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SEQUENTIALITY OF PRODUCTS OF SPACES WITH POINT-COUNTABLE k-NETWORKS

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ABSTRACT. We use the Continuum Hypothesis to prove a theorem on sequentiality of products of Fréchet spaces with point-countable k-networks. Examples are constructed to show that under **CH** or **MA+¬CH** our theorem cannot be extended to the wider class of sequential spaces with point-countable k-networks. Example 3.2 shows that the gap in the paper [H] pointed out in [G1] cannot be bridged. We also prove that the main result of [H] is valid for spaces of the sequential order less than or equal to 2 only.

1. INTRODUCTION

Recall that a family γ is a k-network for X if whenever $K \subseteq U \subseteq X$, K is compact and U is open there exists a finite subfamily $\gamma_K \subseteq \gamma$ such that $K \subseteq \cup\gamma_K \subseteq U$. If γ satisfies $|\{\, \xi \in \gamma \mid x \in \xi \,\}| \leq \aleph_0$ for every $x \in X$ it is called point-countable. That point-countable k-networks are of special interest is demonstrated in several papers (see [GMT], [T2], [Fo] and bibliography there). A space X is *sequential* [Fr] if whenever $A \subset X$ and A is not closed, there is a sequence from A converging to a point outside the set A. Sequentiality of products of spaces with point-countable k-networks and spaces closely related to them was studied in [G2], [H], [T1], [T3]. The paper [H] is an attempt to obtain a full characterization for the product of two k-spaces with closed point-countable k-networks to be sequential (recall that a k-network is closed if it consists of closed sets). The following theorem was suggested in [H] assuming **CH** and the existstence of an uncountable measurable cardinal:

(Wrong) Statement 1.1 (CH+MC) *Let X and Y be regular k-spaces with point-countable closed k-networks. Then $X \times Y$ is sequential if and only if one of the three properties below holds:*
(a) X and Y have point-countable bases.
(b) X or Y is locally compact.
(c) X and Y are locally k_ω-spaces.

The statement was then applied to prove similar facts about closed images of metric spaces, quotient s-images of locally compact metric spaces etc. Unfortunately in the final part of the proof for the "only if" implication a wrong assumption was used as it was pointed out in [G1] so the statement may be considered only as a hypothesis. In section 3 we construct assuming **CH** a counterexample to Statement 1.1. The example has a σ-disjoint closed k-network. By Theorem 2.11 it cannot have k-network which is σ-locally finite. Although Statement 1.1 fails to be true in full, the use of **CH** permits us to obtain some of its particular versions. Thus in [H] it was shown using **CH** that if X and Y are as in Statement 1.1 and $X \times Y$ is sequential then either both X and Y are locally σ-compact, or (a) or (b) of Statement 1.1 holds (see [H, lemmas 4–5]). Example 3.1 constructed under **MA+¬CH** shows that **CH** is essential even for such a weak version of Statement 1.1. Proposition 2.6 shows that under **CH** closedness of k-networks in this case may be omitted. Theorems 2.1 and 2.5 strengthen Statement 1.1 in case of Fréchet X and Y. Theorem 2.10 shows that Statement 1.1 is true for spaces of sequential order less than or equal to 2. It is easily seen from the construction of Example 3.2 that the example has sequential order 3 so 2 is the exact boundary for sequential order under which Statement 1.1 remains true. Also it is easy to obtain from Example 3.2 an example of a sequential space of sequential order 2 with (non closed) point-countable k-network such that it is not locally a k_ω-space and its product with the sequential fan is sequential; so one cannot omit closedness of k-network in Theorem 2.10 and the boundary for the sequential order is then given by Theorem 2.1. The question whether Statement 1.1 is false in ZFC naturally raised by the pair of Examples 3.1–3.2 remains open.

All spaces are assumed to be Hausdorff. A space X is *Fréchet* (or *Fréchet-Urysohn*) if whenever $x \in \overline{A}$, there is a sequence from A converging to x. A point $x \in X$ is *a strongly Fréchet point* [M] if whenever $x \in \overline{A_n}$ for every $n \in \omega$, then there is a point $x_n \in \bigcap\{A_k : k \leq n\}$ for every $n \in \omega$ such that the sequence $\{x_n : n \in \omega\}$ converges to x. If A is a subset of a space X, then $[A]^{seq}$ denotes the sequential closure of A, i.e. the set of limits of convergent sequences consisting of points of A. Obviously $A \subseteq [A]^{seq}$. We define $[A]_\alpha$ by induction on $\alpha \in \omega_1 + 1$ as follows: $[A]_0 = A$, $[A]_{\alpha+1} = [[A]_\alpha]^{seq}$ and $[A]_\alpha = \bigcup\{\, [A]_\beta \mid \beta < \alpha \,\}$ for a limit α. One can easily see that $[A]_{\omega_1+1} = [A]_{\omega_1}$, and that a space X is sequential if and only if $\overline{A} = [A]_{\omega_1}$ for every $A \subset X$. Now let us introduce the sequential order of X and denote $so(X) = \min\{\, \alpha \mid [A]_\alpha = \overline{A} \textit{ for every } A \subseteq X \,\}$. For an infinite cardinal number σ we denote by S_σ the space obtained by identifying all the nonisolated points of a disjoint sum of σ convergent sequences to a single point. S_2 and S_3 denote the standard sequential spaces of sequential order 2 and 3 respectively; the first of them is also called Arens' space (see [R], [NT]). The problem with Statement 1.1 occurs when one tries to prove the implication:

(I) $\qquad X \times S_\omega$ *is sequential* $\Rightarrow X$ *is locally a* κ_ω-space

for X having a point-countable closed k-network. On the other hand implication (I) is necessary for Statement 1.1 to be true. Example 3.2 in section 3 shows that (I) is not valid under **CH**. We will need the following easy lemma which is an immediate corollary of [E, Theorem 3.7.14 and Theorem 3.7.25]:

Lemma 1.2. *Let $f : X \to Y$ be a perfect map. Then $X \times Z$*

is a k-space if and only if $Y \times Z$ is a k-space.

The next lemma is folklore.

Lemma 1.3. *Let $f : X \to X'$ be a mapping shrinking a single compact $K \subseteq X$ to a point and let X be a Fréchet space. If X' contains a (closed) copy of S_σ then X also contains a (closed) copy of S_σ.*

2. Products of k-spaces with point-countable k-networks

Using results of [T3] it may be (in ZFC) shown that Statement 1.1 is valid for Fréchet X and Y. Under **CH** we show that the assumption of closedness of the k-network may be omitted. Since every closed image of a metric space is a Fréchet space with a point-countable k-network ([Fo]) the result may be applied to Lašnev spaces too.

