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Polynomial-like Property for Real Quadratic Polynomials

Jacek Graczyk* and Grzegorz Świątek†

Abstract

We prove that all renormalizations of infinitely renormalizable real quadratic polynomials are polynomial-like with the modulus bounded from below by a positive constant, independent of a polynomial. By Douady and Hubbard's Straightening Theorem, this means that the renormalizations on appropriately chosen neighborhoods of small Julia sets are quasiconformally equivalent to real polynomials and the maximal dilatation of the straightening map is universally bounded.

1 Introduction.

1.1 General discussion of the problem

Recent progress both in understanding topological aspects of one-dimensional dynamics (real and complex) and in inventing appropriate analytic tools resulted in the complete or partial

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proofs of a few conjectures which for a long time seemed beyond the reach. The current paper constitutes an important technical part of the proof of *Dense hyperbolicity conjecture* for real quadratic polynomials (see [4]).

In [16] there is a result that renormalizations of real quadratic polynomials of so called combinatorially bounded type are subjected to both real and complex universal *apriori bounds.* The result means both the compactness of the renormalization operator as well as a quasiconformal equivalence, through the work of Douady and Hubbard [1], between the renormalizations of combinatorially bounded type quadratic polynomials and prototype mappings $z^2 + c$. The next important consequence of [16] is that all real infinitely renormalizable polynomials of bounded combinatorics are quasisymmetrically conjugate provided that they are topologically conjugate.

Our paper extends Sullivan's result on compactness of the renormalization operator. A feasible consequence of our work may consist in the discovery of a new phenomenon in the renormalization theory. Namely, it is quite plausible that there exists a non-trivial, i.e. different from a quadratic polynomial, limit of the renormalizations of so called almost parabolic, but not combinatorially bounded, infinitely renormalizable quadratic polynomials. The almost parabolic mappings were discovered in this context in [3] as the only possible exceptions for the statement that an increasing return time of the critical value to the restrictive interval, along the subsequent renormalizations, necessarily yields trivial accumulation points of the renormalization operator. (see [3], Theorem 2).

The current paper in its topological part is based on the method of *inducing* while in the analytical is a combination of real and complex estimates.

The method of inducing in the study of unimodal maps turned out to be especially fruitful. The first description of this method may be found in the work of [7] and next in [6] where an attempt was made to build up a general topological framework of the unimodal dynamics. The clear and systematic implementation of the method of inducing to the study of dynamics of unimodal maps appeared in the paper of [8], were so called *box geometry* and *starting condition* was provided. The former notion describes a non-hyperbolic part of dynamics while the starting condition, if satisfied, would lead to at least exponential decay of *box geometry*.

Another important progress is in the work of [17] (see [13]) on non-renormalizable quadratic polynomials. Yoccoz constructed, using external rays and potential lines, a Markov partition of a neighborhood of the Julia set. Refinement of the initial partition through the pullback construction combined with a computation based on estimates of moduli gives the local connectivity of appropriate Julia sets and the monotonicity conjecture for non-renormalizable quadratic polynomials. In the real case the independent approach in [3] based on induction which comprised both moduli and cross-ratio arguments resulted in a considerable strengthening of Yoccoz result. The moduli of annuli forming an infinite nest in the Yoccoz partition occurred not only be divergent but linearly growing with the number of appropriately counted levels. This resulted in a very strong distortion estimates which "almost" established the complex bounds for renormalization. The only gap consisted again in the "hard" *almost parabolic case*. The current paper fills out this missing point.

A stronger result (proved by a different method) was obtained about the same time by Levin and van Strien [9] for all real infinitely renormalizable polynomials regardless of the degree. An estimate on moduli of polynomial-like renormalizations was provided depending only on the singularity of a critical point. Results similar to ours were also reported by others, see [18] for a survey. The construction of [9] used in conjunction with these bounds give the local connectivity of the Julia sets for all real infinitely renormalizable polynomials of the form $z^\ell + c$. The construction of [9] works also in the case of non-renormalizable unimodal polynomials extending the results on the local connectivity of the Julia sets over all real polynomials of the form $z^\ell + c$. Again the complex bounds were involved in the proof, this time for non-renormalizable polynomials (see also [5]).

1.2 Main theorems

Quadratic-like maps. A holomorphic map of large topological degree may sometimes behave in the small scale like a quadratic polynomial. This intriguing phenomenon is indeed observed while studying renormalizations of rational mappings. One of the most important and the most difficult questions consists in estimating how well this low degree behavior in the small scale approximates the local dynamics. The concept of quadratic-like maps due to Douady and Hubbard turned out to be fruitful in pursuing both qualitative and quantitative aspects of these low degree approximations of rational maps.

Definition 1.1 *Let U and V be open topological discs, and let $\overline{U} \subset V$. Then a proper holomorphic map $g : U \xrightarrow{\text{onto}} V$ of degree 2 will be called* quadratic-like. *We call the modulus $\operatorname{mod}(V \setminus U)$ the* complex bound *of g.*

The filled Julia set $\mathcal{K}(g)$ of g is defined by

$$\mathcal{K}(g) = \bigcap_{i=1}^{\infty} g^{-i}(U).$$

By performing quasiconformal surgery one can conjugate quasiconformally a quadratic-like map on a neighborhood of its filled Julia set with a quadratic polynomial. A precise formulation of this result will follow later when it is needed.

Statement of results. Our result is a generalization of Sullivan's "complex bounds", see [16], for infinitely renormalizable real quadratic polynomials with bounded combinatorics. If f is a real quadratic polynomial with the critical point at c, let us recall that an interval $I = (c-t, c+t), t > 0$, is called *restrictive* if there is an integer $n \geq 2$ so that intervals $I, f(I), \cdots, f^{n-1}(I)$ are disjoint and $f^I \subset I$. Then n will be called the *return time* of I. Then f is called infinitely renormalizable it there is a sequence of restrictive intervals with return times going to infinity.

Theorem 1 *Let f be an infinitely renormalizable real quadratic polynomial. There is a constant $K > 0$ so that if I is a restrictive interval with return time n, then f^n considered as a holomorphic map on some open topological disk $U \supset I$ is quadratic like with complex bound at least K.*

1.3 A few definitions

Definition 1.2 *f is called unimodal if*

- $f(-1) = -1$
- *the mapping f can be written as $h(z^2)$ where h is an orientation-reversing diffeomorphism from $[0,1]$ onto its image $[-1, a]$ where $0 < a < 1$.*

Next, we can classify unimodal mappings according to their topological dynamics or according to the smoothness of h. So, we get the following definitions.

Definition 1.3 *Let $\eta > 0$. A mapping $f : [-1,1] \to [-1,1]$ is said to belong to the class $\mathcal{F}_\eta$ if $f(x) = H(x^2)$ for $x \in [-1,1]$ and is an entire function mapping $\mathbf{R}$ into itself. Moreover, the following restrictions are imposed on H:*

- $H(1) = -1,$
- *the map H has no critical values except in $\mathbf{R} \backslash (-\eta, 1+\eta)$,*

- *there is an inverse branch of H defined in* $\mathbf{R}\setminus(-\eta, 1+\eta)$ *so that* $H^{-1} \circ H$ *is the identity on* $[0,1]$.

Up to normalization, our class $\mathcal{F}_\eta$ is the same as the *Epstein class* which appears in the literature, see [11], page 448. We will talk about *Epstein diffeomorphisms* a little later. The fact that H^{-1} must not increase Poincaré distance is easily translated to H restricted to the real line having non-positive Schwarzian derivative.

Then we also consider the class of mappings $\mathcal{F} = \bigcup_{\eta>0} \mathcal{F}_\eta$. We will refer to the members of $\mathcal{F}$ as *unimodal mappings*.

1.4 Some non-Euclidean geometry

Geodesic neighborhoods. This convenient tool was introduced in [16]. Consider an interval $[x-y, x+y]$, $y>0$, on the real line.

Definition 1.4 *Look at two circles passing through $x-y$ and $x+y$, one centered at $x+it$, and the other at $x-it$ where $t \in \mathbf{R}$. Let $0 \le \alpha \le \pi/2$ be the angle of intersection between the circles and the real line.. Consider the two open discs delimited by these circles. The union of these discs will be called the* geodesic neighborhood *of $[x-y, x+y]$ with angle $\pi-\alpha$, and denoted $\mathcal{D}(\pi-\alpha, [x-y, x+y])$. Likewise, the intersection of these discs will be called the* geodesic neighborhood *of $[x-y, x+y]$ with angle α and denoted $\mathcal{D}(\alpha, [x-y, x+y])$.*

Epstein diffeomorphisms. Let us introduce a class of real-analytic diffeomorphisms.

Definition 1.5 *Let a diffeomorphism h of an interval I_1 onto its image I_2 be called an* Epstein diffeomorphism *if h^{-1} has an analytic continuation to a univalent mapping of $\bigcup_{0<\theta<\pi} \mathcal{D}(\theta, I_2)$ into $\bigcup_{0<\theta<\pi} \mathcal{D}(\theta, I_1)$.*

Properties of Epstein diffeomorphisms. The following observations are based on the contraction property of univalent maps in Poincaré metric. Proofs can be found in [11].

Fact 1.1 *Let h be an Epstein diffeomorphism of an open interval $(a,b) \subset \mathbf{R}$ onto its image (a',b'). Then, for any $\alpha \in (0,\pi)$ the preimage of $\mathcal{D}(\alpha,(a',b'))$ by the analytic continuation of h^{-1} is contained in $\mathcal{D}(\alpha,(a,b))$.*

Fact 1.2 *If h is an Epstein diffeomorphism then its Schwarzian derivative is non-positive.*

1.5 Box mappings

Definition 1.6 *Let $U \ni 0$ be an open subset of $\mathbf{R}$. A map ϕ from U into a bounded subset of $\mathbf{R}$, together with a choice of an interval B', (called* the box*) is called a* real box mapping *if it satisfies the following conditions:*

1. *the connected component of U which contains 0, further called B, has the symmetric form $(-c,c)$ and ϕ restricted to B can be factored as $\phi(x) = h(x^2)$ where h is an Epstein diffeomorphism of some interval $(-\epsilon, c^2)$, $\epsilon > 0$, onto its image, further called B',*

2. *if ζ is the restriction of ϕ to any connected component of U other than B, then ζ is an Epstein diffeomorphism onto its image,*

3. *the boundary of B' is disjoint from U and if ζ is as in item 2., then the boundary of the range of ζ is disjoint from U,*

4. *$B' \ni \overline{B}$ and if ζ is as before, then the range of ζ contains B.*

Although formally a box mapping is a conglomerate of ϕ and B', we will frequently refer to just ϕ as a box mapping, with the understanding that B' is also somehow chosen. Once a box mapping ϕ is given, its restriction to any connected component of its domain will be called a *branch* of ϕ. Items 1. and 2. can be summarized by saying the all branches are monotone, except for the central one which is folding. Item 3. is a "Markov property", which will allow us to compose box mappings among themselves. The last item is to rule out uninteresting cases.

Complex box mappings.

Definition 1.7 *Let $U \ni 0$ be a union of open disjoint topological disks in* $\mathbf{C}$*, embedded so that their boundaries are Jordan curves. A holomorphic map ϕ from U into a bounded subset of* $\mathbf{C}$ *is called a* complex box mapping *if it satisfies the following conditions:*

1. *the connected component of U which contains 0, further called B, is symmetric with respect to 0 and if ψ denotes ϕ restricted to B, then ψ is proper of degree 2 and can be factored as $\phi(x) = h(x^2)$ where h is univalent onto the range of ψ, further called B',*

2. *if ζ is the restriction of ϕ to any connected component of U other than B, then ζ is a univalent onto its image,*

3. *the restriction of ϕ to any connected component of U has a continuous continuation to the closure of this component,*

4. *the boundary of B' is disjoint from U and if ζ is as in item 2., then the boundary of the range of ζ is disjoint from U,*

5. $B' \supset \overline{B}$ *and if* ζ *is as before, then the range of* ζ *contains* B.

This definition is extremely similar to the definition of the real box mapping. The only addition is item 3. which in the real case follows from other assumptions. We define branches as restrictions of ϕ to connected components of U, and we can talk of univalent branches and the central one. Note that the central branch is quadratic-like (comp. Definition 1.1) with complex bound mod $B' \setminus \overline{B}$.

Analytic continuation of real box mappings.

Definition 1.8 *Consider a real box mapping* (ϕ, B'). *If* Φ *a complex box mapping and* Φ *restricted to the real line equals* ϕ, *we say that* Φ *is an* analytic continuation *of* (ϕ, B').

A question naturally arises when does a real-analytic real box mapping have an analytic continuation. This question is relevant to Theorem 1 in the following way. If I is a locally maximal restrictive interval and n is its return time, then it is well known that the end-points of I are a repelling periodic point of f^n and its symmetric. If B is chosen to be a small neighborhood of I, then f^n restricted to B becomes a real box mapping with just one branch. What we claim in Theorem 1 is that this mapping has an analytic continuation. To see the connection more broadly, let us discuss *inducing.*

1.6 Inducing

Inducing generally refers to defining a new mapping piecewise as iterations of an old one. In this paper, we will rely on inducing in two contexts. First, given a unimodal infinitely renormalizable map, we can build its *canonical induced map* which turns out to be a real box mapping with all branches given by some iterates of the original unimodal transformation.

The canonical induced map. It is well known that if a unimodal map $f(0)$ is infinitely renormalizable, then all its periodic orbits are repelling. Also, $f(0) > 0$ in our normalization. Consider the fixed point $q > 0$. Define ϕ_0 to be the first return map of f into $(-q, q)$. One can directly observe that ϕ_0 has one central folding branch and $2k$ symmetric monotone branches mapping over $(-q, q)$ where k could be any non-negative integer. We can also choose $B' = (-q, q)$ and try to show that (ϕ, B') is a real box mapping. This is true when $k > 0$ and there is no trouble verifying the properties required by Definition 1.6. Recall that since $f \in \mathcal{F}$, f has non-positive Schwarzian derivative. The case of $k = 0$ is special and ϕ_0 is not a box mapping with $B' = (-q, q)$ since $B' = B$ contradicts item 4. of the definition.

Inducing by box mappings. We start with a definition.

Definition 1.9 *Let ϕ be box mapping, real or complex. Define its* tempered version *$\mathcal{T}(\phi)$ as a mapping equal to ϕ except on the central domain B, and the identity on B.*

Notice that the tempered map satisfies the Markov property: the boundary of the range of any of its branches is disjoint from the domain.

Definition 1.10 *Let ϕ be a box mapping, real or complex, and $\mathcal{T}(\phi)$ denote its tempered version. A box mapping ϕ' is said to be* directly induced *by ϕ if every branch of ϕ' is equal to a branch of the map*

$$\mathcal{T}(\phi) \circ \cdots (n \text{ times}) \circ \mathcal{T}(\phi) \circ \phi \,.$$

Here, n is a non-negative integer and may vary from one branch of ϕ' to another. The domain of the composition is determined according to the usual convention, and a branch of the composition is its restriction to a connected component of the domain.

