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WEAK NORMALITY IN DOWKER SPACES

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ABSTRACT. It is shown that the product of Rudin's Dowker space with the closed unit interval is weakly normal. This answers a question of Arhangel'skii and gives further evidence for his conjecture that weak normality is preserved under products with a compact second countable factor. Also, an analogue to Dowker's Theorem for weakly normal spaces is proven.

A space X is said to be weakly normal (over R^ω) if for each pair of disjoint closed subsets A and B there is a continuous function $F : X \to R^\omega$ such that $F(A) \cap F(B) = \emptyset$. The concept of weak normality was introduced by A.V. Arhangel'skii in [A] where one may find the basic results on weakly normal spaces.

N denotes the set of natural numbers and throughout this note we let X denote M.E. Rudin's Dowker space [R1]:

$$X = \{x \in \Pi_{i\in N}(\omega_i + 1) : \exists n \in N \ \forall i \in N \ \omega < \operatorname{cof}(x(i)) < \omega_n\}.$$

The box product topology on $\Pi_{i\in N}(\omega_i+1)$ is denoted $\square_{i\in N}\omega_i+1$ and $X \subseteq \square_{i\in N}\omega_i + 1$ inherits the subspace topology. X is zero-dimensional and is a P-space (i.e., countable intersections of open sets are open).

Theorem 1. $X \times [0,1]$ *is weakly normal.*

was motivated by the following conjecture of A.V. Arhangel'skii.

Conjecture 2. (Arhangel'skii)*: If Y is normal, then $Y \times [0,1]$ is weakly normal.*

Even the stronger conjecture that weak normality is preserved under products with a compact second countable space is open.

Conjecture 3. (Arhangel'skii)*: If Y is weakly normal, then $Y \times [0,1]$ is weakly normal.*

Note that it is easy to prove that the product of a weakly normal space with any countable Tychonoff space is weakly normal. In particular, if Y is weakly normal, then $Y \times (\omega + 1)$ is weakly normal.

We will say that a space Y is strongly-Dowker if it is normal, while $Y \times [0,1]$ is not weakly normal. Any stronger topology on a subspace of R^ω is weakly normal (the identity map uniformly separates all pairs of disjoint closed sets). Therefore the Dowker spaces constructed by refining the Euclidean topology on a set of reals are not strongly-Dowker. Likewise it is easy to see that Dowker spaces with the property that any two uncountable closed subsets intersect are not strongly-Dowker. Therefore, the deCaux type Dowker spaces are not strongly-Dowker. See Rudin's survey article [R2] or for a more recent survey of Dowker spaces see [SW]. Recently Z. Balogh has constructed another ZFC Dowker space [B]. We don't know whether this space is strongly-Dowker.

Before we prove Theorem 1.1 need some notation. For any space Y, any set $A \subseteq Y \times [0,1]$ and any $r \in [0,1]$ let

$$A_r = \{y \in Y : (y,r) \in A\}$$

For $i \in N$ let π_i denote the projection map $\pi_i : X \to \omega_i + 1$. For any open set $U \subseteq X$, $t_U \in \Pi_{i \in N}\omega_i + 1$ is defined by

$$\forall i \in N, t_U(i) = sup\{x(i) : x \in U\}.$$

For functions $f, g \in \Pi_{i\in N}\omega_i + 1$, $f < g$ means that $f(i) < g(i)$ for each $i \in N$ and $f \leq g$ means that $f(i) \leq g(i)$ for each $i \in N$. The half open interval $(f, g]$ denotes the set $\{h \in X : f < h \leq g\}$, a basic open subset of X.

Lemma 4. *Let X be Rudin's Dowker space. For any pair of disjoint closed sets A and $B \subseteq X \times [0,1]$ there is a disjoint clopen cover $\mathcal{U}$ of X such that*

(1) $|\mathcal{U}| \leq 2^\omega$.
(2) *For each $U \in \mathcal{U}$ and each $r \in [0,1]$ either $U \cap A_r = \emptyset$ or $U \cap B_r = \emptyset$.*

Proof: The proof is similar to Rudin's proof that X is collectionwise normal. For each $\alpha < \omega_1$ we define a pairwise disjoint clopen cover T_α of X such that

(a) $|T_\alpha| \leq 2^\omega$.
(b) For each $\beta < \alpha < \omega_1$ and each $V \in T_\alpha$, there is a $U \in T_\beta$ such that
  (1) $V \subseteq U$.
  (2) If there is an $r \in [0,1]$ such that both $V \cap A_r \neq \emptyset$ and $V \cap B_r \neq \emptyset$ then $t_U \neq t_V$.
  (3) If for each $r \in [0,1]$ either $U \cap A_r = \emptyset$ or $U \cap B_r = \emptyset$ then $U = V$.