Theorem 2.1 (CH). *Let X and Y be regular Fréchet spaces with point-countable k-networks (in particular Lašnev spaces). Then $X \times Y$ is sequential if and only if one of the three properties below holds:*

(a) *X and Y have point-countable bases (metrizable for Lašnev X and Y).*
(b) *X or Y is a locally compact metrizable space.*
(c) *X and Y are topological sums of $\aleph_0$-k_ω-spaces.*

To prove this we need some lemmas. The following one was proved in [NS]

Lemma 2.2. *Let X be a Fréchet space with a point-countable k-network. If X is not first-countable then it contains a closed subspace homeomorphic to S_ω.*

Before mentioning the next lemma let us recall the definition of the space T as it is described in [G2]. The space is the union $T = \bigcup_{i\in\mathbb{N}} D_i \cup \{x\}$ where every D_i is an infinite countable set. All points of $T \setminus \{x\}$ are isolated. The base of open neighborhoods for the point x consists of sets of the form: $T \setminus$ *finitely many* D_i*'s.*

Lemma 2.3. *Let X be a regular k-space with a point-countable k-network. If X has no point-countable k-network consisting of sets with compact closures then X contains a closed copy of T.*

Proof: Let γ be a point-countable k-network for X. Assume without loss of generality that γ is closed under finite intersections. Put $\gamma' = \{\xi \in \gamma \mid \overline{\xi} \textit{ is compact}\}$. Suppose γ' is not a k-network for X. Then (see [T2, Proposition 1.2(1)]) there exists a sequence $S = \{x_n \mid n \in \omega\}$ such that $x_n \to x$ as $n \to \infty$ and an open set $U \ni x$ so that there is no finite $\gamma_{U,S} \subseteq \gamma'$ such that $\cup\gamma_{U,S} \subseteq U$ and $S \setminus \cup\gamma_{U,S}$ is finite. Let now $\Gamma = \{\gamma_i \mid i \in \omega\}$ be all finite subfamilies of γ such that for each $\xi \in \gamma_i$ holds $\xi \cap S \neq \emptyset$ and $\cup\gamma_i \subseteq U$, $S \setminus \cup\gamma_i$ is finite. It is easy to see that $\overline{\cup\gamma_i}$ is not compact for any $i \in \omega$. Put $\xi_i = \bigcap_{k \leq i}(\cup\gamma_k)$. Since γ is closed under finite intersections, then $\xi_i = \cup\gamma_{j_i}$ for some $\gamma_{j_i} \in \Gamma$. So no $\overline{\xi_i}$ is compact. Using regularity of X and the fact that γ is a k-network one can see that for any open $V \ni x$ there is $n \in \omega$ such that $\overline{\xi_i} \subseteq V$ for $i \geq n$. Now by [GMT, Theorem 4.1] $\overline{\xi_i}$ is not countably compact so for every $i \in \omega$ there is a countable closed discrete subset $D_i \subseteq \overline{\xi_i}$. Without loss of generality one can assume that $D_i \cap D_j = \emptyset$ if $i \neq j$. It is easy to see now that $\bigcup_{i \in \omega} D_i \cup \{x\}$ is homeomorphic to T. $\square$

Lemma 2.4. *Let X be a regular k-space having point-countable k-network γ consisting of sets with compact closures. If X is not locally σ-compact then its image Y under some perfect map contains a closed copy of S_{ω_1}.*

Proof: Consider the following property:

(P) For any compact $K \subseteq X$ there is a countable family $\delta_K = \{\xi_i \mid i \in \omega\} \subseteq \gamma$ such that for any sequence $S = \{x_i \mid i \in \omega\} \subseteq X$ such that $x_i \to x \in K$ as $i \to \infty$ we have $S \cap \bigcup_{i \in \omega} \xi_i \neq \emptyset$.

Let us prove that if (P) does not hold for X then the perfect map $f : X \to Y$ shrinking a compact K violating (P) to a point satisfies the conclusion of the lemma. Let $T_0 = \{ x_i^0 \mid i \in \omega \}$ be an arbitrary nontrivial sequence converging to $x_0 \in K$ and $\gamma_0 = \{ \xi \in \gamma \mid \xi \cap T_0 \neq \emptyset \}$. Suppose we have constructed $T_\alpha = \{ x_i^\alpha \mid i \in \omega \}$ and $\gamma_\beta = \{ \xi \in \gamma \mid \xi \cap T_\alpha \neq \emptyset \text{ for } \alpha \leq \beta \}$ so that

1. $x_i^\alpha \to x_\alpha \in K$ as $i \to \infty$ and
2. $T_\alpha \cap (\cup \gamma_\beta) = \emptyset$ for $\beta < \alpha$

for all $\alpha < \sigma < \omega_1$. Put $\gamma'_\sigma = \{ \xi \in \gamma \mid \xi \cap T_\alpha \neq \emptyset \text{ for } \alpha < \sigma \}$. Obviously $\gamma'_\sigma = \bigcup_{\alpha<\sigma} \gamma_\alpha$. Since γ'_σ is countable due to point countability of γ and K violates (P) one can choose a nontrivial sequence $T_\sigma = \{ x_i^\sigma \mid i \in \omega \}$ such that $x_i^\sigma \to x_\sigma \in K$ as $i \to \infty$ and $T_\sigma \cap \cup\{ \xi \mid \xi \in \gamma'_\sigma \} = \emptyset$. Now one can see that properties 1 and 2 take place for all $\alpha \leq \sigma$. So by induction we have constructed T_α for all $\alpha < \omega_1$ so that 1 and 2 take place. Suppose not that there exists an increasing sequence of ordinals $\{ \alpha_n \mid \alpha_{n+1} > \alpha_n \}$ and a set of points $R = \{ x_{i_n}^{\alpha_n} \mid i_n \in \omega \} \subseteq S = \{ x_i^\alpha \mid i \in \omega, \alpha < \omega_1 \}$ such that $x_{i_n}^{\alpha_n} \to x' \in X$. Since γ is a k-network there is $\xi \in \gamma$ such that $\xi \cap R$ is infinite. But this contradicts condition 2 because the condition and the construction of γ_β imply that each $\xi \in \gamma$ intersects no more than one T_α. Thus for any increasing sequence of ordinals $\{ \alpha_n \mid \alpha_{n+1} > \alpha_n \}$ any set of the form $R = \{ x_{\alpha_n}^{i_n} \mid i_n \in \omega \} \subseteq S$ is a closed discrete subset of X. Then obviously conditions 1–2 and the construction of γ_β imply that $T_\alpha \cap T_\beta = \emptyset$ if $\alpha \neq \beta$. This and the previous fact and nontriviality of T_α imply that $f(S)$ is a closed subset of Y homeomorphic to S_{ω_1}.

