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WHEN IS ONE GRAPH THE WEAKLY CONFLUENT IMAGE OF ANOTHER?

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ABSTRACT. A graph is a continuum which can be written as the union of finitely many arcs, the intersection of any pair of which is at most two points. A continuous map of continua $f : X \to Y$ is said to be weakly confluent provided that if A is a subcontinuum of Y, then some component of $f^{-1}[A]$ maps onto A under f. C. A. Eberhart, J. B. Fugate, and G. R. Gordh proved that the weakly confluent image of a graph is a graph and, moreover, the set of branchpoints of the range is covered by the image of the set of branchpoints of the domain. This paper extends these results by showing that if $f : X \to Y$ is a weakly confluent map of graphs and K is a subcontinuum of Y, then there is a subcontinuum A of X such that $\mathrm{ord}(A, X) \geq \mathrm{ord}(K, Y)$ and $f[A] = K$. A method for determining if a given continuous map of graphs is weakly confluent and for determining if there is a weakly confluent map from one given graph to another given graph is developed.

1. INTRODUCTION

The goal of this paper is to answer the question posed by the title as a first step in answering the question asked by C. A. Eberhart, J. B. Fugate, and G. R. Gordh in [EFG]: Given a graph X, is there an algorithm for listing those graphs which are weakly confluent images of X? We start off with some definitions and a lemma about graphs then move to the definitions of monotone and weakly confluent maps.

Definition 1.1. *A* graph *is a continuum which is the union of finitely many arcs, the intersection of any pair of which is at most two points.*

We use standard terminology for graphs which we include here for completeness:

Definition 1.2. *Let G be a graph and A a subcontinuum of G. The* order of A in$G$, *denoted ord(A,G), is defined to be the least integer n such that for each open set U of G such that $A \subset U$, there is an open set V such that $A \subset V \subset U$ and* $|\mathrm{bd}(V)| = n$. *In the case of degenerate subcontinua, we denote ord $(\{x\}, G)$ by ord(x,G). We say x is a branchpoint of G provided ord$(x,G) \geq 3$. We say x is an endpoint of G provided ord$x,G) = 1$. A vertex set of G is any finite subset of G that contains all of the branchpoints and endpoints of G. An element of a vertex set is called a vertex of G. The edge set of G corresponding to a vertex set V is the collection of the closures of the components of $G - V$. Each of the elements of an edge set is called an edge of G. A subdivision of G is an ordered pair (V, E_V) consisting of a vertex set of G and its corresponding edge set. The standard subdivision of G is the subdivision (V_G, E_{V_G}) whose vertex set consists only of the branchpoints and endpoints of G.*

Remark 1. *In this paper, all graphs have the standard subdivision unless explicitly stated otherwise. In fact, the only exceptions to graphs having the standard subdivision occurs in Theorem 3.3 and its applications.*

Lemma 1.3. *Let G be a graph and A a subcontinuum of G. Then*

$$\mathrm{ord}(A,G) = \sum_{w \in A} [\mathrm{ord}(w,G) - \mathrm{ord}(w,A)].$$

Proof: Let G be a graph and A a subcontinuum of G. First note that if A is degenerate, say $A = \{a\}$, then the equality holds since $\mathrm{ord}(a,A) = 0$. If A is nondegenerate, then we interpret the summation $\sum_{w \in A} [\mathrm{ord}(w,G) - \mathrm{ord}(w,A)]$ to be equal to the finite summation $\sum_{w\in \mathrm{bd}(A)} [\mathrm{ord}(w,G) - \mathrm{ord}(w,A)]$. This is because, if $y \in \mathrm{int}\,(A)$, then we may restrict, without loss of generality, the open sets used to compute $\mathrm{ord}(y,G)$ to those which are subsets of $\mathrm{int}\,(A)$; consequently, $\mathrm{ord}(y,G) - \mathrm{ord}(y,A) = 0$ for any $y \in \mathrm{int}\,(A)$.