The notion of ϕ' being induced *by ϕ is defined by the requirement that either $\phi' = \phi$ or ϕ' is directly induced by some box mapping induced by ϕ.*

Critical completeness.

Definition 1.11 *A box mapping ϕ (real or complex) is called* critically complete, *if and only if the infinite orbit of* 0 *by ϕ is defined, that is, the critical orbit never leaves the domain of ϕ.*

Observe that the canonical induced map is critically complete provided that f is infinitely renormalizable. In every inducing algorithm we will seek to preserve critical completeness, since otherwise the most crucial piece of information about the dynamics of f is lost.

1.7 Sketch of the proof

With the notions introduced so far, we are ready to sketch the proof of Theorem 1. The main step is contained in the following:

Theorem 2 *Let $f \in \mathcal{F}_\eta$ be infinitely renormalizable, and let n be the minimum of return times for all restrictive intervals of f. For every $\eta > 0$ there are numbers $N \geq 3$ and $K > 0$ so that if $n \geq N$, then there is a critically complete real box mapping ϕ induced by the canonical induced map ϕ_0 which has an analytic continuation Φ, in the sense of Definition 1.8. If D and D', respectively, denote the domain and range of the central branch of Φ, then $\bmod (D' \setminus \overline{D}) \geq K$ and every univalent branch of Φ maps over D'.*

We will outline the proof of Theorem 2 later. Once Theorem 2 has been established, we will to proceed to induce on Φ, that is to create box mappings induced by Φ. This will be done according to a specific algorithm the details of which are unimportant. The relevant properties are these: the procedure will furnish a box mapping the central branch of which is f^n, and in the process of inducing the modulus $\bmod (B' \setminus \overline{B})$ decreases by a factor of 4 at most. Hence the central branch of this terminal box mapping will provide that needed quadratic-like map with complex bound at least $K/4$.

To prove Theorem 1, we will pick a locally maximal restrictive interval I with respect to the ordering by inclusion and look either at the original polynomial, if this is the maximal restrictive interval (corresponding to the first renormalization), or the first return map to the next bigger restrictive interval. Denote this map by f. It is well known that $f \in \mathcal{F}_\eta$ where η is a positive constant and that I is the maximal restrictive interval for f. Now the reasoning based on Theorem 2 and given in the preceding paragraph, implies that if the return time of I, in terms of f, is sufficiently large, then the quadratic-like continuation of f^n exists as postulated by Theorem 1.

It remains to consider the case of n bounded. In this case, one has to look further back into the history of I to the lowest return time n_1 where the previous argument applies, or all the way back to the original polynomial. Without loss of generality, we may assume that $n_1 = 1$. Otherwise, knowing already that f^{n_1} is polynomial-like, we use the straightening theorem of Douady and Hubbard to see that f^{n_1} is quasiconformally conjugate to a quadratic polynomial. Then conjugacy send I to a restrictive interval I' for this polynomial and it a quadratic-like continuation can be constructed for the first return map into I', it will be carried over to the original phase space by the conjugacy. The complex bound will be distorted at most by the factor equal to the maximal dilation of the conjugacy.

Hence we reduced the problem to the situation when for an infinitely renormalizable polynomial there is a sequence of intervals

$$[-1, 1] = I_0 \supset I_1 \supset \cdots \supset I_m = I$$

where I_j is a restrictive interval for $j = 1, \cdots, m$ and the return times on I_j *in terms of the first return map* into I_{j-1} are all bounded by n. This is very close to the bounded type case considered in [16] and methods similar to the "sector lemma" are used.

About the proof of Theorem 2. We will present an explicit inducing algorithm starting with the canonical induced mapping ϕ_0. This algorithm is similar to the procedure of [8], but with the "filling-in" step skipped. We will prove a real condition, in terms of the geometry of a real box mapping, which implies that this box mapping induces another one which already has an analytic continuation. Next, we show that this condition is satisfies after some number of steps of the inducing algorithm depending on η only. This is as much as was known after [3] and still much less than Theorem 2 says because one step on the inducing algorithm may mean that the return time of the central branch increases by many iterations of f. One situation in which it happens is a deep close return (case ii. or case iii. in the classification of [8].) We show, however, that a sufficiently deep close return of either type allows one to induce a real box mapping with analytic continuation. Also, the return time of the central branch may greatly increase by one step of the inducing algorithm if the critical value falls into a monotone branch with large return time. This introduces enough expansion to also allow us to induce a mapping with an analytic continuation.

Ordering of the material. In the next section we will prove Theorem 2. In the following section we will infer Theorem 1. That will include a proof in the "almost bounded case" based on a new argument.

2 Analytic Continuation and Inducing

2.1 A criterion for analytic continuation

We will show a fairly general condition under which a real box mapping induces another one with an analytic continuation. If a, b, c, d are consecutively ordered points on the real line, allowing some of them to be equal as long as $d > b$ and $c > a$, then we have the familiar cross-ratio:

$$\mathbf{Poin}(a, b, c, d) := \frac{|b-c||a-d|}{|a-c||b-d|} .$$

In addition, we introduce another one.

$$\mathbf{Cr}(a, b, c, d) := \frac{|a-b||c-d|}{|a-c||b-d|} .$$

Figure 1: explains the setting of Proposition 1 visually.

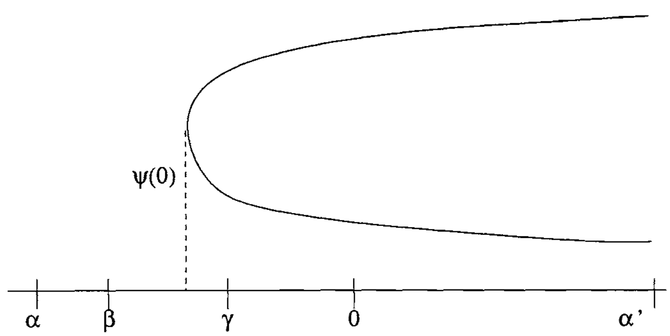

Figure 1: The situation of Proposition 1 in the hyperbolic case

Proposition 1 *Let (ϕ, B') be critically complete real box mapping (see Definition 1.11). Suppose that an open interval $B'' \subset B'$ can be chosen, so that:*

- *B'' is symmetric with respect to* 0 *and its boundary is disjoint from the domain of ϕ,*

- *the critical value of ϕ belongs to the domain ϕ as well as to B'',*

- *the range of any monotone branch of ϕ contains B''.*

Choose α, β, γ to be the endpoints of B', B'' and B, respectively, all on the same side of 0 *as the critical value of ϕ. Let α' be the other endpoint of B'. Suppose that for some $\epsilon > 0$ the conditions*

- $2\mathbf{Poin}(\alpha', \psi(0), \beta, \alpha) \leq 1 - \epsilon$ *in the case when* 0 *is the the range of ψ, or* $2\mathbf{Poin}(\alpha', \gamma, \psi(0), \alpha) \leq 1 - \epsilon$*, which includes the assumption that $\psi(0)$ is between γ and α, in the remaining case,*

- *either the range of the central branch of ϕ does not contain* 0 *(a parabolic return), or* $\mathbf{Cr}(\alpha', -\beta, \beta, \alpha) \geq \epsilon$.

Then for every $\epsilon > 0$ there is a $K > 0$ so that (ϕ, B') induces a critically complete real box mapping which has an analytic continuation Φ (in the sense of Definition 1.8). Moreover, if D is the central domain of Φ and D' its range, then mod $(D' \setminus D) \geq K$ and every univalent branch of Φ maps over D'.

We proceed to prove Proposition 1.

Normalizations and notations. For definiteness, assume that 0 is the minimum of the central branch. This can always be achieved by "flipping" the coordinates. We distinguish between two situations. One is that the critical point is positive, that will be called a *parabolic* case, the other one being referred to as *hyperbolic*. In other words, the hyperbolic case is defined by the requirement that the range of the folding branch contains 0. It will be more convenient to use a, b, c for positive end-points of B', B'', B, respectively, and a' for the negative end-point of B'. Let ψ be the central branch of ϕ but continued analytically.

Lemma 2.1 *Let (ϕ, B') be a real box mapping. Choose the orientation so that the central branch ψ of ϕ has the minimum at 0, and let $B' = (a', a)$. Choose $x \in B'$ so that $x < \psi(0)$. Define G to be $\mathcal{D}(\frac{\pi}{2}, (x, a))$. Then if* $\mathbf{Poin}(a', x, \psi(0), a) \leq \frac{1}{2}$, *then*

$$\psi^{-1}(G) \subset \mathcal{D}(\frac{\pi}{2}, B) .$$

Proof: The key observation is that $\psi^{-1}(G) \subset \mathcal{D}(\frac{\pi}{2}, B)$ if only $\mathbf{Poin}(-a, x, \psi(0), a) \leq \frac{1}{2}$. Indeed, representing $\psi(x) = h(x^2)$ where h is a diffeomorphism with non-positive Schwarzian derivative (it follows from the hypothesis on P), this assumption implies that $|h^{-1}(x, \psi(0))| \leq |h^{-1}(\psi(0), a)|$ while by Fact 1.1 the preimage of G by the analytic continuation of h is contained in the geometric disc with diameter $(h^{-1}(x), h^{-1}(a))$. Now it follows immediately that indeed $\psi^{-1}(G) \subset \mathcal{D}(\frac{\pi}{2}, B)$.

□

Several estimates.

Lemma 2.2 *Let us consider the hyperbolic case. For every $\varepsilon > 0$ there is a positive K so that if* $2\mathbf{Poin}(a', -b, \psi(0), a) < 1 - \varepsilon$ *and* $\mathbf{Cr}(a', -b, b, a) \geq \varepsilon$, *then there is a point y, $-a < y < -b$, so that*

- $\mathbf{Poin}(-a, y, \psi(0), a) \leq \frac{1}{2}$,
- $\mathbf{Cr}(y, -b, b, a) \geq K$.

Proof: As $y \in (a', -b]$, the cross ratio $\mathbf{Poin}(a', y, \psi(0), a)$ is a continuous and decreasing function of y. Let us fix the value of y so that $\mathbf{Poin}(a', y, \psi(0), a) = \frac{1}{2}$. Suppose that the Lemma does not hold. Then we get a sequence of configurations

$$(a'_n, y_n, \psi_n(0), b_n, a_n)$$

which all satisfy the hypotheses yet $\mathbf{Cr}(y_n, -b_n, b_n, a_n)$ tend to 0. We normalize these configurations by linear-fractional mappings M_n so that $M_n(a_n) = 1$, $M_n(-b_n) = 0$ and $M_n(a'_n) = -1$. Then a subsequence n_j is chosen so that

$$M_{n_j}(y_{n_j}) \to Y, M_{n_j}(\psi_{n_j}(0)) \to C \text{ and } M_{n_j}(b_{n_j}) \to B .$$

Since $C \geq 0$ and $Y \leq 0$, we are allowed to consider

$$\mathbf{Poin}(-1, Y, C, 1) = \lim \mathbf{Poin}(a'_{n_j}, y_{n_j}, \psi_{n_j}(0), a_{n_j}) = \frac{1}{2} .$$

By the same reasoning, $\mathbf{Poin}(-1, 0, C, 1) \leq 1 - \varepsilon$. We infer that $Y < 0$. On the other hand,

$$\mathbf{Cr}(-1, 0, B, 1) \geq \varepsilon$$

from where $B < 1$. It follows that

$$\mathbf{Cr}(Y, 0, B, 1) > 0$$

but this contradicts our original choice of the sequence in which

$$\lim_{n \to \infty} \mathbf{Cr}(y_n, -b_n, b_n, a_n) = 0 .$$

□

There is an similar fact in the parabolic case.

Lemma 2.3 *Let us consider the parabolic case, that is* $\psi(0) \in (c, b)$. *For every* $\varepsilon > 0$ *there is a positive* K *so that if*

$$2\mathbf{Poin}(a', c, \psi_0, a) \leq 1 - \varepsilon ,$$

then there is a point z, $a' < z < c$ *so that*

- $\mathbf{Poin}(-a, z, \psi(0), a) \leq \frac{1}{2}$,
- $\mathbf{Cr}(z, c, b, a) \geq K$.

Proof: The method of the proof is the same as in the preceding lemma, so we will be more brief. Fix $z < c$ so that $\mathbf{Poin}(-a, z, \psi(0), a) = \frac{1}{2}$. Suppose that the lemma does not hold, and choose a sequence of configurations which satisfy the hypotheses yet $\mathbf{Cr}(z_n, c_n, b_n, a_n)$ tends to 0. Normalize them by linear-fractional mappings M_n which send a'_n, c_n, a_n to $-1, 0, 1$, respectively. Choose a subsequence n_j so that

$$M_{n_j}(z_{n_j}) \to Z,\ M_{n_j}(\psi_{n_j}(0)) \to C,\ M_{n_j}(b_{n_j}) \to B\ .$$

Since $2\mathbf{Poin}(-1, Z, B, 1) = 1$ while $2\mathbf{Poin}(-1, 0, B, 1) \leq 1 - \varepsilon$, we conclude that $Z < 0$. But $2\mathbf{Poin}(-1, 0, B, 1) < 1$ also implies $B < 1$. Hence $\mathbf{Cr}(Z, 0, B, 1) > 0$, contradiction.

□

Lemma 2.4 *Consider points $x_1 < x_2 < x_3 < x_4$. Denote*

$$r = \exp\left[mod \left(\mathcal{D}(\frac{\pi}{2}, (x_1, x_4)) \setminus \overline{\mathcal{D}(\frac{\pi}{2}, (x_2, x_3))} \right) \right] .$$

Then

$$\mathbf{Cr}(x_1, x_2, x_3, x_4) = \frac{(r-1)^2}{(r+1)^2} .$$

Proof: Map the points x_1, x_2, x_3, x_4 by a Möbius transformation M chosen so that $M(x_1) = -1$, $M(x_4) = 1$ and $|M(x_2)| = |M(x_3)|$. Then $\mathcal{D}(\frac{\pi}{2}, (x_1, x_4))$ goes to the unit disk while $\mathcal{D}(\frac{\pi}{2}, (x_2, x_3))$ is a disk centered at 0 which must have radius r^{-1} for the modulus to be preserved. Since the cross-ratio is also preserved,

$$\mathbf{Cr}(x_1, x_2, x_3, x_4) = \frac{(1-r^{-1})^2}{(1+r^{-1})^2}$$

and the Lemma follows.