Now let us show that (P) implies that every point $x \in X$ has a σ-compact neighborhood. For a compact $K \subseteq X$ let $\delta'_K = \{ \overline{\xi} \mid \xi \in \delta_K \}$ where δ_K is the family mentioned in (P). Of course every δ'_K is countable and consists of compact sets. Put $\delta_0 = \{\{x\}\}$, $\delta_{i+1} = \bigcup_{K \in \delta_i} \delta'_K$, $\delta = \bigcup_{i\in\omega} \delta_i$, $U = \cup\delta$. It is easy to see that every δ_i is countable and consists of compact sets. So U is σ-compact. Let us show that U is open. Consider the set $F = X \setminus U$. Let $x \in U$ be an arbitrary point. Let $x \in \overline{F}$

then $x \in [F]_\alpha$ for some $\alpha < \omega_1$. Obviously $\alpha > 0$. Then there exists $i \in \omega$ and $K_x \in \delta_i \subseteq \delta$ such that $x \in K_x$. We have that there is a sequence $S = \{ x_i \mid i \in \omega \} \subseteq X$ such that $x_i \to x$ as $i \to \infty$ and $x_i \in [F]_{\beta_i}$, $\beta_i < \alpha$. But by (P) and construction of δ there is $K_{x_n} \in \delta_{i+1} \subseteq \delta$ such that $K_{x_n} \ni x_n \in S$ for some $n \in \omega$. Thus there is $x' \in U$ such that $x \in [F]_\beta$ for $\beta < \alpha$. Repeating the previous argument and using the fact that there are no infinite decreasing sequences of ordinals we come to a conclusion that there is $x'' \in [F]_0 = F$ such that $x'' \in U$. A contradiction. Thus U is open. The proof is complete. $\square$

Proof of theorem 2.1. "If" part is well known. We show "only if". There are three possible cases:

(1) both X and Y are first-countable.

(2) neither X nor Y is first-countable.

(3) one space is first-countable while the other is not.

If 1 holds then by [GMT, Corollary 3.6] both spaces have point-countable bases. If 2 holds then by Lemma 2.2, $X \times S_\omega$ and $Y \times S_\omega$ are sequential. Thus by Lemma 2.3 and the fact that $T \times S_\omega$ is not sequential (see [G2, Lemma 4]), X and Y have point-countable k-networks consisting of relatively compact sets. Since under **CH** $S_{\omega_1} \times S_\omega$ is not sequential (see for example [G2]) one has by Lemma 2.4 and Lemma 1.2 that X and Y are locally σ-compact and thus locally separable. By virtue of [GMT, Proposition 8.8] both spaces are topological sums of $\aleph_0$-spaces. Thus both have closed point-countable k-networks. By Lemma 2.3 X and Y have point-countable k-networks consisting of compacts. By an argument similar to the proof of [GMT, Theorem 5.2] it may be shown that each Fréchet $\aleph_0$-space in X and Y has countable k-network consisting of compacts. Thus every $\aleph_0$-space in X and Y is a k_ω-space.

Suppose that 3 holds. Without loss of generality we may assume that X is first-countable and Y is not. By lemma 2.2 $X \times S_\omega$ is sequential. So by [G2, Lemma 3] X is locally compact. Now by [GMT, Proposition 8.8] X is a topological sum of locally compact $\aleph_0$-spaces. So X is metrizable as a topological sum of metrizable spaces, and the proof is complete. □

We have the following ZFC statement.

Theorem 2.5. *Let X and Y be regular Fréchet quotient s-images of metric spaces. Then $X \times Y$ is sequential if and only if one of the three properties below holds:*
(a) X and Y have point-countable bases.
(b) X or Y is a locally compact metrizable space.
(c) X and Y are topological sums of $\aleph_0$-k_ω-spaces.

Proof: By [GMT, Corollary 6.2] X and Y have point-countable k-networks. As in the proof of Theorem 2.1 we have cases (1)–(3). Cases (1) and (3) are considered by the same argument. We consider case (2). We again have that X and Y have point-countable k-networks consisting of relatively compact sets. To complete the proof let us show that both X and Y are locally σ-compact. Suppose the contrary. As is seen from the proof of Lemma 2.4, there exist continuous mappings $g_1 : X \to X'$ and $g_2 : Y \to Y'$ such that each of them shrinks a single compact to a point and either X' or Y' contains a subspace homeomorphic to S_{ω_1}. Then by Lemma 1.3 either X or Y contains a subspace homeomorphic to S_{ω_1}, which is a contradiction by [GMT, Example 9.2. Thus both X and Y are locally σ-compact and the proof may be completed similarly to the proof of Theorem 2.1. □

Recall that a space X belongs to *class $\mathcal{T}'$* [T1] if it is the union of countably many closed and locally compact subsets X_n such that $A \subseteq X$ is closed whenever $A \cap X_n$ is closed for all $n \in \omega$. It may be seen from the proof above that condition (c) in Theorem 2.1 may be replaced by c′: X and Y are in the class $\mathcal{T}'$. So the following theorem holds:

Theorem 2.6 (CH). *Let X and Y be regular Fréchet spaces with point-countable k-networks. Then $X \times Y$ is sequential if and only if one of the three properties below holds:*

(a) X and Y have point-countable bases.
(b) X or Y is a locally compact metrizable space.
(c) X and Y are in the class $\mathcal{T}'$.

In fact condition (c) in Theorem 2.1 may be strengthened to the following: X and Y are topological sums of $\aleph_0$-k_ω-spaces and in each of them the set of non-first-countable points is closed and discrete.

Using the fact proved in [G2, Lemma 5] that $S^2_{\omega_1}$ is not a k-space one can prove the following ZFC statement by argument similar to the proof of Theorem 2.1.

Theorem 2.7. *Let X be a regular Fréchet space with a point-countable k-network. Then X^2 is sequential if and only if either X has a point-countable base or X is a topological sum of $\aleph_0$-k_ω-spaces.*

Lemmas 2.2–2.4 permit one to obtain some results on products of sequential spaces with point-countable k-networks. By a method similar to the proof of Theorem 2.1 one can get the following proposition:

Proposition 2.8 (CH). *Let X and Y be regular k-spaces with point-countable k-networks. If $X \times Y$ is sequential then at least one of the three properties below holds:*
(a) X and Y have point-countable bases.
(b) X or Y is a locally compact metrizable space.
(c) X and Y are locally σ-compact.

Proof of this theorem uses another canonical space—so called Arens' space S_2 (for the definition of S_2 and discussion of "general sequential" properties of S_ω and S_2 see [NT]) and the obvious analog of Lemma 2.2 which can be proved by a similar argument.

Lemma 2.9. *Let X be a σ-compact sequential space with a point-countable k-network consisting of compacts such that $so(X) \leq 2$. Then X is a k_ω-space.*

Sketch of Proof Let γ be a point countable k-network for X consisting of compacts. Let us prove the following property of X:

(*) for every compact K there is a countable $\gamma' \subseteq \gamma$ such that if $x_n \to x \in K$ as $n \to \infty$ and $x_n \neq x_k$ if $n \neq k$ then there is $\xi \in \gamma'$ such that $|\xi \cap \{ x_n \mid n \in \omega \}| \geq \aleph_0$

Suppose (*) is false. Then let us show the following fact.