We now show that $\mathrm{ord}(A,G) = \sum_{w\in \mathrm{bd}(A)} [\mathrm{ord}(w,G) - \mathrm{ord}(w,A)]$. Let U be an open set in G such that $A \subset U$. For each $w \in \mathrm{bd}(A)$, there is an open set $U_w \subset U$ such that $\mathrm{ord}(w,G) = |\mathrm{bd}\,(U_w)|$ and $\mathrm{ord}(w,A) = |\mathrm{bd}(U_w) \cap A|$. Thus, $\mathrm{ord}(w,G) - \mathrm{ord}(w,A) = |\mathrm{bd}(U_w)| - |\mathrm{bd}(U_w) \cap A| = |\mathrm{bd}(U_w) - A|$. Let $T = \mathrm{int}\,(A) \cup \bigcup_{w\in \mathrm{bd}(A)} U_w$. Clearly, $T \subset U$. Moreover, $|\mathrm{bd}(T)| = \sum_{w\in \mathrm{bd}(A)} |\mathrm{bd}(U_w) - A| = \sum_{w\in \mathrm{bd}(A)} [\mathrm{ord}(w,G) - \mathrm{ord}(w,A)]$. Therefore, $\mathrm{ord}(A,G) = \sum_{w\in \mathrm{bd}(A)} \mathrm{ord}(w,G) - \mathrm{ord}(w,A)]$. □

Definition 1.4. *A continuous map of continua $f : X \to Y$ is called* monotone *provided for every subcontinuum A of Y, $f^{-1}[A]$ is a subcontinuum of X. A continuous map of continua $f : X \to Y$ is called* weakly confluent *provided for every subcontinuum A of Y, some component of $f^{-1}[A]$ maps onto A under f. Such a component will be called an* onto component *of $f^{-1}[A]$.*

The following three results come from [EFG]. The theorems numbered here as Theorems 1.5 and 1.6 are called branchpoint covering theorems; this paper will extend Theorem 1.6 to a subcontinuum covering theorem (Corollary 2.9).

Theorem 1.5. *If $f : X \to Y$ is a weakly confluent map from a hereditarily locally connected continuum X onto a continuum Y, then Y is hereditarily locally connected and the branchpoints of Y and covered under f by the closure of the branchpoints in X.*

Theorem 1.6. *Suppose $f : X \to Y$ is a light weakly confluent map from a graph X onto a compactum Y. Then if p is the vertex of an n-od contained in Y, then there is an n-od contained in X with vertex q such that $f(q) = p$.*

Corollary 1.7. *The image of a graph under a weakly confluent map is a graph.*

2. A Subcontinuum Covering Theorem

We start this section with two lemmas about the number of disjoint "large" subcontinua of a continuum. Then we define comparable components for two subcontinua in the range of a weakly confluent map and prove two lemmas about comparable components; the second of which will be used in the proof of Lemma 2.7. Lemma 2.7 then leads directly to the subcontinuum covering theorem.

Lemma 2.1. *Let $\varepsilon > 0$ and let X be a graph. There exists a positive integer N such that if $\{K_i\}_{1 \leq i \leq N}$ is a collection of subcontinua of X each of diameter greater than ε, then two of the K_i's have a non-empty intersection.*

Proof: Let $\varepsilon > 0$ and let X be a graph. There is a finite collection $\{C_q\}_{1 \leq q \leq m}$ of subcontinua of X such that for each q, $\operatorname{diam}(C_q) < \varepsilon$ and $\bigcup_{1 \leq q \leq m} C_q = X$ from [N, Theorem 8.4]. Let $\{K_i\}_{1 \leq i \leq n}$ be a collection of disjoint subcontinua of X each with diameter greater than ε. Each K_i must intersect at least two distinct C_q's. Further, no K_i is a subset of any C_q. Thus, if for some i and some q, $K_i \cap C_q \neq \emptyset$, then $K_i \cap \operatorname{bd}(C_q) \neq \emptyset$. Hence, each C_q meets at most $|\operatorname{bd}(C_q)|$ K_i's; this is a finite number since X is a graph. Therefore, n must be no greater than $\sum_{1 \leq q \leq m} \frac{|\operatorname{bd}(C_q)|}{2}$. Let N be a positive integer greater than this sum. □

Lemma 2.2. *Let $f : X \to Y$ be a continuous surjection of graphs and let $\varepsilon > 0$. Then, there exists a positive integer N*

such that if K is a subcontinuum of Y and diam$(K) > \varepsilon$ then $|\{A : A$ is an onto component of $f^{-1}[K]\}| \leq N$.