□

Proof of Proposition 1 in the hyperbolic case. Let us consider a box mapping ϕ_1 as follows. Restrict the domain of the central branch of ϕ to $B_1 = \psi^{-1}(B'') \cap \mathbf{R}$. Then on $B \setminus B_1$ define $\phi_1 := \phi \circ \phi$. This means that monotone branches of ϕ are pulled into B by the "laps" of the central branch. Leave $\phi_1 = \phi$ outside B. It is clear that ϕ_1 is a box mapping, and moreover is directly induced by ϕ in the sense of Definition 1.10. Observe that ϕ_1 remains critically complete if ϕ was so, indeed this type of inducing causes no loss of infinite orbits. We will proceed to show that ϕ_1 has an analytic continuation.

Choose point y from Lemma 2.2 and consider the disk $D_1 := \mathcal{D}(\frac{\pi}{2}, (y, a))$. Also, consider $D_2 := \mathcal{D}(\frac{\pi}{2}, B'')$. If $r = \exp(\bmod (D_1 \setminus \overline{D_2}))$, then by Lemmas 2.2 and 2.4

$$\frac{(r-1)^2}{(r+1)^2} \geq K$$

where $K > 0$ only depends on ε. Hence, $\bmod (D_1 \setminus \overline{D_2}) \geq K_1 > 0$ where K_1 depends only on ε. By Lemma 2.1,

$$\psi^{-1}(D_2) \subset \psi^{-1}(D_1) \subset \mathcal{D}(\frac{\pi}{2}, B) \subset D_2 \ .$$

Moreover,

$$\bmod (D_2 \setminus \psi^{-1}(D_2)) \geq \bmod (\psi^{-1}(D_1) \setminus \psi^{-1}(\overline{D_2})) =$$

$$= \frac{1}{2} \bmod (D_1 \setminus \overline{D_2}) \geq K_1/2 \ .$$

So we can take D_2 to be the range of the central branch of ϕ. As for analytic continuation of monotone branches of ϕ_1, take one of them, ζ, with domain U and range V. Then ζ^{-1} is well-defined on $\mathcal{D}(\frac{\pi}{2}, V)$ and maps it into $\mathcal{D}(\frac{\pi}{2}, U)$ by virtue of ζ being an Epstein diffeomorphism. Make $\zeta^{-1}(\mathcal{D}(\frac{\pi}{2}, V))$ the domain of the analytic continuation of ζ. This can be done for any monotone branch ζ. Notice also that $V \supset B''$ which implies that all univalent branches map over D_2.

This way Φ was defined. In checking the conditions of Definition 1.7 only item 4. requires an argument. Notice, that in our construction the domains of all complex branch are contained in the geodesic neighborhoods of angle $\frac{\pi}{2}$ of their respective real domains, while the ranges are equal to the geodesic neighborhoods of the same angle of the real ranges. Hence item 4. follows from item 3. of Definition 1.6 applied to ϕ_1. Hence Φ is an analytic continuation of ϕ_1 in the sense of Definition 1.8, and the proof of Proposition 1 has been achieved in the hyperbolic case.

Proof in the parabolic case. Let us denote by Δ_p the connected component of the domain of ϕ which contains $\phi(0)$, and by χ_p the branch of ϕ defined on Δ_p. As a consequence of Definition 1.6 and the hypotheses of Proposition 1, we have $\Delta_p \subset B'' \setminus \overline{B}$. Let $\mathcal{T}(\phi)$ be the tempered map of ϕ and define ϕ_1 by replacing the central branch of ϕ with $\mathcal{T}(\phi) \circ \phi$ and leaving ϕ unchanged elsewhere. The central branch of ϕ_1 is $\chi_p \circ \psi$. Then construct ϕ_2 by restricting the domain of the central branch of ϕ_1 to $\psi^{-1}(\chi_p^{-1}(B))$ and replacing this central branch on the rest of its domain by the composition with ϕ_1. Like in the hyperbolic case, this process of inducing caused no loss of infinite orbits, and hence the property of ϕ being critically complete was inherited by ϕ_1.

We will proceed to construct an analytic continuation Φ of ϕ_2.

Choose the point z from Lemma 2.3 and define $D_1 := \mathcal{D}(\frac{\pi}{2}, (z, a))$. Also, let $D_2 := \mathcal{D}(\frac{\pi}{2}, (c, b))$. As a consequence of χ_p being an Epstein diffeomorphism, we get

$$\psi^{-1}(\chi_p^{-1}(\mathcal{D}(\frac{\pi}{2}, B))) \subset \psi^{-1}(\mathcal{D}(\frac{\pi}{2}, \Delta_p)) \subset \psi^{-1}(D_2) \, .$$

By Lemma 2.1,

$$\psi^{-1}(D_2) \subset \psi^{-1}(D_1) \subset \mathcal{D}(\frac{\pi}{2}, B) \, .$$

Moreover,

$$\text{mod}\ (\mathcal{D}(\frac{\pi}{2}, B) \setminus \psi^{-1}(\chi_p^{-1}(\overline{\mathcal{D}(\frac{\pi}{2}, B)}))) \geq \frac{1}{2}\ \text{mod}\ (D_1 \setminus \overline{D_2})$$

which is bounded away from 0 solely in terms of ε by Lemmas 2.3 and 2.4.

The construction of analytic continuations for monotone branches and the final check of the required properties are the same as in the hyperbolic case and omitted.

2.2 Real inducing algorithm

As the next step in the proof of Theorem 2, we introduce an algorithm for getting induced mappings beginning from the canonical induced map ϕ_0. The maps obtained in this way, beyond being real box mappings, will have extra properties making it easier to apply Proposition 1. We call them "type III" real box mappings.

2.3 A few technical tools

Negative Schwarzian. Here is the fundamental inequality about the distortion of the cross-ratio **Poin** by diffeomorphisms with negative Schwarzian derivative.

If $a < b < c < d$, then

$$\mathbf{Poin}(a, b, c, d) := \frac{|d-a||b-c|}{|c-a||d-b|} .$$

Fact 2.1 *Diffeomorphisms with negative Schwarzian derivative expand the cross-ratio* **Poin**

$$\mathbf{Poin}(a, b, c, d) < \mathbf{Poin}(f(a), f(b), f(c), f(d)) .$$

Contraction principle of the Poincaré length is closely related with the cross-ratio expanding property.

Fact 2.2 *Diffeomorphisms with negative Schwarzian contract the Poincaré metric.*

Type III of real box mappings.

Definition 2.1 *A real box mapping* (ϕ, B') *with an additionally specified open interval* B'' *is of* type *III if the following hold:*

- B'' *is symmetric with respect to* 0,
- $B \subset B'' \subset B'$, *and the range of every monotone branch contains* B'',
- B'' *contains the critical value of* ϕ, *i.e.* $\phi(0)$.

To describe the *type III inducing* algorithm, suppose that a type III box mapping given by (ϕ, B', B'') is given, with central domain B. Let us assume for the entire construction that ϕ is critically complete (see Definition 1.11), moreover, that the orbit $\phi^n(0)$ accumulates at 0.

Close and non-close returns. If $\phi(0) \in B$ we say that the box mapping is showing a *close return*, otherwise we talk about a non-close return. The close returns can further be classified as *terminal* if the orbit $\phi^n(0)$ forever remains in B, and non-terminal. If a return is non-terminal, (close or not), we define its *depth* as

$$E := \min\{i = 1, \cdots : \phi^i(0) \notin B\} .$$

The staircase construction. Suppose that (ϕ, B', B'') shows a close return with return time E. Choose $0 \leq m < E$. Consider the interval B^m as the set of points in B that stay in B under m iterations of ϕ. For example the critical value is in B^{m-1} but not in B^m and $B^0 = B$. Define a new mapping ϕ_1 as follows. Outside of B, ϕ_1 and ϕ are the same. Also, inside B^m they are the same, i.e. the central branch of ϕ_1 is the same as the central branch of ϕ restricted to B^m. For k between 1 and m, define ϕ_1 on $B^{k-1} \setminus B^k$ as ϕ^{k+1}. That is, points get iterated k times by the central branch, which is just enough to throw them out of B, and then again mapped by ϕ. This has the effect of pulling the branches of ϕ into $B \setminus B_m$.

To complete the definition of ϕ_1 as a box mapping, we need to specify B'. If the central branch of ϕ was $h(x^2)$, for ϕ_1 we need to trim the domain of h on the side of the "legs" of the central branch which have been shortened, or formally speaking to comply with item 1. of Definition 1.6, but we may leave B' unchanged on the other side (the "head" of the branch). Call this interval B'_1. Thus, (ϕ_1, B'_1, B^{m-1}) is a type III real box mapping.

When $m = E - 1$ we will refer to this as the *full staircase construction*. After a full staircase construction, we necessarily get a box mapping with a non-close return.

Notice that the staircase construction will not lose any infinite orbit, and hence it results in a critically complete box mapping. It is also easy to see that $\phi_1^n(0)$ is recurrent.

Critical filling. Let (ϕ, B', B'') be a real box mapping which shows a non-close return. This means that the critical value is in the domain of some monotone branch. Denote this branch with χ_p and its domain with Δ_p. The objective of the first stage of our construction is to make sure that $\chi_p(\phi(0)) \in B$. This may not hold initially, of course. Then consider the tempered map $\mathcal{T}(\phi)$ and obtain ϕ_1 by changing ϕ on the Δ_p only, namely replacing it with $\mathcal{T}(\phi) \circ \chi_p$. Now (ϕ_1, B') remains a box mapping (though (ϕ, B', B'') is no longer a type III box mapping). The critical value is now in the domain of some monotone branch χ_1 of ϕ_1 which maps at least onto B''. It may already be that $\chi_1(\phi(0)) \in B$. Otherwise, construct ϕ_2 by replacing χ_1 with $\mathcal{T}(\phi) \circ \chi_1$ and leaving ϕ_1 unchanged elsewhere. If this construction is continued, we claim that after a finite number of steps ϕ_0 lands in the domain of a branch χ_k

of ϕ_k, χ_k maps at least over B'' and $\chi_k(\phi(0)) \in B$. Otherwise the process could continue to infinity, and then it would follow that the orbit $\phi^n(0)$ forever avoids B, which contradicts our standing assumption.

Let V denote the range of χ_k. Then we can construct a type III box mapping (φ, V, B) by making φ equal to ϕ_k outside of B, and replacing it with $\phi_k \circ \phi_k$ on B. This (φ, V, B) is the outcome of the *critical filling* of (ϕ, B', B''). Similarly to the previous step, we observe that the critical filling also results in a critically complete box mapping with a recurrent critical orbit.

Type III inducing step. Given an arbitrary type III box mapping making a non-terminal return, we can first subject to the full staircase construction, provided that ϕ makes a close return, or do nothing if ϕ makes a non-close return. This gives a type III box mapping ϕ_1. Then critical filling is administered to ϕ_1 and that gives the final result as type III box mapping. This procedure is called the *type III inducing step* and as argued before it will result in a critically complete and recurrent mapping provided that ϕ had these properties.

The induced sequence. Let us come back to the canonical induced map $(\phi_0, (-q, q))$. It can also be viewed as a type III real box mapping. We can apply to it a sequence of type III inducing steps until a terminal return occurs. Notice that the canonical induced map is critically complete and recurrent provided that f was infinitely renormalizable. Indeed, the only orbits that escape the domain of ϕ_0 are those of pre-periodic points. On the other hand, since f is infinitely renormalizable, the orbit of 0 is recurrent. The orbit of 0 under ϕ_0 consists of all returns of the orbit by f to $(-q, q)$, and so is also recurrent. The type III inducing step preserves both properties. Thus the type III inducing steps can be repeated until a terminal return is encountered.

Terminal returns. The possibility to induce ends when some ϕ shows a terminal return. Terminal returns for box mappings induced by ϕ_0 are intimately related to first returns of the maximal restrictive interval of f to itself.

Lemma 2.5 *Let ϕ be a box mapping induced by the canonical induced map ϕ_0. If ϕ is terminal, and its central branch is f^n, then the central domain contains a restrictive interval of f and n is the return time for this restrictive interval.*

Proof: Take preimages of the central domain B by the central branch ψ. They form a nesting sequence of symmetric intervals and their intersection must be a symmetric interval I. We have $\psi(I) \subset I$. To show that I is the restrictive interval for f, we need to show that none of the intermediate images of I by f intersects I. Suppose $f^k(I) \cap I \neq \emptyset$. But the end-points of I are accumulation points of preimages of the end-points of B. For any box mapping induced by ϕ_0, end-points of the domain of any branch are eventually mapped into q by f. But this is a contradiction, since the restrictive interval cannot contain q.

□

How about the other way: if f is known to be renormalizable, will this be detected by inducing in the form of a terminal return?

Lemma 2.6 *Suppose that f is renormalizable with the return of its restrictive interval I equal to n. Let ϕ be a critically complete real box mapping induced by the canonical induced map. Then the central domain of ϕ contains I and the central branch is equal to f^m with $m \leq n$.*

Proof: Again we observe that the end-points of the central domain B of ϕ are eventually mapped into q, hence $I \subset B$. If $m > n$, then the central branch would not be unimodal.

□

It follows that by considering the sequence of box mappings induced by the canonical induced map of f through a type III inducing steps we will encounter a terminal return. Otherwise the number of iterations of f would grow indefinitely in contradiction to Lemma 2.6. Then Lemma 2.5 tells us that indeed a restrictive interval sits in B with the first return map given by the central branch. So this must be the maximal restrictive interval.

2.4 Metric analysis of the induced sequence

Now, we are in position to formulate the main geometric property of the real inducing procedure. Loosely speaking, we have always two possibilities: either the sequence of maps induced by ϕ_0 through consecutive type III steps end after a bounded number of steps by meeting a terminal return, or one can induced a box mapping with an analytic continuation that satisfies the demands of Theorem 2.

Proposition 2 *Let $f \in \mathcal{F}_\eta$ be infinitely renormalizable. Then for every $\eta > 0$ there are numbers K and N so that either after applying n, where $n < N$, type III inducing steps to $\phi_0, (-q, q), (-q, q)$ a mapping showing a terminal return for ϕ_n with $n < N$, or there is a critically complete real box mapping ϕ induced by ϕ_0 which has an analytic continuation as a complex box mapping Φ, and mod $(D' \setminus \overline{D}) \geq K$ where D and D' are the domain and range of the central branch of Φ, respectively. In addition, every univalent branch of Φ maps over D'.*

Definition 2.2 *Given a type III box mapping* (ϕ, B', B'') *we will introduce its* box parameters a, b, c *as follows:* $a = dist(0, \partial B')$, $B'' = (-b, b)$ *and* $B = (-c, c)$.

Returning to Proposition 1 we see that if $2\mathbf{Poin}(-a, c, b, a) < 1-\epsilon$ and $\frac{b}{a} < 1-\epsilon$ with $\epsilon > 0$, than the hypotheses are satisfied with some $\varepsilon > 0$ determined by ϵ. Then Proposition 1 implies that a real box mapping can be induced by ϕ with an analytic continuation satisfying the bound $\mod (D' \setminus \overline{D}) \geq K$ with K determined by ϵ.