Fact 1. *There is a map $t : \omega_1 \times \omega \to X$ such that $t(\alpha, n) \to x_\alpha \in K$ as $n \to \infty$, $x_\alpha \neq x_\beta$ if $\alpha \neq \beta$, $t(\alpha, n) \neq t(\alpha, k)$ if $n \neq k$, K is compact and $|\{ \alpha \mid |\xi \cap \{ t(\alpha, n) \mid n \in \omega \}| \geq \aleph_0 \}| \leq 1$ for any $\xi \in \gamma$.*

Proof of fact 1: Let K be a compact violating property (*). If $S \subseteq X$ is a countable set then let us denote $\gamma(S) = \{ \xi \in \gamma \mid \xi \cap S \neq \emptyset \}$. Construct by induction on $\alpha < \omega_1$ the points $t(\alpha, n)$, $n \in \omega$ so that:

(1) $t(\alpha, n) \to x_\alpha \in K$ as $n \to \infty$
(2) $t(\alpha, n) \neq t(\alpha, k)$ if $n \neq k$
(3) put $S_\alpha = \{ t(\beta, n) \mid \beta < \alpha, n \in \omega \} \cup \{ x_\beta \mid \beta < \alpha \}$; then for any $\xi \in \gamma(S_\alpha)$ $|\xi \cap \{ t(\alpha, n) \mid n \in \omega \}| < \aleph_0$

Let $\{ t(0, n) \mid n \in \omega \}$ be an arbitrary sequence such that $t(0, n) \to x_0 \in K$ as $n \to \infty$ and $t(0, n) \neq t(0, k)$ if $n \neq k$. Such sequence obviously exists. Suppose we have already constructed the points $t(\beta, n)$ for all $\beta < \alpha$, $n \in \omega$ such that they satisfy conditions (1)–(3). Since $\gamma(S_\alpha)$ is a countable family and K violates property (*) there exists a sequence $\{ t(\alpha, n) \mid n \in \omega \}$ such that $t(\alpha, n) \to x_\alpha \in K$ as $n \to \infty$, $t(\alpha, n) \neq t(\alpha, k)$ if $n \neq k$ and for any $\xi \in \gamma(S_\alpha)$ $|\xi \cap \{ t(\alpha, n) \mid n \in \omega \}| < \aleph_0$. So properties (1)–(3) hold. Now prove that $t : \omega_1 \times \omega \to X$ and points x_α satisfy what is required. It follows from (3) and γ being closed k-network that $x_\alpha \neq x_\beta$ if $\alpha \neq \beta$. Suppose now that $\xi \in \gamma$ and $|\xi \cap \{ t(\alpha, n) \mid n \in \omega \}| \geq \aleph_0$. Consider an arbitrary $\beta > \alpha$. Then $\xi \in \gamma(S_\beta)$ and thus $|\xi \cap \{ t(\beta, n \mid n \in \omega \}| < \aleph_0$ by (3). It easily follows that $|\{ \alpha \mid |\xi \cap \{ t(\alpha, n) \mid n \in \omega \}| \geq \aleph_0 \}| \leq 1$ for any $\xi \in \gamma$. So fact 1 holds. $\square$

Let us denote $\gamma_\alpha = \{\, \xi \in \gamma \mid |\xi \cap \{\, t(\alpha, n) \mid n \in \omega \,\}| \geq \aleph_0 \,\}$. It follows from fact 1 that $\gamma_\alpha \cap \gamma_\beta = \emptyset$ if $\alpha \neq \beta$. Suppose $x \in U \subseteq X$ and U has the property: for any $\{x_n\}$ such that $x_n \to x$ as $n \to \infty$ there is $k \in \omega$ such that $x_k \in U$. We shall say for brevity that U is an o-neighborhood of x. Consider the set

$$\Lambda = \{\alpha < \omega_1 | \text{ there is finite } \delta_\alpha \subseteq \gamma_\alpha \text{ such that } \cup\delta_\alpha \text{ is an } o\text{-neighborhood of some } t(\alpha, n_\alpha)\}$$

Now prove the following fact.

Fact 2. Λ *is countable.*

Proof of fact 2: Suppose Λ is uncountable. Let $X = \bigcup_{i \in \omega} K_i$ where every K_i is compact. Then every K_i is metrizable and it easily follows that if $t(\alpha, n_\alpha) \in K_i$ for some $i \in \omega$, $\alpha \in \Lambda$ then $\cup\delta_\alpha \cap K_i$ has the non empty interior in the topology of K_i. It easily follows that there is uncountable $\Lambda' \subseteq \Lambda$ and a compact $K_n \subseteq X$ such that for any $\alpha \in \Lambda'$ the set $\cup\delta_\alpha \cap K_n$ has the non empty interior in the topology of K_n. But K_n is separable. Then there is a point $x \in K_n$ such that $x \in (\cup\delta_\alpha)^\circ_{K_n}$ for any $\alpha \in \Lambda'' \subseteq \Lambda'$ where Λ'' is uncountable. It follows that $x \in \xi_\alpha \in \delta_\alpha \subseteq \gamma_\alpha \subseteq \gamma$ for any $\alpha \in \Lambda''$. But $\xi_\alpha \neq \xi_\beta$ if $\alpha \neq \beta$ which contradicts the point-countability of γ. So Λ is countable. $\square$

Combining fact 1 and fact 2 one can construct an injection $s : \omega^2 \to X$ such that:

1) $s(n,k) \to x_n$ as $k \to \infty$

2) $x_n \to x$ as $n \to \infty$

3) $x \neq x_n \neq x_k \notin s(\omega^2)$ if $n \neq k$

4) $|\{\, n \mid |\xi \cap \{\, s(n,k) \mid k \in \omega \,\}| \geq \aleph_0 \,\}| \leq 1$ for any $\xi \in \gamma$

5) if $\xi_i \in \gamma$, $i \leq m$, $|\xi_i \cap \{\, s(n,k) \mid k \in \omega \,\}| \geq \aleph_0$ then $\bigcup_{i \leq m} \xi_i$ is not an o-neighborhood for any $s(n,k)$, $k \in \omega$.

Let us show that for any $n \in \omega$ there is an injection $s^n : \omega^2 \to X$ such that:

6) $s^n(m,i) \to s(n, k_m)$ as $i \to \infty$, $k_{m+1} > k_m$

7) $s^n(\omega^2) \cup \{\, s^n(n,k_m) \mid m \in \omega \,\} \cup \{x_n\}$ is a closed subset of X homeomorphic to S_2

First introduce some notation. If $S \subseteq X$ is countable then $\gamma(S)$ is also countable so set $\gamma(S) = \{\, \xi^n(S) \mid n \in \omega \,\}$. If the points $s^n(l,j)$, $l < m$, $j \in \omega$ have been already constructed then denote $S_m = \{\, s^n(l,j) \mid l < m, j \in \omega \,\} \cup \{\, s(n,k_l) \mid l < m \,\}$. For any $m \in \omega$ we choose a sequence $T_m = \{\, s^n(m,i) \mid i \in \omega \,\}$ such that $s^n(m,i) \neq s^n(m,j)$ if $i \neq j$, $T_m \subseteq X \setminus \bigcup_{i,j<m} \xi^i(S_j)$ and $s^n(m,i) \to s(n,k_m)$ as $i \to \infty$ for some $k_m \in \omega$. It is easily seen that 5) and closedness of elements of γ implies that if $\gamma' \subseteq \gamma$ is finite then there is $k \in \omega$, $k > \max\{\, k_i \mid i < m \,\}$ such that $\bigcup\gamma'$ is not an o-neighborhood for $s(n,k)$. Using this it is easy to construct T_m to satisfy the requirement. Let us show 7). It is enough to prove that any set of the form $\{\, s^n(m_k,i_k) \mid k \in \omega \,\}$ where $m_{k+1} > m_k$ is a closed discrete subset of X. Suppose not. Then we may assume without loss of generality that $s^n(m_k,i_k) \to y \in X$ as $k \to \infty$. Choose $\xi \in \gamma$ such that $\xi \cap \{\, s^n(m_k,i_k) \mid k \in \omega \,\}$ is infinite. Then $\xi = \xi^i(S_j)$ for some $i,j \in \omega$. But $s^n(m_k,i_k) \notin \bigcup_{i,j<m_k} \xi^i(S_j)$ thus $s^n(m_k,i_k) \notin \xi^i(S_j)$ if $m_k > \max\{i,j\}$. This contradicts the fact that $\xi \cap \{\, s^n(m_k,i_k) \mid k \in \omega \,\}$ is infinite and $m_{k+1} > m_k$.

