alpha$  is in  $C$ . With  $C$  and  $D$  as in the statement of the claim, it quickly follows that  $C$  and  $D$  are not disjoint. This proves the claim.

The following is immediate by Fact 3.1 and (3) above.

**Fact 3.5.** *Let  $C$  be an unbounded subset of  $M$  in  $P$ . If  $C$  is closed and is disjoint from  $\bigcup_{\beta < \alpha} I_\alpha$ , then there is some  $\varepsilon > 0$  and some  $\gamma$  in  $\omega_1$  such that  $C$  is a subset of  $M(\gamma, \varepsilon)$   $\square$*

Now let us prove that  $P$  is normal:

Let  $C$  and  $D$  be any two disjoint closed subsets of  $P$ . Either at least one of  $C$  or  $D$  is bounded in  $P$ , or both are unbounded. Suppose that  $C$  is bounded and is a subset of  $P_\alpha$ .  $P_{\alpha+1}$  is homeomorphic to  $R$  and is therefore normal, so there are disjoint open sets  $U$  and  $V$  in  $P_{\alpha+1}$  such that  $C$  is contained in  $U$ ,  $U$  is a subset of  $P_\alpha$ , and  $D \cap P_{\alpha+1}$  is contained in  $V$ . Then  $U$  and  $V \cup (P - \overline{P_\alpha})$  are disjoint open sets containing  $C$  and  $D$  respectively.

Now suppose that both  $C$  and  $D$  are unbounded in  $P$ . By Claim 3.4, at least one of  $C$  and  $D$  meets only countably many  $I_\alpha$ , so we may assume that  $C$  does not meet any  $I_\beta$  for  $\alpha < \beta$ . By the metrizability of  $P_{\alpha+2}$  there are disjoint open sets  $U$  and  $V$ , such that  $U$  contains  $C \cap P_{\alpha+1}$  and  $\bar{U}$  is disjoint from  $D$ , and  $V$  contains  $D \cap P_{\alpha+1}$  and  $\bar{V}$  is disjoint from  $C$ . If  $D$  meets only countably many  $I_\beta$ , say  $I_\beta \cap D$  is empty for all  $\beta < \gamma$ , then without loss of generality  $\gamma = \alpha$  and, since  $M$  is normal, there are disjoint open  $U'$  and  $V'$  containing  $C - P_\alpha$  and  $D - P_\alpha$  respectively. If  $D$  meets uncountably many of the  $I_\beta$ , then Fact 3.5 above implies that there is some  $\varepsilon$  such that  $C$  is a subset of  $M(\gamma, \varepsilon)$ . Again we may assume that  $\gamma = \alpha$ .  $M(0, \varepsilon)$  is an open, normal subspace of  $P$ , so there are open sets  $U'$  and  $W$  containing  $C - P_\alpha$  and  $(D - P_\alpha) \cap M(\alpha, \varepsilon)$ . Let  $V' = P \cup \{p \in P_\alpha : p \notin M(0, \varepsilon)\}$ . In either case  $(U \cup U') - \bar{V}$  and  $(V \cup V') - \bar{U}$  are disjoint open sets containing  $C$  and  $D$  respectively.  $\square$

**3.6 Example.** ( $\clubsuit^*$ ) *There is an anti-Dowker manifold which contains a Dowker manifold as a subspace.*

*Proof:* Let  $(Y, \mathcal{T})$  be the anti-Dowker space constructed in Section 2. Give  $P$  the topology  $\mathcal{T}'$  generated as described above when the  $I_\alpha$  are inserted inductively along  $\omega_1$  with respect to  $\mathcal{T}$ . Let  $Q$  denote this manifold. Since  $Y$  is a closed subspace of  $Q$ ,  $Q$  is not normal.

Let  $\{D_n\}$  be a decreasing sequence of closed sets with empty intersection. By Claim 3.4, Fact 3.1 and the argument from 2.5 (with the appropriate notational alterations) it is easy to see that, for some  $n$ ,  $\{\alpha : D_n \cap I_\alpha \neq \emptyset\}$  is bounded in  $\omega_1$ . Hence, there is an  $n \in \omega$  and  $\alpha \in \omega_1$  such that  $D_n$  is a subset of  $M' = M \cup \bigcup_{\beta < \alpha} I_\beta$ . Since  $M'$  is homeomorphic to  $M$ , which is countably paracompact, we see that  $Q$  is countably paracompact.  $\square$

Example 3.8 describes a manifold  $\Pi$  with Dowker square. To simplify the proof that  $\Pi^2$  is normal, we replace  $X \times \omega_1$  by

$X \times L$  in Theorem 3.3 of [GNP] to get:

**3.7 Theorem.** *Let  $X$  be a normal, countably paracompact,  $\omega_1$ -compact space of countable tightness.  $X \times L$  is normal.*

The proof follows (modulo appropriate modifications) the proof of Theorem 3.3 [GNP] with the notions of  $\omega_1$ -continuous and  $\omega_1$ -continuous closure, introduced there, replaced by: A collection of subsets  $\mathcal{H} = \{H_a : a \in A\}$  of a space  $X$ , indexed by a subset  $A$  of  $L$ , is said to be  *$L$ -continuous* if  $x$  is an element of  $H_a$  whenever  $x \in \overline{\bigcup\{H_b : b \in (y, l + \varepsilon) \cap A\}}$  for all  $\varepsilon > 0$  and  $y$  in  $L$ . For a collection of subsets  $\mathcal{Z} = \{Z_a : a \in L\}$  of  $X$ , if  $Z'_a$  is the set  $\bigcap\{\overline{\bigcup\{Z_a : a \in (y, l + \varepsilon)\}} : y < l, \varepsilon > 0\}$  for all  $a \in L$ , then the collection  $\{Z'_a : a \in L\}$  is said to be the  *$L$ -continuous closure* of  $\mathcal{Z}$ .