The T_α are defined by induction on $\alpha < \omega_1$. Let $T_0 = \{X\}$ and suppose that for each $\beta < \alpha$, T_β has been defined.

Case 1: α is a limit.
For each $x \in X$ and each $\beta < \alpha$ let $U_x(\beta)$ be the unique element of T_β containing x. For each $x \in X$ let $U_x = \bigcap_{\beta<\alpha} U_x(\beta)$. Since X is a P-space, if we let

$$T_\alpha = \{U_x : x \in X\}$$

then T_α is a pairwise disjoint clopen cover of X. It is easy to verify that the inductive hypotheses (a) and (b) are preserved (see [R]).

Case 2: $\alpha = \beta + 1$.
For each $U \in T_\beta$ we define T_U, a disjoint clopen cover of U of size 2^ω such that for each $V \in T_U$, U and V satisfy (1)–(3) of inductive hypothesis (b). Having done this we will let $T_\alpha = \bigcup\{T_U : U \in T_\beta\}$. If for each $r \in [0,1]$ either $U \cap A_r = \emptyset$ or $U \cap B_r = \emptyset$, then we must let $T_U = \{U\}$. So suppose that U

intersects both A_r and B_r for some $r \in [0,1]$ and consider t_U. As in [R1] our proof depends on the cofinalities of the range values of t_U. If there is an $i \in N$ such that $\mathrm{cof}(t_U(i)) = \omega$, then we fix such an i and fix an increasing sequence of ordinals $(\alpha_n)_{n\in\omega}$ cofinal in $t_U(i)$. Let

$$T_U = \{U \cap (\pi_i^{-1}((\alpha_n, \alpha_{n+1}])) : n \in \omega\}$$

Since $t_V(i) < t_U(i)$ for each $V \in T_U$, the family T_U is as required.

Therefore we may assume that $\mathrm{cof}(t_U(i)) > \omega$ for each $i \in N$. One may take care of this case as in Rudin's proof of Lemma 5 in [R1]. However, the following elementary submodel proof distills the essential closing off argument in that proof. The survey [D] is a good reference for the reader unfamiliar with elementary submodel techniques. Fix θ a regular cardinal large enough so that any relevant properties are absolute for $V, H(\theta)$ ($\theta = 2^{2^{\aleph_\omega}}$ suffices). For each $n \in N$ fix an elementary submodel $\mathcal{M}_n \prec H(\theta)$ of size ω_n containing X, A, B, U and anything else relevant. We also require that $\mathcal{M}_n$ is *ω-covering*, i.e., for each countable subset $D \subseteq \mathcal{M}_n$ there is a countable $E \in \mathcal{M}_n$ such that $D \subseteq E$. Define a function h_n on N by

$$h_n(i) = \sup(t_U(i) \cap \mathcal{M}_n).$$

Notice that for each $i \in N$,

$$\omega < \mathrm{cof}(h_n(i)) \leq \omega_n.$$

The first inequality holds since $\mathcal{M}_n$ is ω-covering, and the second is true since ω_n is both an element and a subset of $\mathcal{M}_n$. Therefore $h_n \in X$. Also, for each $i \in N$ if $\mathrm{cof}(t_U(i)) \leq \omega_n$ then $h_n(i) = t_U(i)$. For each $r \in [0,1]$, h_n is not in both A_r and B_r. Therefore there is a finite open cover $\mathcal{W}$ of $[0,1]$ and for each $W \in \mathcal{W}$ a function $g_W < h_n$ such that for each $W \in \mathcal{W}$ either $A \cap ((g_W, h_n] \times W) = \emptyset$ or $B \cap ((g_W, h_n] \times W) = \emptyset$. Note that the g_W's may be chosen so that $g_W(i) \in \mathcal{M}_n$ for each $i \in N$. Now let g_n be defined by $g_n(i) = \max\{g_W(i) : W \in \mathcal{W}\}$. Then $g_n < h_n$ and for each $r \in [0,1]$ either $(g_n, h_n] \cap A_r = \emptyset$ or