Real a priori bounds. We begin by proving a bound for ϕ_0. Remember that for ϕ_0 $B' = B''$.

Lemma 2.7 *For every* $\eta > 0$ *there is* $\varepsilon_0 > 0$ *so that if* $f \in \mathcal{F}_\eta$, *then* $c/b < 1 - \varepsilon_0$ *where* c *and* b *are the box parameters of the canonical induced map of* f, (ϕ_0, B', B'').

Proof: Define J_k for $k > 0$ as the set of points in $1 > x > 0$ whose first entry time into the fundamental inducing domain is k. Then $J_{k+1} = f_r^{-1}(-J_k)$ where f_r denotes the right "lap" of f. The derivative of f_r on $(q, 1)$ is greater than 1, since it is so at the endpoints q and 1 and Schwarzian derivative is negative. So $|J_k|$ form a decreasing sequence. Now B is the preimage by $f(x) = h(x^2)$ of this J_{k_0} which contains the critical value of f. At the same time, $B' \setminus B$ is the preimage of the union of J_k for $k < k_0$. The Poincare length of J_{k_0} with respect to $(q, 1 + \eta)$ is less than a uniform constant depending solely on η. Since h^{-1} contracts the Poincare distance and x^2 distorts the ratio by at most squaring, the ratio $|B|/|B'|$ is indeed bounded away from 1 in terms of η only.

□

Relations between box parameters. Let us first review how the "boxes" B, B'' and B' evolve in a type III inducing step. Suppose that (ϕ, B', B'') is a type III box map, B is the central domain of ϕ, next (ϕ_1, B'_1, B''_1) is derived from it by the full staircase construction if ϕ shows a close return, or is equal to (ϕ, B', B'') in the case of a non-close return, and (ϕ_2, B'_2, B''_2) is the final outcome of the type III inducing step. Then we have B''_2 equal to the central domain of ϕ_1, always contained in B, and B'_2 at least as large as B''.

If a, b, c are the box parameters for ϕ, and a', b', c' are the box parameters for ϕ_2, this leads to $a' \geq b$ and $b' \leq c$. Another relation is given by this Lemma.

Lemma 2.8 *Consider a type III box mapping (φ, B', B'') with box parameters a, b, c and choose δ so that $\frac{b}{a} \leq \delta$. Then for every $\delta < 1$ there is a number $\lambda < 1$ so that whenever*

$$\frac{c}{b} \geq \lambda \frac{b}{a} ,$$

then $2\mathbf{Poin}(-a, c, b, a) \leq 1 - \frac{(1-\delta)^2}{5}$.

Proof: Denote $\alpha = \frac{b}{a}$ and $\alpha_1 = \frac{c}{b}$.

$$2\mathbf{Poin}(-a, c, b, a) = 2\mathbf{Poin}(-\frac{1}{\alpha}, \alpha_1, 1, \frac{1}{\alpha}) = \frac{4\alpha(1-\alpha_1)}{(1-\alpha_1\alpha)(1+\alpha)} . \tag{1}$$

Notice that this expression decreases with α_1 increasing as α is kept fixed and increases as α increases and α_1 is fixed. Hence, assuming $\alpha_1 \geq \lambda\alpha$ for some positive λ, we get

$$1 - 2\mathbf{Poin}(-a, c, b, a) \geq 1 - \frac{4\alpha_0(1-\lambda\alpha_0)}{(1-\lambda\alpha_0^2)(1+\alpha_0)} .$$

The difference on the right-hand for $\lambda = 1$ expresses nicely as

$$\left(\frac{1-\alpha_0}{1+\alpha_0}\right)^2 \geq \frac{(1-\alpha_0)^2}{4} .$$

The value of $\lambda < 1$ can be picked for every δ to satisfy the claim of the lemma by continuity.

□

Lemma 2.9 *There are constants $\varepsilon > 0$ and N depending only on the bound ε_0 of Lemma 2.7, and an integer $2 \leq j < N$ so that if φ is the map obtained from the canonical induced map by j consecutive type III inducing steps, then* either

- *φ shows a terminal return,* or
- *if a, b, c are the box parameters of φ, then*

$$2\mathbf{Poin}(-a, c, b, a) < 1 - \varepsilon \quad \textit{and} \quad \frac{b}{a} < 1 - \varepsilon\ .$$

Proof: Assume that for $j < N$ and $j+1$ consecutive type III inducing steps are possible starting from ϕ_0. Let a, b, c be the box parameters of ϕ_j, that is the map obtained from ϕ_0 by j consecutive type III inducing steps.

We will show that there is a constant K depending on ε_0 and an index $j < N$ so that $2\mathbf{Poin}(-a, c, b, a) < 1 - \varepsilon_0$ and $\frac{b}{a} < 1 - K\varepsilon_0^2$, where ε_0 is a constant from Lemma 2.7. Apply Lemma 2.8 to the map ϕ_1 derived from ϕ_0 by one type III inducing step. By Lemma 2.7 we can set $\delta := 1 - \varepsilon_0$. This will give a $\lambda < 1$. Next, choose j as the smallest positive integer so that either $\frac{b_{j+1}}{a_{j+1}} \geq \lambda \frac{b_j}{a_j}$ or $\frac{b_j}{a_j} \leq \frac{1}{4}$ where a_i, b_i, c_i, etc, are the box parameters of the map ϕ_i obtained from ϕ_0 by i consecutive type III inducing steps. Proposition 2.9 follows when we show:

- that j is bounded from above depending only ε_0.
- that the estimates claimed in Proposition 2.9 hold for ϕ_j.

The first statement is immediate since until j is reached $\frac{b_i}{a_i}$ have to decrease with ratio λ. For the second statement, observe first that $\frac{b_j}{a_j} \leq 1 - \varepsilon_0$. Indeed, this estimate held $i := 1$ and the characteristic ratios keep decreasing until j is reached. If $\frac{b_{j+1}}{a_{j+1}} \geq \lambda \frac{b_j}{a_j}$, then the needed estimate for $\mathbf{Poin}(-a_j, c_j, b_j, a_j)$ follows directly from Lemma 2.8 with $K = \frac{1}{5}$. All we are left to do is to prove that $\mathbf{Poin}(-a_j, c_j, b_j, a_j)$ is bounded as needed when $\frac{b_j}{a_j} \leq \frac{1}{4}$.

The case of a small nest. To finish the proof of Lemma 2.8, we recall the estimate (1) which is applied here for $\varphi := \phi_j$, $\alpha := \frac{b_j}{a_j}$ and $\alpha_1 := \frac{b_{j+1}}{a_{j+1}}$. If $\alpha \leq \frac{1}{4}$, the maximal value of the right-hand side is obtained by setting $\alpha_1 = 0$, and this value is $\frac{4}{5}$. In this case, $K = \frac{1}{2}$ will do.

□

Now, Proposition 2 follows from Lemma 2.9 and Proposition 1.

2.5 Deep close returns

If Theorem 2 had been stated with n meaning the number of consecutive type III inducing steps possible from the canonical induced map without meeting a terminal return, then Proposition 2 would have settled the matters already. The meaning of the n in Theorem 2, however, is different: the return time of the restrictive interval which is equal, in view of Lemmas 2.5 and 2.6, to the number of iterations of f composing the central branch when a terminal return is encountered. One case when it is much larger than the number of type III inducing steps leading to the terminal return, is when in the type III inducing process very deep close returns occur. We consider this case now with the conclusion that sufficiently deep close returns lead to an induced box mapping which has an analytic continuation with the suitable bound.

Recall that box parameters were introduced in Definition 2.2.

Proposition 3 *Let (ϕ, B', B'') be a type III critically complete box mapping with box parameters a, b, c. Suppose that it makes a close return with depth E. For every $\epsilon > 0$ there are E_0 and $K > 0$ so that if $\frac{b}{a} \leq 1 - \epsilon$ and $E \geq E_0$, then the there is a critically complete real box mapping φ induced by ϕ which has an analytic continuation Φ. If D and D', respectively, denote the domain and range of Φ, then mod $(D' \setminus \overline{D}) \geq K$.*

Generalities. The proof of this Proposition is very different in the case when ϕ makes a parabolic return, that is 0 is not in the range of the central branch, than when ϕ makes a hyperbolic return. Some notations, however, will be used in both cases. Let $\psi(z) = h(z^2)$ by analytic continuation of the central branch. Remember that h is an Epstein diffeomorphism onto B'. Denote with B^i, $i = 0, \cdots, E-1$ the $i-th$ preimage of B' by ψ, intersected with the real line. Then $B^0 = B$ and B^i are nesting intervals.

We choose the orientation so that 0 is the minimum of the central branch. Let $B' = (\alpha', \alpha)$. Without loss of generality we will assume that $|\alpha'| \leq \alpha$. Otherwise we can always restrict h to the smaller range $(-\alpha, \alpha)$. This will change neither ϕ as a real mapping nor the box parameter a.

Lemma 2.10 *Let (ϕ, B', B'') be a type III box mapping which satisfies the hypotheses of Proposition 3. Then for every $\epsilon > 0$ and every $i = 1, \cdots, E-1$ there is a bound $\lambda_i < 1$ so that*

$$\frac{|B^i|}{|B^{i-1}|} < \lambda_i \; .$$

Proof: Let us first prove this for $i = 1$. Applying h^{-1} to points $\alpha', -b, b, \alpha$, we get

$$\mathbf{Cr}(h^{-1}(\alpha'), h^{-1}(-b), h^{-1}(b), h^{-1}(\alpha)) \geq \mathbf{Cr}(\alpha', -b, b, \alpha) \geq \frac{\epsilon^2}{4} \; .$$

Since the critical value of ψ falls into $(-b, b)$,

$$\frac{|h^{-1}(b)|}{|h^{-1}(\alpha)|} \leq \frac{|h^{-1}(b) - h^{-1}(-b)|}{|h^{-1}(-b) - h^{-1}(\alpha)|} \leq 1 - \frac{\epsilon^2}{4}$$

and we can take $\lambda_1 := \sqrt{1 - \frac{\epsilon^2}{4}}$. To prove the Lemma inductively for a general i, apply the same reasoning replacing B' with B^{i-2}, B'' with B^{i-1}, and ϵ with $1 - \lambda_{i-1}$.

□

The case of a small nest.

Lemma 2.11 *Let (ϕ, B', B'') satisfy the hypotheses of Proposition 3. If $\frac{|B^1|}{a} \leq \frac{1}{30}$ and $E \geq 3$, then the $K > 0$ and a complex box mapping Φ exist as required in the claim of Proposition 3*

In other words, in the proof of Proposition 3, $|B^1|/a$ can be bounded from below by $1/30$ without loss of generality.
Proof: Denote $B^1 = (-a_1, a_1)$. Under the hypothesis of the Lemma, we get

$$\mathbf{Poin}(\alpha', -2a_1, \psi(0), \alpha) \leq \mathbf{Poin}(-a, -2a_1, a_1, a) \leq \frac{1}{5}$$

In view of Lemma 2.1, this means that

$$\psi^{-1}(\mathcal{D}(\frac{\pi}{2}, (-2a_1, \alpha))) \subset \mathcal{D}(\frac{\pi}{2}, B) .$$

Also, $\mathbf{Cr}(-2a_1, -a_1, a_1, \alpha) > \frac{1}{6}$, thus by Lemma 2.4 we have

$$\mathrm{mod}\ (\mathcal{D}(\frac{\pi}{2}, (-2a_1, \alpha)) \setminus \overline{\mathcal{D}(\frac{\pi}{2}, B^1)}) \geq M_1 > 0$$

where M_1 is a constant. Take $D' := \mathcal{D}(\frac{\pi}{2}, B^1)$ and $D := \psi^{-1}(D)$. It follows that $D \subset D'$, moreover, $\mathrm{mod}\ (D' \setminus \overline{D}) \geq M_1/2$. If ϕ_1 is the mapping obtained from ϕ by the staircase construction with just one step (i.e. $m = 1$), then D and D' can be chosen as the domain and range of the central branch for its analytic continuation. The rest, that is the continuation of monotone branches and the checking of the conditions of Definition 1.7, is done as in the proof of Proposition 1, so we skip it. We see that in this case the claim of Proposition 3 holds.

□

From now on the argument diverges for hyperbolic and parabolic returns.

The hyperbolic return. We look at the fixed point $r > 0$ of ψ. Remember that in our notation $\psi(0) < -r < 0$.

Lemma 2.12 *If (ϕ, B', B'') is a type III real box mapping which satisfies the hypotheses of Proposition 3, together with $a/b \leq 10$ and $|\alpha'| \leq \alpha$, then there is a constant $M_2 > 0$ so that $r \geq M_2 a$.*

Proof: This is a compactness argument. Choose a sequence of central branches $\psi_n(z) = h_n(z^2)$ for which the fixed points r_n go to 0. Change the coordinate affinely so that α' becomes -1 and restrict h^n, if need be, so that their real image becomes $(-1, 1)$. This will not cut r_n out since $r_n \in B_n \subset (-1, 1)$. By Montel's theorem, find a subsequence so that h_n^{-1} converge uniformly on compact sets to a map H^{-1}. This H^{-1} cannot be constant, since by our assumption about b/a it maps $(-1, 1)$ onto a non-degenerate interval, hence it is univalent and H restricted to the real line is an Epstein diffeomorphism. Since the derivatives $h_n'^{-1}$ converge uniformly on $[-1/2, 1/2]$, then $h_n'^{-1}(r_n)$ converge to $H^{-1}(0)$. It follows that for n large enough $\psi_n(r_n) < 1$. But $\psi_n^{-1}(1) < 1$ and the negative Schwarzian condition implies that $\psi_n(0) \in (0, r_n)$. So the orbit of 0 by ψ_n will never leave $(0, r_n)$ which is a contradiction with the assumption of Proposition 3 that the depth E is finite.

□

Denote the endpoint of B^i contained in $I\!R_+$ by a_i. We will show that the sequence a_i approaches r exponentially fast with the uniform rate. In particular it will imply that the eigenvalue of r is uniformly greater than 1.

Lemma 2.13 *For any a_i, $i > 2$, the following inequality holds*

$$\frac{|a_i - r|}{|a_1 - r|} \leq \mathbf{Poin}(-r, 0, a_{i-1}, a_1) \ \frac{|a_{i-1} - r|}{|a_1 - r|} \, .$$

Proof: Observe that

$$\frac{|a_i - r|}{|a_1 - r|} = \frac{|a_i|}{|a_1|} \mathbf{Poin}(0, r, a_i, a_1) \tag{2}$$

which, by expanding cross-ratio property, is smaller than

$$\frac{|a_i|}{|a_1|} \mathbf{Poin}(\psi(0), r, a_{i-1}, a_0) \, .$$

The assumption that the map ϕ is not terminal, $\psi(0) < -r$, and algebra imply that

$$\mathbf{Poin}(\psi(0), r, a_{i-1}, a_0) \leq \frac{|a_{i-1}| - r}{|a_1 - r|} \ \frac{|a_1 - (-r)|}{|a_{i-1} - (-r)|} \tag{3}$$

Clearly, $|a_i| < |a_{i-1}|$. Combine (2) and (3) together and replace in the resulting inequality $|a_i|$ by $|a_{i-1}|$. The Lemma follows.