**3.8 Example.** ( $\clubsuit^*$ ) *There is a countably paracompact manifold  $\Pi$  which has Dowker square.*

*Proof:* For  $i \in 3$  let  $(W, \mathcal{T}_i)$  be the spaces constructed in 2.7. Let  $P_i^*$  denote the set  $P^*$  endowed with the topology  $\mathcal{T}'$  generated as described above when the  $I_\alpha$  are inserted inductively along  $\omega_1$  with respect to  $\mathcal{T}_i$ . Let  $\Pi^*$  be the disjoint sum of  $P_0^*$  and  $P_1^*$ . Let  $\Pi$  be the manifold formed by identifying (point-wise) the subset  $\{0, 0\} \times (0, 1)$  of the subset  $M$  of  $P_0$  with the corresponding subset of the  $M$  of  $P_1$ . The quotient map  $\rho : \Pi^* \rightarrow \Pi$  is a closed map and induces a closed map from  $\Pi^{*2}$  to  $\Pi^2$ .

Since  $Z$  is a closed subspace of  $\Pi^*$ ,  $\Pi^2$  is not countably paracompact. Since normality is preserved by closed maps, it is enough now to show that each  $P_i$  is countably paracompact and that, for  $i, j \in 2$ ,  $P_i \times P_j$  is normal.

For convenience, let us denote by  $P_{\alpha, i}$  the subspace  $P_\alpha$  of  $P_i$ .

**Claim 3.9.** *Each  $P_i$  is countably paracompact.*

*Proof of Claim:* Let  $\{D_n\}$  be a decreasing sequence of closed subsets of  $P_i$  with empty intersection. By Fact 3.5 and by Claim 2.9, there are  $n_0 \in \omega$ ,  $\alpha \in \omega_1$  for which  $D_n$  is a subset

of  $M' = M \cup \bigcup_{\beta < \alpha} I_\beta$  for all  $n \geq n_0$ . Since  $M'$  is an open, countably paracompact subspace of  $P_i$ , we see that  $P_i$  is also countably paracompact.

**Claim 3.10.** *For any  $i, j \in 3$ ,  $P_i \times P_j$  is normal.*

*Proof of Claim:* Let  $C$  and  $D$  be disjoint closed subsets of  $P_i \times P_j$ . By Fact 3.2 there are  $\alpha, \beta \in \omega_1$  and  $\varepsilon, \delta > 0$  such that at most one of  $C$  and  $D$  meets

$$T = \{(x, y) \in P_i \times P_j : \alpha, (1 - \varepsilon) \preceq x \text{ and } \beta, (1 - \delta) \preceq y\}.$$

Since  $T$  is a closed subset of  $P_i \times P_j$ , it is enough to show that the complement of  $T$  is normal. The complement of  $T$  is the closed image of the disjoint union of the spaces

$$\begin{aligned} A &= \{x \in P_i : \alpha, (1 - \varepsilon) \succ x\} \times P_j && \text{and} \\ B &= P_i \times \{y \in P_j : \beta, (1 - \delta) \succ y\}. \end{aligned}$$

It is enough to show that each of  $A$  and  $B$  is normal.

Let  $k \in 3$ . Since the subspace  $M' = M \cup \bigcup_{\beta < \alpha} I_\beta$  of  $P_k$  is homeomorphic to  $M$ ,  $A$  is homeomorphic to  $M \times P_j$  and  $B$  to  $P_i \times M$ . By 4.13 of [Pr],  $P_k \times (0, 1)$  is normal (and hence countably paracompact by the result of Rudin and Starbird mentioned in the introduction). By the proof of Claim 2.8,  $P_k$  is  $\omega_1$ -compact, so  $P_k \times (0, 1)$  is  $\omega_1$ -compact.  $P_k \times (0, 1)$  is a manifold so has countable tightness. Now  $A$  and  $B$  are homeomorphic to  $P_k \times (0, 1) \times L$  ( $k = i, j$ ) so it follows from from Theorem 3.7 that they are both normal.  $\square$

## QUESTIONS

In [Bs3] Bešlagić constructs a perfectly normal space with Dowker square assuming  $\diamond$ . Rudin (see [N2]) has shown that perfectly normal manifolds are metrizable assuming  $\text{MA} + \neg\text{CH}$ , and from 4.14 [N2] and 3.22 [Pr] we have the proposition:  $(\text{MA} + \neg\text{CH})$  If  $X$  is a locally compact, collectionwise Hausdorff, perfectly regular space, then  $X^2$  is paracompact. Is there a perfectly normal manifold which has Dowker square? If  $X$  is a normal, countably paracompact space and  $X^2$  is normal,

must  $X^2$  be countably paracompact assuming  $MA + \neg CH$ ? What if  $X$  is also perfect? Does  $MA + \neg CH$  imply the existence of a Dowker manifold (or even a locally compact Dowker space)? The results of [Bg1] put severe restrictions on such spaces. Does  $\clubsuit$  imply the existence of a Dowker manifold? Is there an hereditarily normal Dowker manifold? Monotonically normal spaces are countably paracompact (see [Ru2]). The Sorgenfrey line is a  $GO$ -space and is therefore monotonically normal, but its square is not normal. (It is also Lindelöf.) Is there a monotonically normal (or Lindelöf) space which has Dowker square? Is there a Dowker space  $X$  such that  $X^2$  (or  $X^n$  for every  $n \in \omega$ ) is Dowker? (Such a space cannot contain a copy of a convergent sequence.)

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