Using 4) it is easy to construct by induction a sequence $\{(n_i,l_i)\}_{i\in\omega}$ so that $n_{i+1} > n_i$ and

$$\{\, s(n_{m+1},k) \mid k > l_{m+1} \,\} \cap \Big(\bigcup_{i,j<n_m} \xi^i(\{\, s(j,k) \mid k \in \omega \,\}) \Big) = \emptyset$$

Considerations similar to the proof of 7) give that the set $\{\, s(n_i,k) \mid i \in \omega, k > l_i \,\} \cup \{\, x_{n_i} \mid i \in \omega \,\} \cup \{x\}$ is a closed subset of X homeomorphic to S_2. Using 6), 7) and the proved above one can construct three injections $t_1 : \omega \to X$, $t_2 : \omega^2 \to X$ and $t_3 : \omega^3 \to X$ so that:

(a) $t_i(\omega^i) \cap t_j(\omega^j) = \emptyset$ if $i \neq j$, $x \notin \bigcup_{i\leq 3} t_i(\omega^i)$
(b) $t_3(m,n,k) \to t_2(n,k)$ as $m \to \infty$
(c) $t_2(n,k) \to t_1(n)$ as $k \to \infty$
(d) $t_1(n) \to x$ as $n \to \infty$

(e) for any $n \in \omega$ the set $\{ t_3(m,n,k) \mid m,k \in \omega \} \cup \{ t_2(n,k) \mid k \in \omega \} \cup \{t_1(n)\}$ is a closed subset of X homeomorphic to S_2

(f) the set $\{ t_2(n,k) \mid n,k \in \omega \} \cup \{ t_1(n) \mid n \in \omega \} \cup \{x\}$ is a closed subset of X homeomorphic to S_2

Using (e), (f) and compactness of elements of γ one can easily prove:

(g) for any $\xi \in \gamma$ there is $n(\xi) \in \omega$ such that $\xi \cap \{ t_2(n,k) \mid n > n(\xi), k \in \omega \} = \emptyset$

(h) for any $\xi \in \gamma$, any $n \in \omega$ there is $k(\xi) \in \omega$ such that $\xi \cap \{ t_3(m,n,k) \mid k > k(\xi), m \in \omega \} = \emptyset$

Now consider the family $\gamma(\bigcup_{i \leq 3} t_i(\omega^i) \cup \{x\}) = \{\xi_n\}_{n \in \omega}$ It is easy to choose $n_i \in \omega$, $k_i \in \omega$ using g. and h. so that:

8) $n_{i+1} > n_i$

9) $(\bigcup_{j<i} \xi_j) \cap (\{ t_3(m,n_i,k) \mid k > k_i, m \in \omega \} \cup \{ t_2(n_i,k) \mid k > k_i \} \cup \{t_1(n_i)\}) = \emptyset$

Now using routine considerations, properties (a)–(h), 8) and 9) one can easily show that the set $\bigcup_{i \in \omega}(\{ t_3(m,n_i,k) \mid k > k_i, m \in \omega \} \cup \{ (t_2(n_i,k) \mid k > k_i \} \cup \{t_1(n_i)\}) \cup \{x\}$ is a closed subset of X homeomorphic to S_3.

Thus if $so(X) \leq 2$ then (*) takes place. It follows easily from (*), sequentiality and σ-compactness of X and the fact that γ consists of compact sets that X is a k_ω-space. □

Using Lemma 2.9 and Proposition 2.8 we obtain the following theorem (we use Lemma 2.2, Lemma 2.3 and the fact that $T \times S_\omega$ is not sequential proved in [G2] to show that we can assume X and Y to have point-countable k-networks consisting of compacts).

Theorem 2.10 (CH). *Let X and Y be regular k-spaces with point-countable closed k-networks such that $so(X) \leq 2$ and $so(Y) \leq 2$. Then $X \times Y$ is sequential if and only if one of the three properties below holds:*
(a) X and Y have point-countable bases.
(b) X or Y is a locally compact metrizable space.
(c) X and Y are locally k_ω-spaces. □

It is not possible to change (c) in the above theorem to 'are topological sums of k_ω-$\aleph_0$-spaces' (see [GMT, Example 9.3] for details). Let us note without proof the following theorem.

Theorem 2.11. *Let X and Y be regular k-spaces with σ-locally finite k-networks. Then $X \times Y$ is sequential if and only if one of the three properties below holds:*
(a) X and Y have point-countable bases.
(b) X or Y is a locally compact metrizable space.
(c) X and Y are topological sums of $\aleph_0$-k_ω-spaces.

3. Examples

Example 3.1 (MA+¬CH). There is a space Y with a point-countable closed k-network such that Y is not locally σ-compact and $Y \times S_\omega$ is sequential.

Proof: Let $B \subseteq \boldsymbol{I} = [0,1]$ be an arbitrary subset of cardinality ω_1. Let $\{ S_b \mid b \in B \}$ be a family of spaces so that each of S_b has unique nonisolated point x_b and is homeomorphic to a convergent sequence. Finally let Y be the space obtained by identifying x_b and $b \in B \subseteq \boldsymbol{I}$. It is easy to see that the space obtained from Y by identifying $\boldsymbol{I}$ to a point is homeomorphic to S_{ω_1}. **MA+¬CH** implies that $S_{\omega_1} \times S_\omega$ is sequential (see [G2]). The obvious quotient map $p : Y \to S_{\omega_1}$ is perfect. So $Y \times S_\omega$ is also sequential by Lemma 1.2. Obviously Y is not locally σ-compact. □

Example 3.2 (CH) There is a σ-compact space Γ_B with a point-countable closed k-network such that Γ_B is not locally a k_ω-space and $\Gamma_B \times S_\omega$ is sequential.