□

Lemma 2.14 *The ratio $|a_2|/|a_1|$ is smaller than $K < 1$ and the bound K depends on $\alpha(\varphi)$ only.*

Proof: The diffeomorphism h identifies the triples

$$(0, |a_2|^2, |a_1|^2) \xrightarrow{h} (\psi(0), |a_1|, |a_0|)$$

and distorts distances only by a bounded amount. In our situation, $\psi(0) < 0$ and $|\psi(0)| < |a_1|$ which yield the claim of the Lemma.

□

The analytic continuation. From Lemmas 2.12 and 2.14 it follows that the cross-ratio $\mathbf{Poin}(-r, 0, a_{i-1}, a_1)$ is smaller than $1-\delta$ and $\delta > 0$ depends only on $\alpha(\varphi)$. Now Lemma 2.13 asserts that the sequence a_i tends to r uniformly fast. "Uniformly" means that the rate depends only on ϵ from Proposition 3. Thus, for a bounded i the cross-ratio $2\mathbf{Poin}(\alpha', -a_i, \psi(0), a_i)$ gets smaller than $\frac{1}{3}$. Since i is bounded, $\mathbf{Cr}(\alpha', -a_i, a_i, a_{i-1})$ is bounded away from 0 in terms of ϵ.

So if we construct φ from ϕ by the staircase construction with $m = i$, assuming $i < E - 1$, then we can apply Proposition 1 to $(\varphi, (\alpha', a_{i-1}), (-a_i, a_i))$ and derive the claim Proposition 3 right away.

The parabolic case. Let us change the coordinates by an affine map so that B^1 goes to $(-1, 1)$ and 0 is the minimum of the central branch. We will keep the old notations irrespective of the change of coordinates having taken place. By Lemma 2.11, a and b may be assumed to be bounded by 30.

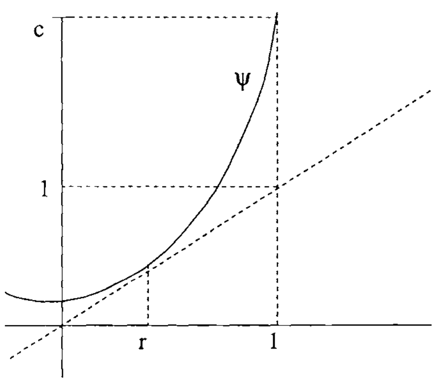

Figure 2: A deep parabolic return

The diffeomorphism h^{-1}, has bounded distortion on $h^{-1}(B)$ by the real Köbe Lemma, for a statement see for example [2], page 237. The bound depends solely on ϵ. Hence, the derivative of h^{-1} is bounded away from 0 on B with the bound depending only on ϵ. Now consider the quantity $\psi(x) - x$ and look for its minimum of B^1. Let r denote a point where this minimum is attained. The change of scale and the choice of r are illustrated on Figure 2.

The minimum of $\psi(x) - x$ is in fact unique, but for the argument we can take r as any minimum of there is more than one. Surely, $r > 0$ and $\psi'(r) = 1$ unless $r = 1$. We observe that for some E_1 and some $\eta_1 > 0$, both determined by ϵ, if $E \geq E_1$, then $r \in (\eta_1, 1 - \eta_1)$. The lower bound on r follows even without choosing E_1, simply because out previous remark that h' is bounded on $(0,1)$ depending on ϵ, and so if r were too close to 0, then the condition $\psi'(r) = 1$ would be violated. For the upper bound, observe that $\psi(1) - 1 \geq \lambda_1$ where $\lambda_1 > 0$ is chosen depending on ϵ by Lemma 2.10. Again because of the upper bound on the derivative of ψ on $(-1, 1)$, one can choose an $\eta_1 > 0$ so that $\psi(x) - x > \frac{\lambda_1}{2}$ is $1 - x < \eta_1$. On the other hand, $\psi(r) - r < \frac{1}{E-5}$ since the interval $(0,1)$ must accommodate $E-3$ consecutive images of 0 by ψ. So by making E_1 large enough we get the desired conclusion.

An interesting estimate is obtained by looking at the orbit

$$\psi(0), \cdots, \psi^{E-2}(0) .$$

Observe that near r the points are close together and advance more or less in equal steps, because the derivative is close to 1. Here is a formal expression of this idea.

Lemma 2.15 *For every $\epsilon > 0$, every $\eta_2 > 0$ and N_0, there is an E_2 so that if $E \geq E_2$, then there are N_0 adjacent intervals $(\psi^j(r), \psi^{j+1}(r))$, .. $\psi^{j+N_0-1}(r), \psi^{j+N_0}(r))$, $1 \leq j \leq E - N_0 - 5$, so that the ratio of lengths of any two of them, not necessarily adjacent, is less than $1 + \eta_2$.*

Proof: Assume that $E > E_1$. Without loss of generality, make $\eta_2 < 1$. First choose $\eta_3 \in (0, \eta_1)$ so that $(1 + \eta_2)^{-1} < (\psi'(x))^{N_0+1} < 1 + \eta_2$ as long as $r - \eta_3 < x < r + \eta_3$. This can be achieved so that η_3 depends only on ϵ and the choice of η_2 and N_0 because the real Köbe Lemma implies that the distortion of ψ is very small on small neighborhoods of r. We noticed already that $\psi(r) - r < \frac{2}{E}$ for $E > 10$. It follows that for $n \leq N_0 + 1$ we have $\psi^n(r) - r \leq \frac{4n}{E}$ as long as $\psi^{n-1}(r) < r + \eta_3$. The last condition can be satisfied by induction if we take $E > 4(N_0 + 1)/\eta_3$. We take j to be the smallest so that $\psi^j(0) \geq r$ and we see that the claim of Lemma 2.15 holds.

□

Asymptotic estimates for box geometry. Adopt the notation $c_n := \psi^n(0)$.

Lemma 2.16 *For every N and $\epsilon > 0$ there is a sequence $o(E)$ which tends to 0 as E grows to infinity so that for every $n \leq N$ the following reverse recurrence relation holds*

$$c_n^2 < \frac{(1 + c_1) \cdot c_{n+1}}{c_1 + c_{n+1}} \frac{n}{n+1} \cdot \frac{2c_{n+1}}{1 + c_{n+1}} + o(E) \, .$$

In particular, for E large enough, depending on ϵ, we have that $c_1 < 0.64$ and $c_2 < 0.788$.

Proof: Since $h^{-1}(1) < 1$, we see that c_n^2 is smaller than

$$\mathbf{Poin}(h^{-1}(-1), 0, |c_n|^2, h^{-1}(1)|) \ ,$$

where h is a diffeomorphism, $\psi(x) = h(x^2)$, with the image ranging over B'. We "push forward" this cross-ratio by h. The property of expanding cross-ratios and simple algebraic transformations give

$$c_n^2 < \mathbf{Poin}(0, c_1, c_{n+1}, 1) \frac{2c_{n+1}}{1 + c_{n+1}} \, .$$

By a direct calculation

$$\mathbf{Poin}(0, c_1, c_{n+1}, 1) = \frac{(1 + c_1) \cdot c_{n+1}}{c_1 + c_{n+1}} \mathbf{Poin}(0, c_1^2, c_{n+1}^2, 1) \, .$$

Take $N_0 = N + 2$ and some $\eta_2 > 0$ in Lemma 2.15 and assume that $E \geq E_2$. Then map the configuration $0, c_1^2, c_{n+1}^2$ by $\psi^{j-1} \circ h$ (where j is also taken from Lemma 2.15). Also, choose w in such a way that $\psi^{j-1} \circ h(w) = 1$. Note that $w < 1$. This gives

$$\mathbf{Poin}(0, c_1^2, c_{n+1}^2, 1) < \mathbf{Poin}(c_j, c_{j+1}, c_{j+1+n}, 1) \ .$$

By choosing E very large, with fixed N and ϵ, we may cause the ratio $\frac{1-c_j}{1-c_{j+1}}$ to be arbitrarily close to 1. On the other hand, by Lemma 2.11,

$$\frac{c_{j+n+1} - c_{j+1}}{c_{n+j+1} - c_j} \leq \frac{n}{n+1}(1 + \eta_2)$$

where η_2 can be made arbitrarily small at the expense of increasing E. This concludes the proof of the reverse recurrence relation.

To get the explicit estimates for c_1 and c_2, note that the right-hand side increases, thus giving poorer estimates, as c_{n+1} increases. Also, it increases as c_1 decreases. So we can always replace c_1 by a lower estimate and c_{n+1} by an upper estimate and get a correct bound for c_n.

Set $N = 3$ in the Lemma and use the formula for $n = 3$ with $c_4 = 1$ and $c_1 = 0$. This results in $c_3 < \frac{\sqrt{3}}{2} < 0.87$ for E large enough. Next, use the formula for $n = 2$ with $c_3 = 0.87$ and $c_1 = 0$. This gives $c_2 < 0.788$. Finally, use the formula for $n = 1$ with $c_1 = 0.64$ and $c_2 = 0.788$. This gives $c_1 < 0.64$. This result can be interpreted to mean that assuming $c_1 \geq 0.64$ leads to a contradiction, hence $c_1 < 0.64$.

□

A tool for constructing quadratic-like mappings. Let Ψ denote the principal inverse branch of the quadratic root, defined on $\mathbf{C} \setminus (-\infty, 0]$. Let G be Joukovsky's map defined on $\mathbf{C} \setminus [-1, 1]$ by $G(z) = z + \sqrt{z^2 - 1}$.

Lemma 2.17 *If $-3 \le a \le 0$, then*

$$\Psi(\mathcal{D}(\frac{\pi}{2},(a,1)) \subset \mathcal{D}(\frac{2\pi}{3},(-1,1)) .$$

Proof: This Lemma could certainly be established by a direct calculation using polar coordinates. However, we will show a different coordinate change, which we will also extensively use in the future.

Let M be the Möbius map $\frac{z}{1-z}$. This map fixes the real line, fixes 0, and sends 1 to infinity. If $x \neq y$ let $A_{x,y}$ be the affine map sending x to 0 and y to 1. Denote $M(A_{a,1}(0))$ by a_1 and observe that $0 \le a_1 \le 3$. Consider the transformation

$$\Psi_1 = M \circ A_{-1,1} \circ \Psi \circ A_{a,1}^{-1} \circ M^{-1} .$$

Ψ_1 is defined except on $[-1, a_1]$, sending that in a univalent fashion onto the complement of $\mathcal{D}(\frac{\pi}{2},(-1,1))$ and extending continuously to -1 and 1. By looking at the action of Ψ_1 at three convenient points, e.g. -1, a_1 and ∞, we conclude that

$$\Psi_1(z) = G(A(z))$$

where A is chosen so that it takes $(-1, a_1)$ onto $(-1, 1)$. The image of $\mathcal{D}(\frac{2\pi}{3},(-1,1))$ by $M \circ A_{-1,1}$ is the set

$$\mathcal{G} = \{z : \ |\arg z| < \frac{2\pi}{3}\}$$

where the principal branch of the argument with values in $(-\pi, \pi]$ is used, and likewise the image of $\mathcal{D}(\frac{\pi}{2},(a,1))$ by $A \circ M \circ A_{a,1}$ is a half-plane, $\Re z > A(0)$. We check that $A(0)$ varies between $-1/2$ and 1 in dependence on a.

What we need to prove becomes: for $-\frac{1}{2} \le \alpha \le 0$, the image by the Joukovsky's map of the set $\{z : \ \Re z > \alpha\}$ is contained in $\mathcal{G}$. Clearly, we may now assume $\alpha = -\frac{1}{2}$. Consider the map $T(y) = G(-\frac{1}{2} + iy)$. For reasons of symmetry, it is enough to restrict our attention to $y > 0$. The limit as $y \searrow 0$ is $e^{2\pi i/3}$. On checks that

$$\frac{dT}{dy} = \frac{iT(y)}{\sqrt{-.75 - y^2 - iy}} . \tag{4}$$

Here, the appropriate branch of the square root is the one which maps into the upper half-plane. Hence

$$\arg\left(\frac{\frac{dT}{dy}}{T(y)}\right) \in (-\pi/2, 0) .$$

This means that as y grows, the trajectory $T(y)$ "curves up" and the needed inclusion follows.

□

Construction of analytic extensions. Let us call ϕ_1 the mapping obtained from ϕ by the full staircase construction. The central domain of ϕ_1 is $B^{E-1} = (-a_{E-1}, a_{E-1})$. It can be considered a type III box mapping with $B'' = B^{E-2} = (-a_{E-2}, a_{E-2})$. The ordering of points is

$$0 < a_{E-1} < c_1 < a_{E-2} < c_2 .$$

Let χ denote the monotone branch of ϕ_1 the domain of which contains the critical value. Let V be the range of χ. Then also consider the mapping φ obtained from ϕ_1 the critical filling. Our construction will be split depending on whether φ shows a terminal return or not.

The terminal case. Suppose that φ shows a terminal return. We may assume without loss of generality that $a_{E-1} \geq 0.33$. Otherwise apply Proposition 1 to $(\varphi, V, (-a_{E-1}, a_{E-1}))$. Since $V \supset (-1, 1)$, we easily check that the metric hypotheses of Proposition 1 are satisfied regardless of the position of $\varphi(0)$

as long as the return is hyperbolic or terminal. So the claim of Proposition 3 follows.