Proof: By Lemma 2.1, it suffices to show that there is a $\delta > 0$ such that if K is a subcontinuum of Y and $\operatorname{diam}(K) > \varepsilon$ then every onto component A of $f^{-1}[K]$ has diameter no less than δ. This, however, is an immediate consequence of the uniform continuity of f. □

Definition 2.3. *Let $f : X \to Y$ be a weakly confluent map of continua. A pair of subcontinua $A \subset B$ of Y is said to have* comparable components *provided there are onto components C_A and C_B of $f^{-1}[A]$ and $f^{-1}[B]$, respectively, such that $C_A \subset C_B$.*

Lemma 2.4. *Let X be a hereditarily locally connected continuum. Let $f : X \to Y$ be a weakly confluent map and $A \subsetneq B$ subcontinua of Y. Then there is a subcontinuum C of Y such that $A \subsetneq C \subset B$ and the pair A and C has comparable components.*

Proof: Let $B_1 = B$. For $i \geq 2$, choose subcontinua B_i of Y such that $A \subsetneq B_i \subset B$ and $A = \lim B_i$. (The limit is in the sense of [N, Definition 4.9].) For each i, let C_i be an onto component of $f^{-1}[B_i]$. Without loss of generality, we may suppose that the sequence $\{C_i\}$ converges to a continuum D in X. Note that $f[D] = f[\lim C_i] = \lim f[C_i] = \lim B_i = A$. Since X has no continua of convergence by [N, Theorem 10.4], there is a positive integer k such that $D \cap C_k \neq \emptyset$. Hence, $D \subset C_k$ since $f[C_k] = B_k$ and $f[D] = A \subset B_k$. Therefore, let $C = B_k$. □

Definition 2.5. *Let X be a space. We denote the set of components of X by Comp (X).*

Lemma 2.6. *Let X be a hereditarily locally connected continuum and Y a graph. Let $f : X \to Y$ be a weakly confluent map. Let $A \subsetneq B$ be subcontinua of Y. Then there is a subcontinuum C of Y such that*

(1) $A \subsetneq C \subset B$,

(2) *the pair A and C has comparable components, and*
(3) $|Comp(C - A)| \geq |Comp(B - A)|$.

Proof: The proof is the same as for Lemma 2.4 except, for $i \geq 2$, we choose B_i so that B_i intersects each component of $B - A$. Hence, $|\text{Comp}(B_i - A)| \geq |\text{Comp}(B - A)|$. □

Lemma 2.7. *Let $f : X \to Y$ be a weakly confluent map of graphs. Suppose K is a subcontinuum of Y such that for all $y \in K$, $ord(y, Y) - ord(y, K) \leq 1$. Then, there is an onto component A of $f^{-1}[K]$ such that for all $y \in K$,*

(2.1)

$$\sum_{w \in f^{-1}[y] \cap A} [\text{ord}(w, X) - \text{ord}(w, A)] \geq \text{ord}(y, Y) - \text{ord}(y, K).$$

Proof: Let $f : X \to Y$ be a weakly confluent map of graphs and suppose K is a subcontinuum of Y such that for all $y \in K$, $\text{ord}(y, Y) - \text{ord}(y, K) \leq 1$. We first note that as in the proof of Lemma 1.3, we interpret the summation to be the finite summation $\sum_{w \in f^{-1}[y] \cap \text{bd}(A)} [\text{ord}(w, X) - \text{ord}(w, A)] \geq \text{ord}(y, Y) - \text{ord}(y, K)$. Secondly, the inequality is clearly satisfied if K is degenerate, so, for the rest of the proof, we suppose $\text{diam}(K) > 0$.

Assume the conclusion is false; that is, assume that there is a subcontinuum K of Y such that for all $y \in K$, $\text{ord}(y, Y) - \text{ord}(y, K) \leq 1$ but for each onto component A of $f^{-1}[K]$, there is an $x \in K$ such that $\text{ord}(x, Y) - \text{ord}(x, K) = 1$, while $\sum_{w \in f^{-1}[x] \cap A} [\text{ord}(w, X) - \text{ord}(w, A)] = 0$. We seek a contradiction.