$(g_n, h_n] \cap B_r = \emptyset$. We now claim that there is a function g'_n satisfying the previous statement which is an element of $\mathcal{M}_n$. To see this, use ω-covering to fix a countable set $D \in \mathcal{M}_n$ containing g_n and let g'_n be defined by $g'_n(i) = sup\{\alpha < t_U(i) : (i, \alpha) \in D\}$ for each $i \in N$. Then $g'_n \in \mathcal{M}_n$ and since each $h_n(i)$ has uncountable cofinality $g_n(i) \leq g'_n(i) < h_n(i)$ for each $i \in N$. We need one more bit of notation: for each $Y \subseteq X$ and each $n \in N$ let $(Y)^n = \{f \in Y : \forall i \in N \text{ cof}(f(i)) \leq \omega_n\}$. Notice that $h_n(i) = t_U(i)$ whenever $\text{cof}(t_U(i)) \leq \omega_n$ and that $[h_n(i), t_U(i)) \cap \mathcal{M}_n = \emptyset$ whenever $\text{cof}(t_U(i)) > \omega_n$. Therefore $\mathcal{M}_n \cap \{f \in (X)^n : g'_n < f \leq t_U\} = \mathcal{M}_n \cap \{f \in (X)^n : g'_n < f \leq h_n\}$ and

$$\mathcal{M}_n \models \forall r \in [0,1] \text{ either } (g'_n, t_U] \cap (A_r)^n = \emptyset \text{ or } (g'_n, t_U] \cap (B_r)^n = \emptyset.$$

By elementarity this statement is true. Therefore, if we let $g = sup(g'_n)_{n \in N}$, then $\forall r \in [0,1]$ either $(g, t_U] \cap A_r = \emptyset$ or $(g, t_U] \cap B_r = \emptyset$. Now we are ready to define T_U. For each $S \subseteq N$ let

$$U_S = \{x \in U : x(i) \leq g(i) \iff i \in S\}.$$

Then $T_U = \{U_S : S \subseteq N\}$ is a disjoint clopen cover of U. Clearly $t_U \neq t_{U_S}$ for each nonempty $S \subseteq N$ and $U_\emptyset = (g, t_U] \cap U$. Furthermore for each $r \in [0,1]$ either $U_\emptyset \cap A_r = \emptyset$ or $U_\emptyset \cap B_r = \emptyset$. Therefore T_U is as required.

We now define the clopen cover $\mathcal{U}$ from the sequence of covers $(T_\alpha)_{\alpha < \omega_1}$. For $x \in X$ and $\alpha < \omega_1$ fix $U^x_\alpha \in T_\alpha$ such that $x \in U^x_\alpha$. If $\beta < \alpha$ then $t_{U^x_\alpha} \leq t_{U^x_\beta}$. If in addition U^x_α meets both A_r and B_r for some $r \in [0,1]$, then there is an $i \in N$ such that $t_{U^x_\alpha}(i) < t_{U^x_\beta}(i)$. By well foundedness, for each $x \in X$ there is an $\alpha_x < \omega_1$ such that $U^x_{\alpha_x} \cap A_r \neq \emptyset$, then $U^x_{\alpha_x} \cap B_r = \emptyset$ whenever $r \in [0,1]$. Therefore $U^x_\alpha = U^x_{\alpha_x}$ for each $\alpha \geq \alpha_x$. Letting $\mathcal{U} = \{U^x_{\alpha_x} : x \in X\}$ completes the proof of Lemma 4. $\square$

Proof of Theorem 1: Fix A and B disjoint closed subsets of

$X \times [0,1]$. Fix $\mathcal{U}$ given by Lemma 4. Since $|\mathcal{U}| \leq 2^\omega$, there is a countable point separating family of functions for $\mathcal{U}$. I.e. there is a family $\{g_n : n \in N\}$ such that

(c) for each $n \in N$, $g_n : \mathcal{U} \to 2$, and

(d) for each $U \neq V$ from $\mathcal{U}$ there is an $n \in N$, $g_n(U) \neq g_n(V)$.