Consequently, we now assume $a_{E-1} \geq 0.33$. If V is the range of χ, let

$$D' := \mathcal{D}(\frac{2\pi}{3}, V) .$$

We claim that $\chi^{-1}(D') \subset \mathcal{D}(\frac{\pi}{2}, (-0.34, 1))$ (where χ^{-1} means the analytic continuation). Using our estimate on c_2 from Lemma 2.16 and the assumption that $a_{E-1} \geq 0.33$, we see that the domain of χ is contained in $(0.33, 0.79)$. Hence $\chi^{-1}(D') \subset \mathcal{D}(\frac{2\pi}{3}, (0.33, 0.79))$. In the upper half-plane, $\mathcal{D}(\frac{2\pi}{3}, (0.33, 0.79))$ coincides with the disk centered at $0.56 + 0.115i$ with radius $\frac{0.46}{\sqrt{3}} < 0.27$. By the triangle inequality, if $z \in \mathcal{D}(\frac{2\pi}{3}, (0.33, 0.79))$, then $|z - 0.33| < 0.54$. On the other hand, $\mathcal{D}(\frac{\pi}{2}, (-0.34, 1))$ is just the disk centered at 0.33 with radius 0.67 and hence the inclusion

$$\chi^{-1}(D') \subset \mathcal{D}(\frac{\pi}{2}, (-0.34, 1))$$

follows, moreover

$$\text{mod } (\mathcal{D}(\frac{\pi}{2}, (-0.34, 1)) \setminus \chi^{-1}(D')) \geq M_1 > 0$$

where M_1 is a constant. From Lemma 2.16,

$$\frac{1 - c_1}{1.34} > \frac{1}{4} .$$

and by Lemma 2.17,

$$\psi^{-1}(\mathcal{D}(\frac{\pi}{2}, (-0.34, 1))) \subset \mathcal{D}(\frac{2\pi}{3}, B^{E-1}) \subset D' .$$

Moreover,

$$\text{mod } \left[D' \setminus \psi^{-1}(\chi^{-1}(D'))\right] \geq \frac{M_1}{2} .$$

Hence D' qualifies as the range of the analytic continuation of the central branch of φ in order to satisfy the requirements of

Proposition 3. By the assumption about a terminal return, the central branch alone is critically complete, so can just ignore the remaining ones.

Non-terminal case. We start with a computation. Using the upper bound $c_1 < 0.64$ from Lemma 2.16, we obtain the following Fact.

Fact 2.3 *Either* $\boldsymbol{Poin}(-1, a_{E-1}, c_1, 1) < 0.49$ *or* $a_{E-1} < 0.4$.

Without loss of generality we may assume that $a_{E-1} < 0.4$ since otherwise we may apply Proposition 1 to ϕ_1 and immediately satisfy the claim of Proposition 3.

Now consider the mapping φ_1 which is either equal to φ if φ makes a non-close return, or is obtained from φ by the full staircase construction otherwise. Let a, b, c be the box parameters of φ_1. We have $a \geq 1$ and $b < 0.4$. If $b \leq 0.33$, then $\mathbf{Poin}(-a, c, b, a) < 0.497$ as long as $c > 0$ and we are done by Proposition 1. Also, if $c \geq 0.1$,

$$\mathbf{Poin}(-a, c, b, a) < \mathbf{Poin}(-1, 0.1, 0.4, 1) < 0.48 \; .$$

In either case we are done by Proposition 1. Now look at the map φ_2 obtained from φ_1 by critical filling. Its box parameters b' and a' satisfy $b' > 0.32$ and $a' < 0.1$. If φ_2 does not show a terminal return, we see that $\mathbf{Poin}(-a', c', b', a') < 0.49$ regardless of c' and we are done by Proposition 1, or otherwise we can follow the argument used to handle the case of φ with the terminal return and $a_{E-1} \leq 0.33$.

Proposition 3 has now been proved in all cases.

2.6 Long return time

For a map $f \in \mathcal{F}_\eta$, let k denote the maximum depth of close parabolic returns for box mappings induced by a sequence of type III steps from the canonical induced map of f.

Proposition 4 *Suppose that the map ϕ_n in the induced sequence $(\phi_i)_{i=2}^n$ is terminal and $n < N$, where N is a constant of* Proposition 2. *For every k and $\eta > 0$, we claim the existence of bounds $M(k,\eta)$ and $K(\eta) > 0$ so that the following holds. If the return time of the restrictive interval is larger than $M(k,\eta)$, then there exist a number $K(\eta)$ and a critically complete real box mapping ϕ induced by the canonical induced map, so that Φ has an analytic continuation as Φ, and mod $(D' \setminus \overline{D}) \geq K(\eta)$ where D and D' denote the domain and range of the central branch of Φ, respectively. In addition, all univalent branches of Φ map over D'.*

Let us define the *induced sequence*

$$(\phi_i, B_i', B_i'')$$

of type III real box mappings so that $(\phi_0, (-q,q), (-q,q))$ is the canonical induced map, and ϕ_{i+1} is obtained from ϕ_i by a type III inducing step. Let c_i, b_i, a_i be the correspondent box parameters, see Definition 2.2.

We will start with the following observation. Even though a box mapping in an induced sequence is not derived from its predecessor as the first return map to the central domain, certainly not for close returns, it shares one important property with the latter.

Lemma 2.18 *Every branch of the box mapping from the induced sequence $(\phi_j)_{i=1}^n$ has the property that no intermediate images of the domain of the branch enter the central domain.*

Proof: This is an easy inductive argument. For the canonical induced map the property holds, then have to come back to the staircase construction and the critical filling to see that the newly created branches also satisfy this property.

□

The next lemma essentially expresses the compactness of finite induced sequences.

Lemma 2.19 *Let ϕ_n be a terminal real box mapping in the induced sequence $(\phi_i)_{i=2}^n$, $n < N$. For any number $\delta > 0$, there exist integers $M(k, \delta, \eta)$ and $i \leq n$ so that if the return time of the restrictive interval of f is larger than $M(k, \delta, \eta)$ then either*

- $\frac{c_n}{a_n} < \delta$ *or*
- *there exists* $2 \leq i \leq n$ *so that* $\frac{b_i}{a_i} < \delta$.

Proof: Let $\mathcal{F}_\eta(k)$ be the subclass of $\mathcal{F}_\eta$ such that the depths of almost parabolic points of period less than the return time of the restrictive interval are less than k. We begin by observing that $\mathcal{F}_\eta(k)$ is a normal family in the $C^{2,1}$ topology. Indeed, all members of this family are in the form $h_f(z-1/2)^2$. Diffeomorphisms h_f are of negative Schwarzian derivative and uniformly η-extendible. It is a well known fact the Schwarzian derivative of an ϵ-extendible iterate of a one-dimensional map with finitely many polynomial-type singularities is bounded from below uniformly in terms of ϵ (see a proof of a very similar estimate in [2].) Thus the normality follows.

Suppose the Lemma is false. Then the lengths of the central boxes B_i,$1 < i \leq n$, would be larger than $L(f)\delta^i$, where $L(f)$ stands for the length of the fundamental inducing interval. It is not a hard fact (see [6]) that L is bounded away from zero in terms of η only. Consider a limit g of maps from $\mathcal{F}_\eta(k)$ which have increasing return times of the restrictive intervals while the lengths of central boxes remain bounded away from 0. Now one can easily see that g has a homterval, i.e., an interval on which all iterations of g are monotone. By a theorem of [12], we conclude that g must have a non-repelling, thus neutral cycle. We also notice that g continues to expand cross-ratios, thus by [15] this neutral orbit is unique and the critical point is in the immediate basin of one point, say p. Now carry out the inducing process for g. The critical point and p will always stay together in the central branch, since branches in the inducing construction are separated by preimages of the fixed point. Next, Fact 2.18 says that for any branch, no intermediate images enter the central domain. From this observation it follows that return times on the central branch in the inducing process for g cannot jump over the period of p. Thus, after finitely many steps an induced map is obtained which exhibits a close return (which must be parabolic, i.e. the image of the real central branch does not cover the critical point). Now, if we take a map f from the sequence which allegedly contradicts the claim of the lemma which is very close to g in the C^2 topology, the construction is conducted in the same way for f, since the course of the construction only depends on where the critical value falls with the respect to the mesh built up by finitely many preimages of the fixed point. The map f will show a parabolic return, but will recover from it after a large number of inducing steps B. Since it takes a long time for the critical value to escape the central domain, and this time can be made arbitrarily large by choosing f close enough to g, we can obtain a map f with an arbitrary long escaping time, contradiction.

□

An induced complex box mapping. Without loss of generality, see Proposition 4 and Proposition 1, we may assume that all characteristic ratios $\alpha(\phi_i)$ are less than $1 - \varepsilon_0$, $\varepsilon_0 > 0$ is a function of η only. Let $\delta = \frac{1}{4}$. Then, by Lemma 2.19, there exists $1 < i \leq n$ so that $2\mathbf{Poin}(-a_i, c_i, b_i, a_i) < \frac{4}{5}$. Proposition 4 then follows from Proposition 1.

2.7 Conclusion

To prove Theorem 2, all we need is to see that all cases have been covered. By Proposition 2, we can assume without loss of generality that after fewer than N type III inducing steps starting from the canonical induced map a terminal return will be encountered. This N depends only on η. Proposition 3 allows to reduce the situation further by assuming that the depths of all close returns which occur in this inducing sequence are bounded by some E_0 depending only on η. This information can be plugged into Proposition 4, making $k := E_0$, which implies Theorem 2 in the remaining case.

2.8 Complex box mappings

We finish this session by relating Theorem 2 to our main objective, Theorem 1. Namely, we will show that if a box mapping can be induced which has an analytic continuation with a bound on $\operatorname{mod}(D' \setminus \overline{D})$, then by further inducing a quadratic-like map can be obtained. More precisely, this quadratic-like map will appear as the central domain of some induced box mapping showing a terminal return.

Proposition 5 *Let $f \in \mathcal{F}$ be renormalizable and critically recurrent. Suppose that ϕ is a critically complete real box mapping, induced by the canonical induced map of f, with an analytic continuation as a complex box mapping Φ. If D and D' denote the domain on range of the central branch of Φ, respectively, assume that $\operatorname{mod}(D' \setminus \overline{D}) \geq K > 0$ and assume that all univalent branches of Φ map over D'. Then the first return map of the maximal restrictive interval of f has an analytic continuation which is quadratic-like with complex bound at least $K/8$, in the sense of Definition 1.1.*

From Proposition 5, we see that Theorem 1 is implied by the claim of Theorem 2. The proof of Proposition 5 will be obtained by analyzing an inducing algorithm for complex box mappings.

Holomorphic inducing: Step A - filling in. Suppose that ϕ is a complex box mapping according to Definition 1.7. Assume also that ϕ is critically complete and that the critical orbit is recurrent. Let B denote the central domain of ϕ. If z belongs to the domain of ϕ, define the *landing time* of z, denoted by $e(z)$ as follows

$$e(z) = \min\{i = 0, 1, \cdots : \phi(z) \in B\}$$

allowing the value of ∞. Then put ϕ_z equal to $\phi(z)$ if $z \in B$ and $\phi^{e(z)}(z)$ otherwise provided that $e(z)$ is finite. Such points z for which the landing time is infinite, are outside the domain of ϕ_1.

Evidently, this gives a complex box mapping ϕ_1 with univalent branches all mapping onto B. Observe that ϕ_1 remains critically complete with a recurrent critical orbit. This is even though the filling-in loses some infinite orbits of ϕ, namely those with infinite landing time. But every point of the critical orbit must have a finite landing time by the assumption about recurrence.

If ϕ is an analytic continuation of some real box mapping φ, then the filling-in can be restricted to the real line and hence ϕ_1 is an analytic continuation of some real box mapping induced by φ.

Type I and type II complex box mappings. A complex box mapping is said to be of *type I* if and only if all univalent branches map onto the central domain. It is called a *type II* map if all univalent branches map onto the range of the central branch. For example, the mapping ϕ_1 obtained in the outcome of filling-in is necessarily of type I.

Filling-in of a type II map. A typical example of filling-in occurs if ϕ is a type II holomorphic box mapping. Each branch ζ of ϕ_1 is a restriction of

$$\overline{\zeta} = \zeta_n \circ \cdots \circ \zeta_1$$

where ζ_i are branches of ϕ. In that context, ζ_1 is called the *parent branch* of ζ, and the domain of ζ_1 is called the *parent domain.* Certainly, the domain of ζ is compactly contained in its parent domain. Notice also that $\overline{\zeta}$ naturally maps onto B' even though ζ by definition maps onto B. Hence, every univalent of a type I holomorphic box mapping arising from a type II holomorphic box mapping has a univalent *dynamical extension* onto B'. If we pick another branch of ϕ_1, say η, it may be that $\overline{\eta} = \zeta_{n+k} \circ \cdots \circ \overline{\zeta_1}$ or that $\overline{\eta} = \zeta_{n-k} \circ \cdots \circ \zeta_1$. In the first case, we say that η is *subordinate* to ζ, in the second case ζ is *subordinate* to η, and in the remaining case will say that they are *independent.*

Step B - critical filling. Now suppose that a holomorphic box mapping ϕ is given. Let us assume that it is critically complete and the critical orbit is recurrent. We will only apply the Step B to type I complex box mapping, though the construction is more general. Let ϕ_0 be the tempered map of ϕ, see Definition 1.9. Then define Φ by changing ϕ on the central domain only, where we set $\Phi = \phi_0 \circ \phi$. This Φ is the outcome of the Step B applied to ϕ. This evidently preserves the property of ϕ being critically complete and recurrent. The central branch of Φ has the form $\zeta \circ \psi$ where ψ is the central branch of ϕ and ζ is either a univalent branch of ϕ, or the identity restricted to B. Again, if ϕ was an analytic continuation of some box mapping φ, then Φ is an analytic continuation of a box mapping induced by φ.

Again, the case of most interest to us is when ϕ is a type I holomorphic box mapping. In that case, Φ is a type II holomorphic box mapping. According to whether the critical value of ϕ is in the central domain of ϕ or not, we describe the situation as either a *close* or a *non-close return*. Among close returns, we can distinguish terminal and non-terminal ones, just like in the case of real inducing. However, the distinction between parabolic and hyperbolic returns vanishes, not only in the case of a close return, but for all non-terminal returns. This causes the complex inducing to be combinatorially simpler than real inducing, with fewer cases to consider.

Inducing steps for type I box mappings. We will now define a *simple inducing step* for type I holomorphic box mappings. If ϕ is such a mapping, the simple inducing step is defined to be Step B followed by Step A. As remarked above, the outcome will be a type I holomorphic box mapping. We make a distinction between a *close* and *non-close* return for ϕ, depending on whether the critical value of ϕ is in the central domain of ϕ. The simple inducing step is defined provided that Step B is defined, i.e. the critical value of ϕ is in the domain of ϕ.

Now we define the *type I inducing step.* It takes a holomorphic type I box mapping ϕ which makes a non-terminal return. The type I inducing step is defined recursively so that it is equal to the simple inducing step if ϕ makes a non-close return, and is equal to the type I inducing step applied to ϕ_1 obtained by the simple inducing step for ϕ otherwise. In other words, the type I inducing step is an iteration of simple inducing steps continued until the first non-close return occurs. Observe that a non-close return must appear eventually by the assumption that the return is not terminal.

2.9 Separation

Conformal moduli. Some of the facts about conformal moduli used in our proofs are very simple indeed. By an annulus, synonymous with "ring domain" we mean any open region of the plane homeomorphic to the punctured plane.

Canonical mapping. A classical theorem says that every annulus A can be mapped conformally onto the ring $\{z : 0 \leq d < |z| < 1\}$ where d is unique and $-\log d$ is called the *modulus* of the annulus. We often invoke the inverse of this map and call it the *canonical mapping* of A. It follows that the modulus is a conformal invariant.