Proof: Let $S = \{y_i(j)\}_{i,j\in\mathbb{N}} \cup \{\hat{0}\}$. Putting every point $y_i(j)$ isolated and the base of open neighborhoods of $\hat{0}$ consisting of the sets:

$$O(n_1, \dots, n_i, \dots) = \bigcup_{i\in\mathbb{N}} \{ y_j(i) \mid j \geq n_i \} \cup \{\hat{0}\}$$

we get the sequential fan S_ω. Let $\Gamma = \bigcup_{i \in N} \boldsymbol{I}_i$ where $\boldsymbol{I}_i = [0,1]$. The obvious projection $\pi : \Gamma \to \boldsymbol{I}_0$ will be needed. Let $B \subseteq \boldsymbol{I}_0 = \boldsymbol{I} = [0,1]$. By Γ_B we denote the set Γ equipped with the maximal topology such that every subspace $\boldsymbol{I}_i \subseteq \Gamma_B$ has its usual euclidean topology and for every $b \in B$ the set $\pi^{-1}(b)$ is a countable compact with unique nonisolated point $b \in \boldsymbol{I}_0 = \boldsymbol{I}$. Then $P^i(j) = \Gamma \times (\{ y_k(j) \mid k > i \} \cup \{\hat{0}\})$ and $\Pi_k : P^{-1}(k) \to \Gamma$ is the obvious projection.

To construct the example violating (I) and Statement 1.1 it suffices to show that there exists uncountable $B \subseteq \boldsymbol{I}$ such that any set $C \subseteq \Gamma_B \times S_\omega$ such that $C \cap (\boldsymbol{I}_i \times S_\omega)$ is closed in the usual topology of $\boldsymbol{I}_i \times S_\omega$ either has no cluster point in $\boldsymbol{I}_0 \times \{\hat{0}\}$ or contains a sequence converging to a point $x \in \boldsymbol{I}_0 \times S_\omega$. Indeed consider the space Y obtained from Γ_B by shrinking the set $I_0 \subseteq \Gamma_B$ to a point and corresponding perfect maps: $p : \Gamma_B \to Y$ and $p \times id_{S_\omega} : \Gamma_B \times S_\omega \to Y \times S_\omega$. By Lemma 1.2 $Y \times S_\omega$ is sequential if and only if $\Gamma_B \times S_\omega$ is sequential. At every point except $y_0 = (p(I_0), \hat{0})$ the space $Y \times S_\omega$ is locally a product of a compact and a sequential space. Thus $Y \times S_\omega$ is not sequential if and only if there is a set $C'' \subseteq Y \times S_\omega \setminus \{y_0\}$ such that C'' is closed in $Y \times S_\omega \setminus \{y_0\}$, $\overline{C''} \ni y_0$ and there is no sequence in C'' converging to y_0. For every such C'' one can choose an open and closed neighborhood O of $\hat{0}$ in S_ω so that $(p(\boldsymbol{I}_0) \times O) \cap C'' = \emptyset$. Put $C' = Y \times O \cap C''$. Now if $\Gamma_B \times S_\omega$ is not sequential then we can find $C' \subseteq Y \times S_\omega$ with the properties listed above. Using closedness of $p \times id_{S_\omega}$ one can show that $C = (p \times id_{S_\omega})^{-1}(C')$ has the following properties: $C \subseteq \Gamma_B \times S_\omega$, C has a cluster point in $I_0 \times \{\hat{0}\}$, $C \cap (\boldsymbol{I}_i \times S_\omega)$ is closed in the usual topology of $\boldsymbol{I}_i \times S_\omega$ and there is no sequence in C converging to any point $x \in I_0 \times S_\omega$. Also it is easy to see that uncountability of B implies that Γ_B is not locally a k_ω-space.

Let $\boldsymbol{C}$ be the class of all sets $C \subseteq \Gamma \times S_\omega$ such that $C \cap (\boldsymbol{I}_i \times S_\omega)$ is closed in the usual topology of $\boldsymbol{I}_i \times S_\omega$ and $C \cap (\boldsymbol{I}_0 \times S_\omega) = \emptyset$. It may be checked that $|\boldsymbol{C}| = 2^{\aleph_0}$. Thus by the Continuum Hypothesis $\boldsymbol{C} = \{C_\alpha\}_{\alpha \in \omega_1}$. Let μ be the Lebesgue measure on

$\boldsymbol{I}$. We will not prove the measurability of certain sets since it will be obvious in all the cases.

We construct by induction on $\alpha \in \omega_1$ two families of sets $\boldsymbol{I} \supseteq \cdots \supseteq S_\alpha \supseteq S_{\alpha+1} \supseteq \ldots$ and $\cdots \subseteq B_\alpha \subseteq B_{\alpha+1} \subseteq \cdots \subseteq \boldsymbol{I}$ such that every B_α is countable and $B_{\alpha+1} \setminus B_\alpha \neq \emptyset$, $((\bigcup_{\alpha<\omega_1} B_\alpha) \setminus B_\beta) \subseteq S_\beta$, $\mu(S_\beta) = 1$ and every $C_\alpha \in C$ either has a cluster point in $\boldsymbol{I}_0$ in the topology of $\Gamma_{B_\alpha} \times S_\omega$ or is closed in $\Gamma_{S_\alpha} \times S_\omega$.

Suppose that $\{B_\alpha\}_{\alpha<\beta}$ and $\{S_\alpha\}_{\alpha<\beta}$ are already constructed. Let us consider the set $C^1 = C_\beta \cap \Gamma \times \{\hat{0}\}$. Put $S^1 = \bigcap_{\alpha<\beta} S_\alpha$.

A) If there exists a point $a \in S^1$ such that $|\pi^{-1}(a) \times \{\hat{0}\} \cap C^1| = \aleph_0$ then let

$$(*) \qquad S_\beta = S^1, \quad B_\beta = \textstyle\bigcup_{\alpha<\beta} B_\alpha \cup \{a\} \cup \{x\}, \text{ where } x \in S_\beta \setminus \bigcup_{\alpha<\beta} B_\alpha$$

Otherwise one can choose $C^2 \supseteq C^1$ and $S^2 \subseteq \boldsymbol{I}, \mu(S^2) = 1$ such that $C^2 \subseteq (\Gamma \setminus \boldsymbol{I}_0) \times \{\hat{0}\}$ and $C^2 \cap \boldsymbol{I}_i \times \{\hat{0}\} = C_i^2$ is a finite union of open intervals and $C^{[2]} = \bigcup_{i\in\mathbb{N}}[C_i^2]$ is such that $U = \Gamma \setminus C^{[2]}$ is an open neighborhood of $\boldsymbol{I}_0$ in Γ_{S^2}, where $[C_i^2]$ is the closure of C_i^2 in the usual topology of $\boldsymbol{I}_i \times \{\hat{0}\}$. Such C^2 may be constructed in the following way. Every set $C^1 \cap (\boldsymbol{I}_i \times \{\hat{0}\})$, $i \in \mathbb{N}$ is compact. Choose $C_i^2 \supseteq (C^1 \cap (\boldsymbol{I}_i \times \{\hat{0}\}))$ so that C_i^2 is a union of finitely many open intervals, $\mu([C_i^2] \setminus (C^1 \cap (\boldsymbol{I}_i \times \{\hat{0}\}))) < 1/2^i$ and put $C^2 = \bigcup_{i\in\mathbb{N}} C_i^2$. It is easy to see that $C^2 \supseteq C^1$. Then define

$$(1) \qquad S^2 = \boldsymbol{I}_0 \setminus \{a \in \boldsymbol{I}_0 | \; |\pi^{-1}(a) \cap C^{[2]}| = \aleph_0\}$$