For each point $q \in \text{bd}(K)$, let T_q be an $\text{ord}(q, Y)$-od about q in Y such that each edge of T_q is a proper subset of an edge of Y. Let L be the subcontinuum of Y formed by unioning the T_q 's and K. Thus, $\text{Comp}(L - K)| = \text{ord}(K, L) = \text{ord}(K, Y) = \text{ord}(L, Y)$. By Lemma 2.6, there is a subcontinuum L_1 of Y such that

(1) $K \subsetneq L_1 \subset L$,
(2) the pair K and L_1 has comparable components, and
(3) $|\text{Comp}(L_1 - K)| \geq |\text{Comp}(L - K)|$.

Note that 1 and 2 imply that L_1 is constructed in the same manner as L but with, potentially, smaller T_q's. Since, by the construction of L, $|\text{Comp}(L_1 - K)|$ cannot be greater than $|\text{Comp}(L - K)|$, we obtain the equation $|\text{Comp}(L_1 - K)| = |\text{Comp}(L - K)|$. Let A_1 and B_1 be a pair of comparable onto components of $f^{-1}[K]$ and $f^{-1}[L_1]$, respectively.

By the assumption that the conclusion is false, there is an $x_1 \in K$ such that the left-hand side of inequality 2.1 is zero for A_1 and the right-hand side is one. Since the summation is zero, $f^{-1}(x_1)$ cannot meet the boundary of A_1. Therefore, $f^{-1}(x_1) \cap A_1 \subset \text{int}(A_1)$. Hence, $f^{-1}(x_1) - A_1$ is closed; consequently, there are open sets U_1 about A_1 and V_1 about $f^{-1}(x_1) - A_1$ whose closures do not intersect. Without loss of generality, we may suppose that U_1 is connected and that $\text{bd}(U_1) \subset B_1 - f^{-1}[K]$.

There is, therefore, a component of $L_1 - f[\text{bd}(U_1)]$ containing K . Applying Lemma 2.6 to this component and K, we obtain a subcontinuum L_2 of L_1 such that $\text{ord}(K, L_2) = \text{ord}(K, L_1)$, $K \subsetneq L_2 \subset L_1$, and the pair K and L_2 has comparable components, say A_2 and B_2 respectively.

Claim. $A_2 \neq A_1$.

Proof of Claim: Assume $A_2 = A_1$. The only preimages of x_1 in B_2 are in $\text{int}(A_1)$ since the rest are in V_1, $A_1 \subset U_1$, and $f^{-1}[L_2] \cap \text{bd}(U_1) = \emptyset$. However, $\text{ord}(x_1, L_2) - \text{ord}(x_1, K) = \text{ord}(x_1, Y) - \text{ord}(x_1, K) = 1$. Let J be the component of $L_2 - K$ whose closure intersects K at x_1. Since J is open in L_2, $f^{-1}|_{B_2}[J]$ is open in B_2. Moreover, $f^{-1}|_{B_2}[L_2 - J]$ is also open in B_2 since int $(L_2 - J) = (L_2 - J) - \{x_1\}$ and $f^{-1}|_{B_2}(x_1) \subset \text{int}(A_1)$. Therefore, the connected set B_2 is the union of two disjoint open sets. This is a contradiction. Hence, $A_2 \neq A_1$.

By the assumption that the conclusion is false, we obtain a point $x_2 \in K$ such that the left-hand side of inequality

2.1 is zero for x_2 and A_2 while the right-hand side is 1 for x_2. Repeating the arguments above, we obtain a subcontinuum L_3 of L_2 and comparable onto components A_3 and B_3 of $f^{-1}[K]$ and $f^{-1}[L_3]$, respectively, such that no two of A_1, A_2, and A_3 are equal. In fact, we can repeat these arguments inductively to obtain infinitely many distinct onto components $A_1, A_2, A_3, \ldots$of $f^{-1}[K]$. This contradicts Lemma 2.2 since $\operatorname{diam}(K) > 0$. □

We now consider the case of a general subcontinuum of the range.