Modulus and holomorphic covers. Let A and A' be annuli and $f : A \rightarrow A'$ be a holomorphic cover of degree k. Then $\text{mod } A' = k \text{ mod } A$. This is also a straightforward consequence of the existence of the canonical map. By the canonical maps, f can be lifted to a cover g of one round ring onto another. g can be continued to a conformal map of the punctured plane onto itself by a sequence of reflections, hence g is just $z \rightarrow e^{it}z^k$ and the claim follows.

Super-additivity. We will say that two annuli C_1 and C_2 are *nesting* provided that they are disjoint and one fits into the bounded connected component of the complement of the other one. Then we define a commutative operation $C_1 \oplus C_2$ on pairs of nesting annuli which results into the smallest annulus containing both C_1 and C_2.

The *super-additivity* of the modulus means that

$$\text{mod } (C_1 \oplus C_2) \geq \text{ mod } C_1 + \text{ mod } C_2 \; .$$

This follows from Lemma 6.3 on page 35 in [10].

Another estimate. The final estimate concerns the situation when a mapping from one annulus onto another is holomorphic and proper but not a cover (has critical points.) Figure 3. illustrates the situation.

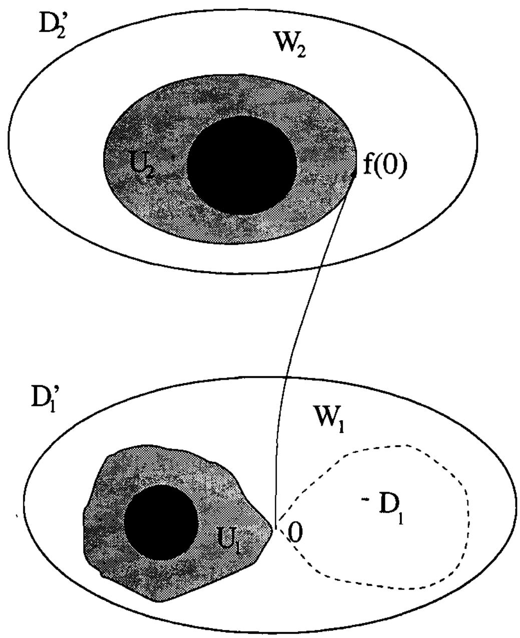

Figure 3: The setting of Lemma 2.20. Similarly colored regions correspond by f. The critical point 0 is on the boundary of U_1. The disc D_1 is the union of the two shaded areas on the lower level. Notice the twin preimage of D_2 denoted by $-D_1$.

Lemma 2.20 *Consider a topological disc D_1', a ring domain $U_1 \subset D_1'$, and the topological disc D_1 determined as the union of U_1 with the bounded connected component of its complement. Denote $W_1 := D_1' \setminus \overline{D_1}$. Then look at a holomorphic mapping f in the form $h(z^2)$ with h univalent, f from D_1' onto a topological disc D_2', proper of degree 2, and assume that f is univalent in D_1. Define $D_2 := f(D_1)$, $W_2 := D_2' \setminus \overline{D_2}$ and $U_2 := f(U_1)$.*

Choose non-negative numbers σ_1 and σ_2 so that

$$\sigma_2 \leq \; mod\, U_2 \quad and$$

$$\sigma_1 \leq \ mod\, U_2 + \ mod\, W_2 \ .$$

Then $mod\, U_1 + \ mod\, W_1 \geq \frac{1}{2}(\sigma_1 + \sigma_2)$.

Proof: Consider the canonical mapping H from the ring $\{z : d < |z| < 1\}$ onto W_2. If c is the critical point of f, let $t = |H^{-1}(c)|$. This splits W_2 into two annuli, $V_i = H(\{z : d < |z| < t\})$ and $V_o = H(\{z : t < |z| < 1\})$. We have mod $W_2 =$ mod $V_i +$ mod V_o. There is an annulus U_i surrounding U_1 which is mapped by f conformally onto V_i and the annulus U_o mapped onto V_o by f as a degree 2 cover. Note that $U_o \oplus U_i = W_1$. So,

$$\text{mod}\ (U_1 \oplus W_1) \geq \ \text{mod}\ U_2 + \ \text{mod}\ V_i + \frac{1}{2}\ \text{mod}\ V_o \geq \qquad (5)$$

$$\geq \ \text{mod}\ \frac{1}{2}(\ \text{mod}\ U_2 + \ \text{mod}\ W_2) + \frac{1}{2}\ \text{mod}\ U_2 \geq \frac{1}{2}(\sigma_1 + \sigma_2)\ .$$

□

Separation bounds.

Definition 2.3 *Let ϕ be a type I holomorphic box mapping. Let p be a univalent branch of ϕ. We say that a univalent map p_e is the* canonical extension *of p if and only if*

- *the domain of p_e is disjoint from B and $\partial B'$, but contains the domain of p,*
- *p_e is an analytic continuation of p and maps onto B'.*

Let us assume that ϕ is a type I complex box mapping and that every univalent branch D of ϕ has a univalent extension mapping over B'. Let $\mathcal{E}_D$ denote this continuation of the branch D. As usual, B is the domain of the central branch of ϕ and B' is the range of the central branch.

Separating annuli. The *separating annuli* for D are any five annuli $A_i(D)$, $i = 1, \cdots, 4$, and $A'(D)$, either open or degenerated to Jordan curves, which satisfy the conditions listed below.

- All annuli are contained in B'.
- The complement of $A_2(D)$ contains B in its bounded component and the domain of $\mathcal{E}_D$ in its unbounded component.
- Given $A_2(D)$, $A_1(D)$ is defined as the intersection of B' and the unbounded connected component of the complement of $\overline{A_2(D)}$.
- The annulus $A'(D)$ is uniquely determined as the set-theoretical difference between the domains of the canonical extension of D (see Definition 2.3) and D.
- The complement of $A_3(D)$ contains $A'(D)$ in its bounded component and B in its unbounded component.
- Given $A_3(D)$, $A_4(D)$ is defined as the intersection of B' with the unbounded connected component of the complement of $\overline{A_3(D)}$.

Separation symbols. Remain in the same set-up, i.e. assume that ϕ is a type I holomorphic box mapping derived by filling-in from a type II holomorphic box mapping and that D is the domain of a univalent branch of ϕ.

Definition 2.4 *A* separation symbol $s(D)$ *for* D *is a choice of separating annuli as described above together with a quadruple of numbers* $s_i(D)$ *for* $i = 1, \cdots, 4$ *so that the following inequalities hold:*

$$s_2(D) \leq \textit{mod}\, A_2(D) \quad \textit{and}$$

$$s_1(D) \leq \text{ } mod\, A_2(D) + \text{ } mod\, A_1(D) \quad and$$

$$s_3(D) \leq \text{ } mod\, A'(D) + \text{ } mod\, A_3(D) \quad and$$

$$s_4(D) \leq \text{ } mod\, A'(D) + \text{ } mod\, A_3(D) + \text{ } mod\, A_4(D)\ .$$

Normalized symbols. We will now impose certain algebraic relations among various components of a separation symbol. Choose a number β, and $\alpha := \beta/2$, together with $\lambda_1(D)$ and $\lambda_2(D)$. Assume that

$$\alpha \geq \lambda_1(D), \lambda_2(D) \geq -\alpha$$

and

$$\lambda_1 + \lambda_2 \geq 0\ .$$

If these quantities are connected with a separation symbol $s(D)$ as follows

$$s_1(D) = \alpha + \lambda_1(D)\ ,$$

$$s_2(D) = \alpha - \lambda_2(D)\ ,$$

$$s_3(D) = \beta - \lambda_1(D)\ ,$$

$$s_4(D) = \beta + \lambda_2(D)\ .$$

we will say that $s(D)$ is normalized with norm β and corrections $\lambda_1(D)$ and $\lambda_2(D)$.

This leads to a definition:

Definition 2.5 *For a type I holomorphic box mapping derived by filling-in, a positive number β is called its* separation index *provided that normalized separation symbols with norm β exist for all univalent branches.*

2.10 Monotonicity of separation

Some terminology. Recall the discussion of filling-in applied to a type II box mapping, in particular the notions of parent branches, subordination of branches, etc. We distinguish the set of "maximal" branches whose defining property is that they are subordinate to no other univalent branch. Domains of maximal branches are mapped by their parent branches directly onto the central domain. Therefore, the domains of canonical extensions of maximal branches mapping onto B' are disjoint. They also cover domains of all branches.

If the return of ϕ_0 is non-close, in the simple inducing step among parent branches we distinguish at most two *immediate branches* which are restrictions of the central branch of ϕ_0 to the preimage of the central domain. All non-immediate parent branches are compositions of the central branch of ϕ_0 with univalent branches of ϕ_0.

We will sometimes talk of branches meaning their domains, for example saying that a branch is contained in its parent branch.

Let ψ be the central branch of ϕ. Let B_1 and B_1' denote the central domain and the range of the central branch of ϕ_1, respectively. Observe that $B_1' = B$ for a non-close return. We will now proceed to build separation symbols with norm β for all univalent domains of ϕ_1. Let b be a univalent branch of ϕ_1 and p denote the parent branch b. Unless p is immediate, we have $p = P' \circ \psi$ where P' is a univalent branch of ϕ_0. Let P be the branch of ϕ whose domain contains the critical value.

Monotonicity of separation indexes. The nice property of separation indexes is that they do not decrease in a simple inducing step. We remark that one could show that they increase at a uniform rate (see [3], Theorem C). For now, we prove

Lemma 2.21 *Let ϕ_1 be derived from a holomorphic type I box mapping ϕ_0 by a simple inducing step. If β is a separation index of ϕ_0, then β is also a separation index of ϕ_1. If ϕ_0 makes a non-close return, then the central branch of ϕ_1 is quadratic-like with complex bound at least $\frac{\beta}{4}$, in the sense of Definition 1.1.*

Proof: Note that it is sufficient to construct the separation symbols for maximal branches. Indeed, consider a separation symbol $s(b)$ for a maximal branch b and let b' be subordinate to b. We claim that we can put $s(b') = s(b)$. We can take $A_1(b') = A_1(b)$ and $A_2(b') = A_2(b)$. Likewise, we can certainly adopt $A_4(b') = A_4(b)$, and $A_3(b')$ can be chosen to contain $A_3(b)$. The annulus $A'(b)$ is the preimage of the annulus $B' \setminus B$ by the parent branch of b. The annulus $A'(b')$ is the preimage of the same annulus by the canonical extension of b, so it has the same modulus. Since the domain of the canonical extension of b' is contained in the parent domain (equal to the domain of the canonical extension of b), the assertion follows.

Non-close returns. Let us assume that ϕ makes a non-close return, that is $P \neq \psi$.

The case of p immediate. Let b denote the maximal branch in p. The new central domain B_1 is separated from the boundary of B by an annulus of modulus at least $(\beta+\lambda_2(D))/2$. The annulus $A_2(b)$ around B_1 will be the preimage by the central branch of $A_3(P) \oplus A'(P)$. Notice that Lemma 2.20 can be applied to this configuration with $U_2 := A_3(P) \oplus A'(P)$, $W_2 := A_4(P)$, $\sigma_1 := s_4(P)$, $\sigma_2 := s_3(P)$. This yields

$$s_1(b) = \frac{\beta + \lambda_2(P)}{2} \text{ and}$$

$$s_2(b) = \frac{\beta - \lambda_1(P)}{2} .$$

Of course, since components of the symbol are only lower estimates, we are always allowed to decrease them if needed. The annulus $A'(b)$ is naturally given as the preimage of the annulus between B_1 and the boundary of B by the central branch, likewise $A_3(b)$ is the preimage of $A_2(P)$.

Since the first two preimages are taken in an univalent fashion, we set

$$s_3(b) = \frac{\beta + \lambda_2(P)}{2} + \alpha - \lambda_2(P) \, .$$

To estimate $s_4(b)$, we apply Lemma 2.20 with $U_2 := A_2(P)$, $W_2 := A_1(P)$. By substituting $\sigma_1 := s_1(P)$ and $\sigma_2 := s_2(P)$, we get

$$s_4(b) = s_3(b) + \frac{\lambda_1(P) + \lambda_2(P)}{2} = \frac{\beta}{2} + \alpha + \frac{\lambda_1(P)}{2} \, .$$

Thus, if we put

$$\lambda_1(b) = \frac{\lambda_2(P)}{2} \, , \; \lambda_2(b) = \frac{\lambda_1(P)}{2} \, ,$$

we get a valid separation symbol with norm β.

In the remaining non-immediate cases, the branch P' is defined by $p = P' \circ \psi$.

P' and P independent. To pick $A_2(b)$, we take the preimage by ψ of $A'(P)$. We claim that its modulus in all cases is estimated from below by $\alpha + \delta$ where δ is chosen as the supremum of $-\lambda_2(b')$ over all univalent branches b' of ϕ_0. Indeed, P is carried onto B by the extended branch, and the estimate is α plus the maximum of $\lambda_1(b')$ with b' ranging over the set of all short univalent domains of ϕ. The assertion follows since $\lambda_1(b') + \lambda_2(b') \geq 0$ for any b'. To estimate $\mod A_1(b)$, use Lemma 2.20 with $U_2 := A'(P)$ and $W_2 := A_3(P) \oplus A_4(P)$. By the hypothesis of induction, the estimates are

$$s_1(b) = \frac{\beta + \lambda_2(P)}{2} \text{ and}$$

$$s_2(b) = \frac{\alpha + \delta}{2} .$$

The annulus $A'(b)$ is determined with modulus at least $s_1(b)$. The annulus $A_3(b)$ will be obtained as the preimage by the central branch of $A'(P')$. This has modulus at least $\alpha + \delta$ in all cases as argued before. The modulus of $A_4(b)$ is estimated using Lemma 2.20 with $U_2 := A'(P')$ and $W_2 := A_3(P') \oplus A_4(P')$. By induction,

$$s_3(b) = \frac{\beta + \lambda_2(P)}{2} + \alpha + \delta \text{ and}$$

$$s_4(b) = s_3(b) + \frac{\beta + \lambda_2(P') - \alpha - \delta}{2} .$$

We put $\lambda_1(b) := \frac{\lambda_2(P)}{2}$ and $\lambda_2(b) := \frac{\alpha - \delta}{2}$. We check that

$$s_3(b) + \lambda_1(b) = \frac{\beta}{2} + \alpha + \lambda_2(P) + \delta \geq \beta - \lambda_2(P) + \lambda_2(P) \geq \beta .$$

In a similar way one verifies that

$$s_4(b) - \lambda_2(b) \geq \beta .$$

Also, the required inequalities $-\alpha \leq \lambda_1(b), \lambda_2(b) \leq \alpha$ and $\lambda_1(b) + \lambda_2(b) \geq 0$ follow directly.