Put $\lambda_i = [C_i^2] \setminus (C^1 \cap \boldsymbol{I}_i)$ then $[C_i^2] = \lambda_i \cup (C^1 \cap \boldsymbol{I}_i)$ and $\mu(\lambda_i) < 1/2^i$. Now let $\Lambda = \bigcup_{i\in\mathbb{N}} \lambda_i$. It is easy to see that $C^{[2]} = \Lambda \cup C^1$ and thus

$$\begin{aligned} \boldsymbol{I}_0 \setminus S^2 &= \{a \in \boldsymbol{I}_0 | \; |\pi^{-1}(a) \cap C^{[2]}| = \aleph_0\} \\ &= \{a \in \boldsymbol{I}_0 | \; |\pi^{-1}(a) \cap \Lambda| = \aleph_0\} \\ &\cup \{a \in \boldsymbol{I}_0 | \; |\pi^{-1}(a) \cap C^1| = \aleph_0\} \end{aligned}$$

Then

$$S^2 = (\boldsymbol{I}_0 \setminus \{ a \in \boldsymbol{I}_0 \mid |\pi^{-1}(a) \cap \Lambda| = \aleph_0 \}) \cap (\boldsymbol{I}_0 \setminus \{ a \in \boldsymbol{I}_0 \mid |\pi^{-1}(a) \cap C^1| = \aleph_0 \})$$

So nonexistence of a point a with the property described in A) gives that

$$S^2 \supseteq (\boldsymbol{I}_0 \setminus \{a \in \boldsymbol{I}_0 | \ |\pi^{-1}(a) \cap \Lambda| = \aleph_0\}) \cap S^1$$

It follows that

$$S^2 \supseteq S^1 \cap (\boldsymbol{I}_0 \setminus \bigcap_{i\in\mathbb{N}} \pi(\bigcup_{j>i} \lambda_j)) = S^1 \cap S'$$

where $\mu(S') = 1-\mu(\bigcap_{i\in\mathbb{N}} \pi(\bigcup_{j>i} \lambda_i))$. Now $\mu(\bigcap_{i\in\mathbb{N}} \pi(\bigcup_{j>i} \lambda_i)) \leq \mu(\pi(\bigcup_{j>i} \lambda_i)) \leq \sum_{j>i} \mu(\lambda_i) \leq 1/2^i$ for any $i\in \mathbb{N}$. It follows that $\mu(S') = 1$ and $\mu(S^2) = 1$.

Then it follows from (1) and closedness of every $C^{[2]} \cap \boldsymbol{I}_i$ that $U = \Gamma \setminus C^{[2]}$ is an open neighborhood of $\boldsymbol{I}_0$ in Γ_{S^2}. So C^2 has all the required properties. Consider the set

$$C^3 = C_\beta \setminus (C^2 \times S_\omega)$$

and put $C^i(j) = C^3 \cap P^i(j)$. Then $\Pi_k(C^i(k)) \cap \boldsymbol{I}_j$ is compact for every $i, j, k\in \mathbb{N}$. Moreover for every $i, k\in \mathbb{N}$ there exists $n_{k,i}\in \mathbb{N}$ such that

$$(2) \qquad \Pi_k(C^{n_{k,i}}(k)) \cap \boldsymbol{I}_j = \emptyset \quad \textit{for every } j \leq i$$

This follows from those easy to observe facts that $\boldsymbol{I}_k \times \{\hat{0}\}$ is compact, $C^3 \cap \Gamma \times \{\hat{0}\} = \emptyset$ and $C^{i+1}(k) \subseteq C^i(k)$.

B) If for some $i, k\in \mathbb{N}$ there exists $a \in S^1 \cap S^2$ such that $|\pi^{-1}(a) \cap \Pi_k(C^i(k))| = \aleph_0$ then let

(**) $\qquad S_\beta = S^1 \cap S^2, \quad B_\beta = \bigcup_{\alpha<\beta} B_\alpha \cup \{a\} \cup \{x\},$ *where* $x \in S_\beta \setminus \bigcup_{\alpha<\beta} B_\alpha$

In other case for every $i \geq 1$ fix $n_i\in \mathbb{N}$ such that

(a) $\Pi_i(C^{n_i}(i)) \cap \boldsymbol{I}_j = \emptyset$ for every $j \leq i$

(b) $\mu(\boldsymbol{I} \setminus (\pi(\Pi_i(C^{n_i}(i))))) > 1 - 1/2^i$.

To get (b) it is enough to note that $\bigcup_{j\in\mathbb{N}}(\boldsymbol{I} \setminus (\pi(\Pi_k(C^j(k)))) \supseteq S^1 \cap S^2$ by (2), nonexistence of a and the fact that $\mu(S^1 \cap S^2) = 1$, $\pi(\Pi_k(C^{j+1}(k)))) \subseteq \pi(\Pi_k(C^j(k))))$. Let now $C = \bigcup_{i\in\mathbb{N}} \Pi_i(C^{n_i}(i))$. By (a) $C \cap \boldsymbol{I}_j$ is compact for every $j\in \mathbb{N}$. Let S^3 be the set of all points $a \in \boldsymbol{I}$ such that $\pi^{-1}(a) \cap C$ is finite. Then $\Gamma \setminus C$ is an open neighborhood of $\boldsymbol{I}_0 \times \{\hat{0}\}$ in Γ_{S^3}.

Consider an arbitrary real $0 <c < 1$. Find $k\in \mathbb{N}$ such that $\mu(R) > c + (1-c)/2$, where

$$R = \boldsymbol{I} \setminus (\pi(\bigcup_{i\geq k} \Pi_i(C^{n_i}(i))))$$

Such k exists due to (b)

For every $i \leq k$ let us find m_i such that $\mu(T_i) < (1-c)/2k$, where

$$T_i = \pi(\Pi_i(C^{n_i}(i)) \setminus \bigcup_{j\leq m_i} \boldsymbol{I}_j)$$

Such m_i always exists because nonexistance of $a \in S^1 \cap S^2$ having properties listed in B) implies that

$$S^1 \cap S^2 \subseteq \boldsymbol{I}_0 \setminus \bigcap_{i\in\mathbb{N}} \pi(\Pi_k(C^{n_k}(k) \setminus \bigcup_{j\leq i} \boldsymbol{I}_j)) \tag{3}$$

Denoting $\lambda_i = \pi(\Pi_k(C^{n_k}(k) \setminus \bigcup_{j\leq i} \boldsymbol{I}_j))$ one can rewrite (3) as $\bigcap_{i\in\mathbb{N}} \lambda_i \subseteq \boldsymbol{I}_0 \setminus (S^2 \cap S^2)$. Now using the fact that $\lambda_{i+1} \subseteq \lambda_i$, the previous formula and the fact that $\mu(S^1 \cap S^2) = 1$ it is easy to find required m_i.