Theorem 2.8. *Let $f : X \to Y$ be a weakly confluent map of graphs. There exists a weakly confluent map $g : X \to Y$ such that if K is a subcontinuum of Y, there exists an onto component A of $g^{-1}[K]$ such that for all $y \in K$,*
(2.2)

$$\sum_{w \in g^{-1}[y] \cap A} [\operatorname{ord}(w, X) - \operatorname{ord}(w, A)] \geq \operatorname{ord}(y, Y) - \operatorname{ord}(y, K).$$

Proof: Let $f : X \to Y$ be a weakly confluent map of aphs. For each branchpoint v of Y, let T_v be a simple $\operatorname{ord}(v, Y)$-od in Y with branchpoint v and satisfying that the intersection of T_v with the vertex set of Y is $\{v\}$. Let $m : Y \to Y$ be a monotone map which collapses each T_v to its corresponding $\{v\}$. Let $g = m \circ f$. Since g is the composition of weakly confluent maps, it is itself weakly confluent. We now show that g has the desired properties. Note that once again we interpret the summation to be the finite summation of w's in the boundary of A.

Let K be a subcontinuum of Y. Then $L = m^{-1}[K]$ is a subcontinuum of Y with the properties:

(1) For $z \in L$, $\operatorname{ord}(z, Y) - \operatorname{ord}(z, L) \leq 1$ and
(2) For $y \in K$, there are at least $[\operatorname{ord}(y, Y) - \operatorname{ord}(y, K)]$ points $z \in L$ such that $\operatorname{ord}(z, Y) - \operatorname{ord}(z, L) = 1$ and $m(z) = y$.

By Lemma 2.7, there is an onto component A of $f^{-1}[L]$ such that for $z \in L$,
(2.3)

$$\sum_{w \in f^{-1}(z) \cap A} [\operatorname{ord}(w, X) - \operatorname{ord}(w, A)] \geq \operatorname{ord}(z, Y) - \operatorname{ord}(z, L)$$

Moreover, A is an onto component of $g^{-1}[K]$. Thus, for $y \in K$,

$$\sum_{w \in g^{-1}(y) \cap A} [\operatorname{ord}(w, X) - \operatorname{ord}(w, A)]$$

$$\geq ([\operatorname{ord}(y, Y) - \operatorname{ord}(y, K)]) \left(\sum_{\substack{x \in f^{-1}(z) \cap A \\ m(z) = y}} [\operatorname{ord}(x, X) - \operatorname{ord}(x, A)] \right)$$

$$\geq \operatorname{ord}(y, Y) - \operatorname{ord}(y, K).$$

The first inequality comes from the fact that, for each $y \in K$, there are at least $[\operatorname{ord}(y, Y) - \operatorname{ord}(y, K)]$ points z in L such that $m(z) = y$ and $\operatorname{ord}(z, Y) - \operatorname{ord}(z, L) = 1$. The second inequality comes from Inequality 2.3 and the fact that $\operatorname{ord}(z, Y) - \operatorname{ord}(z, L) = 1$. □

Corollary 2.9. *Let $f : X \to Y$ be a weakly confluent map of graphs. There exists a weakly confluent map $g : X \to Y$ such that if K is a subcontinuum of Y, there exists an onto component A of $g^{-1}[K]$ such that ord$(A, X) \geq$ ord(K, Y).*

Proof: Let $f : X \to Y$ be a weakly confluent map of graphs. Let K be a subcontinuum of Y. Let $g = m \circ f$ as in the proof of Theorem 2.8 and let A be one of the onto components of $g^{-1}[K]$ satisfying the conclusion of that theorem. Then, $\operatorname{ord}(A, G) = \sum_{x \in K} \left[\sum_{w \in g^{-1}(x) \cap A} [\operatorname{ord}(w, X) - \operatorname{ord}(w, A)] \right]$ from Lemma 1.3 since $g[A] = K$. From Theorem 2.8, we obtain

$$\sum_{x\in K}\left[\sum_{w\in g^{-1}(x)\cap A}[\operatorname{ord}(w,X)-\operatorname{ord}(w,A)]\right]$$
$$\geq \sum_{x\in K}[\operatorname{ord}(x,Y)-\operatorname{ord}(x,K)].$$

The right-hand side of this inequality equals $\operatorname{ord}(K,Y)$ by Lemma 1.3 □

An example of the application of this theorem is now given.

Example 2.10. Consider the graphs in Figure 1. We will show that neither is the weakly confluent image of the other. We first attempt to construct a map going from Z (right) to A (left). Note that in Z, any subcontinuum of order 5 must contain two adjacent edges from the edges labelled x, y, and z.