For example, if for each $s \in 2^{<\omega}$ $f_s : 2^\omega \to 2$ is defined by $f_s(x) = 1$ iff $\forall i < |s|, x(i) = s(i)$, then $\{f_s : s \in 2^{<\omega}\}$ is a countable point separating family for 2^ω.
Note that each g_n defines a partition of X into two clopen sets $\bigcup g_n^{-1}(0)$ and $\bigcup g_n^{-1}(1)$. From the family $\{g_n : n \in N\}$ we define for each $n \in \omega$ functions $f_n : X \times [0,1] \to [0,1]$.
$n = 0$: For each $x \in X$ and $r \in [0,1]$ let $f_0((x,r)) = r$.
$n > 0$: For each $x \in X$ and $r \in [0,1]$ let $f_n((x,r)) = g_n(x)$.
Clearly for $n \in \omega$ each f_n is continuous. Now define $F : X \times [0,1] \to [0,1] \times 2^N$ by $F = \Pi_{n\in\omega} f_n$. Then F is continuous.

The next claim completes the proof of Theorem 1.

Claim 5. $F(A) \cap F(B) = \emptyset$.

Proof: Fix $(x,r) \in A$ and $(y,s) \in B$.

Case 1: $r \neq s$.

Then $f_0((x,r)) = r \neq s = f_0((y,s))$, therefore $F((x,r)) \neq F((y,s))$.

Case 2: $r = s$.

Let $U, V \in \mathcal{U}$ such that $x \in U$ and $y \in V$. Since $x \in A_r$ and $y \in B_r$, Lemma 4(2) implies that $U \neq V$. Therefore by (d) there is an $n > 0$ such that $g_n(U) \neq g_n(V)$. This implies that $f_n((x,r)) \neq f_n((y,r))$ and therefore $F((x,r)) \neq F((y,r))$. □

A similar proof yields the following lemma.

Lemma 6. *Suppose that Y is weakly normal and suppose that for any two disjoint closed subsets A and B of $Y \times [0,1]$ there is a point finite open cover $\mathcal{U}$ of Y such that*

(1) $|\mathcal{U}| \leq 2^\omega$, *and*

(2) *for each* $U \in \mathcal{U}$ *and each* $r \in [0,1]$, *either* $U \cap A_r = \emptyset$ *or* $U \cap B_r = \emptyset$.

Then $Y \times [0,1]$ *is weakly normal.*

Proof: Since $|\mathcal{U}| \leq 2^\omega$ there is a countable *finitely separating* family of functions for $\mathcal{U}$. I.e., there is a family $\{g_n : n \in N\}$ such that

(e) for each $n \in N$, $g_n : \mathcal{U} \to 2$, and
(f) for each pair of disjoint finite subsets F, $G \subseteq \mathcal{U}$ there is an $n \in N$ such that either $g_n(F) = 0$ and $g_n(G) = 1$ or $g_n(F) = 1$ and $g_n(G) = 0$.

For each n let $F_n = X \setminus \bigcup g_n^{-1}(0)$ and let $G_n = X \setminus \bigcup g_n^{-1}(1)$. Then F_n and G_n are disjoint possibly empty closed subsets of X. As $\mathcal{U}$ is point finite and $\{g_n : n \in N\}$ is finitely separating, (2) implies that for any $r \in [0,1]$ and any pair of points $x \in A_r$ and $y \in B_r$ there is an $n \in N$ such that $x \in G_n$ and $y \in F_n$. By weak normality, for each n there is a continuous function $f_n : Y \to [0,1]$ such that $f_n(G_n) \cap f_n(F_n) = \emptyset$. As in the proof of Theorem 1 this implies that $Y \times [0,1]$ is weakly normal. $\square$

One proof that $X \times [0,1]$ is normal assuming X is normal and countably metacompact entails defining a countable open cover of X that satisfies (2) of Lemma 6 (this involves no assumptions on X–see [E]). Therefore we have the proven the following.

Theorem 7. *If* X *is weakly normal and countably metacompact, then* $X \times [0,1]$ *is weakly normal.*

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