Then every point $a \in R \cap (\boldsymbol{I} \setminus \bigcup_{i\leq k} T_i)$ has the property: $\pi^{-1}(a) \cap C$ is finite. Now $\mu(R \cap (\boldsymbol{I} \setminus \bigcup_{i\leq k} T_i)) > c$ by the choice of T_i, R . Hence $\mu(S^3) = 1$ because $S^3 \supseteq R \cap (\boldsymbol{I} \setminus \bigcup_{i\leq k} T_i)$ and there was no restriction on c. Let

(***) $S_\beta = S^1 \cap S^2 \cap S^3$, $B_\beta = \bigcup_{\alpha<\beta} B_\alpha \cup \{x\}$, *where* $x \in S_\beta \setminus \bigcup_{\alpha<\beta} B_\alpha$

$$U_\alpha = (\Gamma \cap U) \setminus C, \qquad V_\alpha = O(n_1, \dots, n_i, \dots)$$

Finally let $B = \bigcup_{\alpha<\omega_1} B_\alpha$. Suppose that $F \in \boldsymbol{C}$. Then $F = C_\beta$ for some $\beta < \omega_1$. If A) or B) takes place one can choose a sequence in C_β converging to a point of $\boldsymbol{I}_0 \times S_\omega$ in the topology of Γ_{B_β} and thus in the topology of $\Gamma_{\boldsymbol{B}}$. Such sequence is easy to find in $\pi^{-1}(a) \times \{\, y_j(k) \mid j > i \,\}$ (see (*), (**)). If neither A) nor B) takes place then one can find U_β and V_β such that

$$\boldsymbol{I}_0 \times \hat{0} \subseteq U_\beta \times V_\beta, \quad U_\beta \times V_\beta \cap C_\beta = \emptyset$$

and U_β is open in Γ_{S_β}, V_β is open in S_ω. Since B_β is countable, $\Gamma_{B_\beta} \times S_\omega$ is sequential being the product of two k_ω-spaces and one can either choose a sequence in C_β converging to a point of $\boldsymbol{I}_0 \times S_\omega$ or an open U'_β in Γ_{B_β} and open V'_β in S_ω such that $U'_\beta \supseteq \boldsymbol{I}_0$ and $U'_\beta \times V'_\beta \cap C_\beta = \emptyset$. In the last case we have

$$(U_\beta \cup U'_\beta) \times (V_\beta \cap V'_\beta) \cap C_\beta = \emptyset$$

and $(U_\beta \cup U'_\beta) \times (V_\beta \cap V'_\beta)$ is an open neighborhood of $\boldsymbol{I}_0 \times S_\omega$ in $\Gamma_{S_\beta \cup B_\beta} \times S_\omega$ and thus is open in $\Gamma_B \times S_\omega$ since $S_\beta \cup B_\beta \supseteq B$.

Let us now describe in short a procedure that allows to make Γ_B a regular space. Light change in the above construction will make B intersect every Borel set of nonzero measure in $\boldsymbol{I}$. Let now $\{ W_\alpha \mid \alpha < \omega_1 \}$ be all subsets of Γ such that W_α is an open neighborhood of $\boldsymbol{I}_0$ in some Γ_{P_α} where $\mu(P_\alpha) = 1$. For every α it is possible to find P'_α, W'_α such that $P'_\alpha \subseteq P_\alpha$, $\mu(P'_\alpha) = 1$, $\overline{W'_\alpha} \subseteq W_\alpha$ in $\Gamma_{P'_\alpha}$. Now at each step of the above construction one lets $S'_\beta = S_\beta \cap P'_\beta$ and then uses S'_β instead of S_β in further steps. Let us note that then $B \setminus P'_\beta$ is countable for any $\beta < \omega_1$.

Let now W be an arbitrary neighborhood of $\boldsymbol{I}_0$ in Γ_B. The set $K = \boldsymbol{I}_0 \setminus \bigcap_{i \in \mathbb{N}} \pi((\Gamma_B \setminus W) \setminus \bigcup_{k \leq i} \boldsymbol{I}_k)$ is Borel and $B \subseteq K$. Thus $\boldsymbol{I}_0 \setminus K$ is Borel and has measure zero since it does not intersect B. Thus $W = W_\alpha$ for some $\alpha < \omega_1$. Now $\overline{W'_\alpha} \subseteq W_\alpha = W$ in $\Gamma_{P'_\alpha}$. Since $B \setminus P'_\alpha$ is countable there exists W''_α such that W''_α is an open neighborhood of $\boldsymbol{I}_0$ in $\Gamma_{B \setminus P'_\alpha}$ and $\overline{W''_\alpha} \subseteq W_\alpha$ in $\Gamma_{B \setminus P'_\alpha}$. Now $W' = W'_\alpha \cup W''_\alpha$ is an open neighborhood of $\boldsymbol{I}_0$ in $\Gamma_{B \cup P'_\alpha}$ and thus in Γ_B and it is easy to see that $\overline{W'} \subseteq W$ in Γ_B. So for every open neighborhood W of $\boldsymbol{I}_0$ in Γ_B there exists open W' such that $W' \supseteq \boldsymbol{I}_0$ and $\overline{W'} \subseteq W$ in Γ_B. This is enough to prove the regularity of Γ_B. $\square$

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References

[E] R. Engelking, General Topology, Heldermann Verlag, 1989

[Fo] L. Foged, *A characterization of closed images of metric spaces*, Proc. Amer. Math. Soc., **95** (1985).

[Fr] S. P. Franklin, *Spaces in which sequences suffice*, Fund. Math., **57** (1965), 102–114.

[G1] G. Gruenhage, *MR 93i:54018* Math. Reviews, **i** 1993.

[G2] ——, *k-spaces and products of closed images of metric spaces*, Proc. Amer. Math. Soc., **80** (1980), 478–482.

[GMT] G. Gruenhage, E. Michael, Y. Tanaka *Spaces determined by point-countable covers*, Pacific J. Math., **113** (1984), 303–332.

[H] Chen Huai Peng, *The products of k-spaces with point countable closed k-networks*, Topology Proc., **15** (1990), 63–82.

[M] E. Michael, *A quintuple quotient quest*, Gen. Top. and Appl., **2** (1972), 91–138.

[NS] T. Nogura and A. Shibakov, *Sequential order of product spaces*. Topology and Appl., **65** (1995), 271-285.

[NT] T. Nogura and Y. Tanaka, *Spaces which contain a copy of S_ω or S_2 and their applications*, Topology and Appl., **30** (1988), 51–62.

[R] M. Rajagopalan, *Sequential order and spaces S_n*, Proc. Amer. Math. Soc., **54** (1976), 433–438.

[T1] Y. Tanaka *A characterization for the product of k- and $\aleph$-spaces and related results*, Proc. Amer. Math. Soc., **59** (1976), 149–155.

[T2] Y. Tanaka, *Point-countable covers and k-networks*, Topology Proc., **12** (1987), 327–349.

[T3] Y. Tanaka, *Point-countable k-systems and products of k-spaces*, Pacif. J. Math., **101** (1982), 199–208.

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