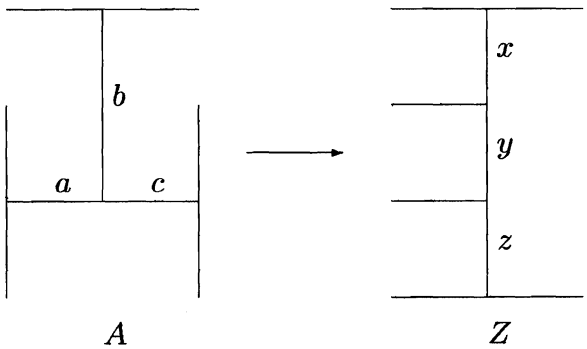

Figure 1

Thus, given any three such subcontinua, the intersection of some pair must contain either the edges x and y or the edges y and z. According to Corollary 2.9, if there were a weakly confluent map $f : Z \to A$, there must be three distinct subcontinua of order at least 5; one each for $a \cup b$, $b \cup c$, and $a \cup c$. Thus, without loss of generality, the onto components of $a \cup b$ and $b \cup c$ both contain $x \cup y$. Thus, x and y must both map to b. However, every subcontinuum of order at least four in Z contains at least one of the edges x, y, and z. There must be a subcontinuum of order at least four to be an onto component of $f^{-1}[a]$, $f^{-1}[b]$, and $f^{-1}[c]$ from Corollary 2.9. But, with x and y both mapping to b, there is not enough room in Z for both $f^{-1}[a]$ and $f^{-1}[c]$ to have onto components. Thus there is no weakly confluent map from Z to A.

We now attempt to find a weakly confluent map $g : A \to Z$. Any subcontinuum of order at least four in A contains at least one of the edges a, b, or c. Thus, the onto components for $f^{-1}[x]$, $f^{-1}[y]$, and $f^{-1}[z]$ each contain a different one of the edges a, b, and c; otherwise there would not be enough room in A to find three distinct subcontinua of order at least four that could serve as onto components for $f^{-1}[x]$, $f^{-1}[y]$, and $f^{-1}[z]$ as required by Corollary 2.9. However, now there is a continuity problem as the middle branchpoint of A is forced to try to map to both of the middle branchpoints of Z.

3. A Labelling Theorem

The goal of this section is to provide a theorem (Theorem 3.3) relating a weakly confluent map of graphs to a vertex-edge function (or labelling) between the same two graphs.

Lemma 3.1. *Let $f : X \to Y$ be a weakly confluent map of graphs. There is another weakly confluent map $g : X \to Y$ such that if K is a subcontinuum of Y which is irreducible about a subset of the branchpoints of Y, then there is an onto component A of $g^{-1}[K]$ satisfying the condition: For each edge E of Y not contained in K but incident to a point $y \in bd\ (K)$, there is an arc $I \subset X$ and a point $x \in A$ such that x is an endpoint of I, $I \cap A = \{x\}$, $g(x) = y$, and $g[I] \subset E$.*

Proof: Let $f : X \to Y$ be a weakly confluent map of graphs. Let $g = m \circ f$ as in the proof of Theorem 2.8. Let K be a subcontinuum of Y which is irreducible about some subset of the branchpoints of Y and E an edge of Y not contained in K but incident to a point $y \in$ bd (K). There is a point $z \in m^{-1}(y) \cap$ bd$(m^{-1}[K])$ such that $z \in m^{-1}[E]$. Let A be an onto component of $g^{-1}[K]$ which satisfies the conclusion of Theorem 2.8.

If no arc I as described above exists, for every $p \in f^{-1}(y)$ such that $\text{ord}(p, X) - \text{ord}(p, A) \geq 1$ and for each $\varepsilon > 0$, every arc J with p as one endpoint and diameter less than ε satisfies $f[J] \cap (E - \{y\}) \neq \emptyset$ and $f[J] \cap (Y - E) \neq \emptyset$. Thus, the f-preimage of the arc between z and y would have infinitely many distinct onto components. This contradicts Lemma 2.2. □

Lemma 3.2. *Let $f : X \to Y$ be a weakly confluent map of graphs. Then, there is another weakly confluent map $g : X \to Y$ which has the property that if E is an edge of Y, then $g^{-1}[E]$ has finitely many components whose images under g intersect $int(E)$.*

Proof: Let $f : X \to Y$ be a weakly confluent map of graphs and let $g = m \circ f$ as in the proof of Theorem 2.8. Assume E is an edge of Y such that there are infinitely many components of $g^{-1}[E]$ which intersect $\text{int}(E)$. The boundary points of each of these components must map to the endpoints of E, so infinitely many of them must map to the same endpoint, say p, of E. Without loss of generality, p is not an endpoint of Y since, if it were, $g^{-1}[E]$ would have infinitely many components contradicting Lemma 2.2.

Consider the arc $m^{-1}(p) \cap E$. Since there are infinitely many components of $g^{-1}[E]$ intersecting $\text{int}(E)$ with boundary points mapping to p under f, the preimage of this arc under f has infinitely many onto components contradicting Lemma 2.2. □

Remark 2. *We have chosen the same alternative weakly confluent map in the proofs of Theorem 2.8, Lemma 3.1, and Lemma 3.2. So, given a weakly confluent map $f : X \to Y$ of graphs, we may suppose, without loss of generality, that it satisfies the conclusions of those three results.*

We now consider the case of nondegenerate graphs without branchpoints (that is, arcs and simple closed curves). The only nondegenerate weakly confluent images of an arc or a simple closed curve are arcs and simple closed curves [EFG, Corollary I.2] and every nondegenerate graph maps weakly confluently onto either an arc or a simple closed curve. The next theorem considers the case when both domain and range graphs have at least one branchpoint. As this theorem is highly technical, examples will be supplied to illustrate the theorem.

Theorem 3.3. *Let X and Y be graphs which have at least one branchpoint. There exists a weakly confluent map $f : X \to Y$ if and only if there is a subdivision of X and a function φ from a subset D of the vertices and edges of X to the set of the vertices and edges of Y such that:*

(1) *if $x \in D$ is a vertex of X, $\varphi(x)$ is a vertex of Y;*
(2) *the function φ preserves incidence of vertices and edges, adjacency of edges, and adjacency of vertices; and*
(3) *for every subcontinuum K of Y irreducible about a subset of the branchpoints of Y, there is a set $A \subset D$ of edges of X (if K is degenerate, A may consist of a single vertex of X) such that:*
   1. *$\bigcup_{e \in A} e$ is connected,*
   2. *$\bigcup_{e \in A} \varphi(e) = K$, and*
   3. *for each edge M of Y not in K but incident to a point k in the boundary of K, there is an edge $B \in D$ and an endpoint b of B such that $(\bigcup_{e \in A} e) \cap B = \{b\}$, $\varphi(b) = k$, and $\varphi(B) = M$.*

Proof: Let X and Y be graphs which have at least one branchpoint.

Sufficiency: Suppose a labelling φ exists. We construct a map $f : X \to Y$ as follows:

Step 1: For $x \in D$ such that x is a vertex of X, let $f(x) = \varphi(x)$.

Step 2: For $e \in D$ such that e is an edge of X, if $\varphi(e)$ is a vertex of Y, then let f map e constantly to that vertex. Otherwise, if e is an arc, let f map e onto $\varphi(e)$ such that the endpoints of e map to the endpoints of $\varphi(e)$ subject to the assignments from Step 1. If e is a loop, let f map e onto $\varphi(e)$ such that the boundary point of e maps to a boundary point of $\varphi(e)$ subject to the assignments from Step 1.

Step 3: Extend f continuously to $X - D$.

By its construction and conditions 1 and 2 of φ, f is continuous. We now show that f is weakly confluent. Let L be a subcontinuum of Y. Then, L is either contained in the interior of some edge of Y or is the union of some subcontinuum K of Y which is irreducible about a subset of the branchpoints of Y and, possibly, some subedges of Y that intersect K at one of their boundary points. In the first case, since every edge of Y is covered by some edge of X under f, L is covered by some subcontinuum of X. In the second case, condition 3 of φ guarantees that L is covered by the $\bigcup_{e\in A} e$ from that condition and the B's from part c of that condition.

Necessity: Let $f : X \to Y$ be a weakly confluent map of graphs. Without loss of generality, we may suppose that f satisfies the conclusions of Theorem 2.8, Lemma 3.1, and Lemma 3.2. For each subcontinuum K of Y irreducible about a subset of the branchpoints of Y, let A_K be an onto component of $f^{-1}[K]$ satisfying the conclusion of Theorem 2.8. According to Lemma 3.2, it is possible to choose a finite vertex set V of X such that the standard vertex set is contained in V; for each A_K, $\text{bd}(A_K) \subset V$; and every edge in the corresponding edge set maps into one edge or vertex of Y under f.

Let D be the union of the edge set corresponding to V and the union of the boundaries of the A_K's. For each $x \in V \cap D$, let $\varphi(x) = f(x)$. For each edge e of X in D, let $\varphi(e)$ be the edge or vertex it maps to under f. Since f sends the boundary points of the A_K's to vertices of Y, condition 1 holds. Condition 2 holds since f is continuous. Parts a and b of condition 3 hold since each A_K is the union of a collection of edges of X

and $f[A_K] = K$. Lemma 3.1 says condition 3c holds for the A_K's. □

We provide two examples of the labelling techniques from Theorem 3.3. The first is an example of a labelling which satisfies the conditions and, thus, a weakly confluent map can be constructed between the graphs in that example. The second is a demonstration of how to show that no labelling, and consequently no weakly confluent map, exists for two given graphs.

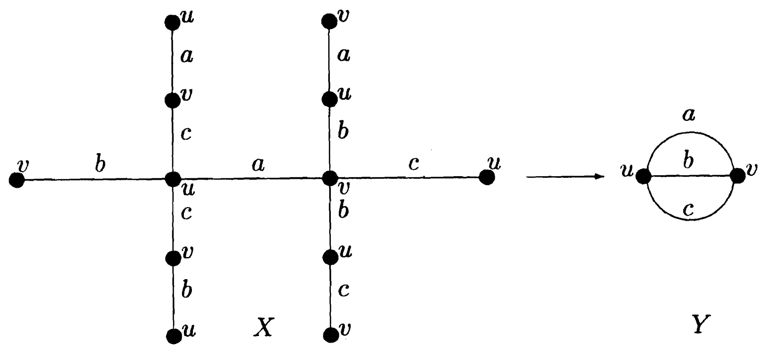

Figure 2

Example 3.4. Consider the graphs in Figure 2. The labelling on Y is for identification purposes; the labelling on X represents a function φ as in Theorem 3.3. We check to see if this labelling on X satisfies the conditions of Theorem 3.3. It is easily verified that conditions 1 and 2 both hold. There are 5 subcontinua of Y that are irreducible about a subset of the branchpoints of Y: $\{u\}$, $\{v\}$, and the edges a, b, and c. For $\{u\}$, the point of order 4 labelled u covers u under f; $\{u\}$ has three edges, a, b, and c , coming off of it in Y and the order four point has three edges coming off it, one each labelled a, b, and c. Thus condition 3 is satisfied for the subcontinuum $\{u\}$ of Y. The subcontinuum $\{v\}$ is checked in a similar fashion.

Now consider the edge a of Y. This edge is covered by the middle horizontal edge labelled a in X. There are two points in the boundary of the edge a in Y, namely u and v. The order-four points labelled u and v in X map to these points respectively. Each of the points u and v in bd(a) have two edges coming off of them: b and c. But, each of the order-four points in X have two edges coming off of them: one labelled b and one labelled c. Thus, condition 3 holds for edge a.

The edge b is covered by the middle half of the rightmost vertical arc of X; call the union of these two edges A. The edge b in Y has two boundary points, u and v, and each of these has two edges coming off of it: a and c. The order-four point in bd(A) in X is labelled v and has an edge coming of it labelled a and one labelled c. The two endpoints of A are both labelled u; the upper one has an edge coming off of it labelled a, the lower an edge labelled c. Thus, condition 3 holds for the edge b. A similar argument shows that condition 3 holds for the edge c in Y. Thus the map f constructed as in the proof of Theorem 3.3 from this labelling is weakly confluent.

Note that in the argument for edge b there were two points in X with one edge coming off of each one mapping to a single point, u, with two edges coming off of it in Y. This is an example of the summation in Theorem 2.8.

Example 3.5 Example 2.10 is actually not only an application of Corollary 2.9 but also a demonstration that no labelling of the type described in Theorem 3.3 exists for the graphs in Example 2.10 and, hence, there is no weakly confluent from either of those graphs to the other.

References

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