P' subordinate to P. This means that some univalent mapping onto B' transforms P onto B and P' onto some P''. Consider $A_2(P'')$ which separates B from P'', and a larger annulus $A_1(P'')$. The preimage of $A_2(P'')$ first by the extended branch and then by the central branch give us $A_2(b)$. The estimates are (using Lemma 2.20 with $U_2 := A_2(P'')$ and $W_2 := A_1(P'')$):

$$s_2(b) = \frac{\alpha - \lambda_2(P'')}{2} \text{ and}$$

$$s_1(b) = \frac{\alpha + \lambda_1(P'')}{2} .$$

The annulus $A'(b)$ is uniquely determined with modulus $s_1(b)$. Then $A_3(b)$ is the preimage of the annulus separating P'' from B. Applying Lemma 2.20 with $W_2 := A_4(P'')$, we get

$$s_3(b) = \frac{\alpha + \lambda_1(P'')}{2} + \beta - \lambda_1(P'') = \beta + \frac{\alpha - \lambda_1(P'')}{2} \text{ and}$$

$$s_4(b) = s_3(b) + \frac{\lambda_1(P'') + \lambda_2(P'')}{2} = \beta + \frac{\alpha + \lambda_2(P'')}{2} .$$

Set

$$\lambda_1(b) = \frac{-\alpha + \lambda_1(P'')}{2} \text{ and}$$

$$\lambda_2(b) = \frac{\alpha + \lambda_2(P'')}{2} .$$

The requirements of a normalized symbol are clearly satisfied.

P subordinate to P'. This situation is analogous to the case of an immediate parent branch considered at the beginning. Indeed, by composing $\psi' := P'' \circ \psi$, where P'' is the canonical extension of P', we get a folding branch with range B'. The domain of ψ' is contained in B and so the separating annuli for any branch of ϕ also separate it from the domain of ψ', and the construction of the separating annuli for ϕ_1 proceeds like in the immediate case except that the preimages are taken by ψ' and not ψ. Because of the inclusion between the domains of ψ and ψ', the estimates can only improve.

A close return. In this case there are no immediate parent branches and we really have only one case to consider. Fix a univalent branch g of ϕ_1, let p be its parent branch, and denote $p = P' \circ \psi$. Consider $A_2(P')$ and $A_1(P')$. Their preimages by the central branch give us $A_2(b)$ and $A_1(b)$, respectively. The estimates are

$$s_2(b) = \frac{\alpha - \lambda_2(P')}{2} \text{ and}$$

$$s_1(b) = \frac{\alpha + \lambda_1(P')}{2} \, .$$

The annulus $A'(b)$ is uniquely determined with modulus at least $s_1(b)$, and $A_3(b)$ will be the preimage by ψ of $A_3(P') \oplus A'(P')$. The modulus of $A_4(b)$ can be estimated by applying Lemma 2.20 with $U_2 := A_3(P')$ and $W_2 := A_4(P')$. The estimates are

$$s_3(b) = \frac{\alpha + \lambda_1(P')}{2} + \beta - \lambda_1(P') = \beta + \frac{\alpha - \lambda_1(P')}{2} \text{ and}$$

$$s_4(b) = s_3(b) + \frac{\lambda_1(P') + \lambda_2(P')}{2} = \beta + \frac{\alpha + \lambda_2(P')}{2} \, .$$

Set

$$\lambda_1(b) = \frac{-\alpha + \lambda_1(P')}{2} \text{ and}$$

$$\lambda_2(b) = \frac{\alpha + \lambda_2(P')}{2} \, .$$

The requirements of a normalized symbol are clearly satisfied. Not quite surprisingly, these are the same estimates we got in the non-close case with P' subordinate to P.

Conclusion. To finish the proof of Lemma 2.21 it remains to show that $\operatorname{mod}(B'_1 \setminus B_1) \geq \frac{\beta}{4}$ when ϕ_0 makes a non-close return. Us usual, define P as the branch of ϕ_0 whose domain contains the critical value of ϕ_0. Since B_1 is the preimage by ψ of the domain of P, the preimage by ψ of $A'(P) \oplus A_3(P)$ is an annulus separating B_1 from the boundary of B'_1. The modulus of this annulus is at least $\frac{\beta - \lambda_1(P)}{2} \geq \beta/4$ which provides the needed estimate.

□

2.11 Getting to terminal mappings

It remains to prove that complex bounds for the mapping which exists by Theorem 2 imply complex bounds for the terminal map.

Let us prove Proposition 5. Suppose that we have a real box mapping φ induced by the canonical induced map of f and that φ has an analytic continuation as a complex box mapping Φ where Φ is as described in the hypotheses of Proposition 5. Let us first apply the filling-in to Φ. This will give a type I complex box mapping ϕ for which every univalent branch has a canonical extension in the sense of Definition 2.3. We claim that $K/2$ is a separation index for ϕ. Indeed, if p is a univalent branch of ϕ, we can set $A_1(p) = D' \setminus \overline{D}$ and $A'(p)$ is determined with modulus at least K. This leads to a separation symbol $(K, 0, K, K)$ for ϕ which is easily modified to a normalized symbol with norm $K/2$.

Let us apply the holomorphic inducing to ϕ for as long as it goes, that is until a terminal return is encountered. Notice that eventually there will be a terminal return. Indeed, each type I inducing step involves composing the central branch with other branches. So the return time of the central branch in terms of iterations of f grows. By Lemma 2.6 this must stop at a terminal return eventually. By Lemma 2.5, when this happens for some ϕ_n, the central branch of ϕ_n is just an analytic continuation of the first return map into the maximal restrictive interval of f.

By the construction of a type I inducing step, ϕ_n was derived by a simple inducing step from some ϕ' which showed a non-close return. Applying the monotonicity of separation indices, we see that ϕ' still has $K/2$ as its separation index, and then applying the second part of Lemma 2.21 we get the claim of Proposition 5.

3 Bounds for Consecutive Renormalizations

3.1 Basic properties

Real renormalization. Suppose that $f \in \mathcal{F}$ is infinitely renormalizable. Let $I_1 \supset \cdots \supset I_n \supset \cdots$ be the sequence of all locally maximal restrictive intervals of f ordered by inclusion. Let $r_1 < \cdots < r_n < \cdots$ be the corresponding first return times. Let A_n be the affine map sending I_n onto $(-1, 1)$ with the orientation chosen so that

$$g_n := A_n \circ f^{r_n} \circ A_n^{-1}$$

belongs to $\mathcal{F}$. That is, the orientation must be chosen so that the preimage of 1 by A_n is *not* contained in the range of f^{r_n}. Of particular importance is g_1 further called the *first renormalization.* Note that r_i is a divisor of r_{i+1} for every i and then r_{i+1}/r_i will be called *relative return times.*

The basic fact is the following:

Fact 3.1 *Let f be an infinitely renormalizable quadratic polynomial. Then there is a constant $\eta > 0$ so that for every n, $g_n \in \mathcal{F}_\eta$.*

Proof: Compare this with Theorem 2.1, items 1 and 1', on page 454 in [11]. This Theorem does not imply Fact 3.1, however its proof on pages 457-458 can be followed literally to get the needed estimate.

□

The case of bounded returns.

Theorem 3 *Let f be an infinitely renormalizable quadratic polynomial. Choose a positive integer n and choose T so that* *the relative return times* r_{i+1}/r_i *for* $i = 1, \cdots, n$ *are all bounded from above by* T. *Then for every* T *there is* $K > 0$ *and* M *so that the mapping* f^{r_n} *restricted to* I_n *has an analytic continuation as a quadratic-like map with complex bound* K. *Moreover, the diameter of the range of this extension divided by the length of* I_n *is bounded by* M.

Derivation of the main Theorem. Let us postpone the proof of Theorem 3 and first derive Theorem 1 from Theorem 2, Proposition 5 and Theorem 3.

Recall that all mappings g_n belong to the same class $\mathcal{F}_\eta$ with $\eta > 0$. Applying Theorem 2 and Proposition 5 to mappings g_n, we get the following corollary:
Suppose that for some $n \geq 1$ *we have* $r_n/r_{n-1} \geq N$, *where* N *is determined by* η *in Theorem 2. Then* g_n *can be continued analytically as a quadratic-like map with complex bound* K, *where* K *is a constant.*

Let n_j denote the subsequence of those n which satisfy this condition. We are done with Theorem 1 for first return mappings into intervals I_{n_j}.

A straightening theorem. Let us quote a fact from holomorphic dynamics.

Fact 3.2 *Let* f *be quadratic-like mapping with complex bound* κ. *Suppose that* $f(\overline{z}) = \overline{f(z)}$, *which is also meant to imply that the domain* U *of* f *is symmetric with respect to the real line. For every* $K > 0$ *there are* Q *and* $L > 0$ *so that if* $\kappa \geq K$, *then a real* Q*-quasiconformal homeomorphism exists, defined on an open set* W, *and the following hold:*

- $H(W) \subset U$ *and* $F = H^{-1} \circ f \circ H$ *is a real quadratic polynomial,*
- *the Green function of the Julia set of* F *with respect to* ∞ *is bounded from below by* L *on the complement of* W.

Proof: The straightening theorem was introduced in [1]. The parameters do not appear there, but it is widely acknowledged that the proof furnishes them, see [11].

□

Reduction of the bounded return case to polynomials. Now focus on some n_j. Apply the straightening theorem with $f := g_{n_j}$. This will give us F, W and H. Without loss of generality normalize F so that it belongs to $\mathcal{F}$. Choose some $n_j < m < n_{j+1}$. Then F satisfies the hypothesis of Theorem 3 with $n := m$ and $T := N$. As a consequence, the m-th renormalization of F has a continuation as a quadratic -like map Φ with complex bound K' (here K' is determined by N, hence ultimately on η). Since the range of Φ has bounded diameter by Theorem 3, the value of Green's function of the Julia set of F is bounded on there, say by L_1. If L_1 is less than L from Fact 3.2, we can infer that the range of Φ is contained in W. In that case, the domain and range of Φ can be carried by H into the phase space of $f^{\tau_{n_j}}$. This proves the claim of Theorem 1 with complex bound at least K'/Q.

If L_1 is greater than L. observe that by restricting Φ to the preimage of its domain by Φ, we also get a quadratic-like map. The domain of Φ now becomes the range of this restriction. The complex bound will get divided by 2, but the bound on the Green function will be divided by a power of 2 as well. After restricting the domain of Φ by a bounded number of such operations, we can get its range fit into W, will the complex bound will remain bounded away from 0.

This concludes the proof of our main theorem (Theorem 1), modulo Theorem 3 which gets handled in the rest of this paper.

3.2 Bounded geometry in the real domain

Let us take a quadratic polynomial f which satisfies the hypotheses of Theorem 3 with some parameters n and T. Keep the notations I_1, .., I_n for the sequence of locally maximal restrictive intervals, adding $I_0 := (-1, 1)$. Let us also set $r_0 := 1$. To further facilitate the notation, let us write I_m^k with $0 \le k < r_m$ for the unique k-th preimage of I_m by f which contains $f^{r_m - k}(0)$. In other words, I_m^k is the k-th preimage of I_m taken along the critical orbit. With these notations, let us recall some facts.

Fact 3.3 *Assume that f satisfies the hypothesis of Theorem 3 with parameters n and T. For every T there is a $K_1 > 0$ so that for every $0 < m \le n$ and for every $0 \le k < r_m$ and $0 \le k' < r_{m-1}$, if $I_m^k \subset I_{m-1}^{k'}$, then*

$$\frac{|I_m^k|}{|I_{m-1}^{k'}|} \ge K_1 \; .$$

Also, if $x, y \in I_m^k$, then $(f^k)'(x)/(f^k)'(y) \ge K_1$.

Proof: Theorem 2.1 of page 454 in [11] establishes the independence of K from n for a fixed f, not necessarily a polynomial. However, the proof gives the independence on f in our situation as well.

□

Fact 3.4 *If f is an infinitely renormalizable polynomial, 0 is the critical point, I is the restrictive interval with return time M, then there is a constant $K_2 > 0$ so that*

$$\frac{|f^M(0)|}{|I|} \ge K_2 \; .$$

Proof: This fact follows directly from Fact 3.1, namely that the derivative of the first return map into a restrictive interval is bounded.

□

The pull-back of I_n. We will start talking about inverse branches of mappings. This has, of course, nothing to do with branches of box mappings. Consider the inverse branch of f^{1-r_n} which sends I_n onto $I_n^{r_n-1}$. Let J_n be the largest interval symmetric with respect to 0 to which this inverse branch can be extended as a homeomorphism. One end-point of J_n must be the critical value of f of order less than r_n closest to I_n. It follows that we must find $0 < L < r_n$ so that I_n^L is the closest to I_n and then one end-point of J_n is $f^{r_n-L}(0)$. Let us call this the "foot-end" of J_n and the other end-point of J_n will be designated as the "head-end". Define J_n^k for $0 < k < r_n$ as the k-preimage of J_n by f so that $J_n^k \supset I_n^k$. The distinction between foot and head ends will be inherited so that f^k always maps the foot end to the foot end and the head end to the head end. In addition, define $J_n^{r_n}$ as the full preimage of $J_n^{r_n-1}$ by f. If you try to graph f^{r_n} on $J_n^{r_n}$ after choosing the orientation so that $f^{r_n}(0) > 0$, it is unimodal, and the names of "foot" and "head" ends are justified by where the feet of this graph stand. Clearly $J_n^{r_n}$ is symmetric with respect to 0. The following is a corollary to the bounded geometry.

Lemma 3.1 *We always have $J_n^{r_n} \subset J_n$ and moreover, if Δ is the Hausdorff distance between J_n and $J_n^{r_n}$, then*

$$\frac{\Delta}{|J_n|} \geq K_3$$

where K_3 is a positive constant.

Proof: The end-points of $J_n^{r_n}$ are mapped to 0 by f^L. They must be the closest to 0 with this property. Inside I_n^L there is one such a point. Let us call this point a_1 while a_2 is chosen to be mapped to $f^{r_n}(0)$ by f^L. That is, a_2 is an end-point of J_n while $J_n^{r_n}$ is contained between a_1 and its symmetric image. The mapping f^{r_n} restricted to I_n^L sends a_1 to a_2 and is conjugate to the n-th renormalization of f. The conjugacy puts

a_2 in correspondence to 0 and a_1 in correspondence to $f^{r_n}(0)$. It follows that the range of f^{r_n} restricted to I_n^L ends at a_2 and covers a_1. But when f^{r_n} viewed globally, a_2 is the turning value nearest to $f^{r_n}(0) \in I_n$ and thus a_1 must be between a_2 and I_n, hence between a_2 and 0. This proves that $J_n^{r_n} \subset J_n$.

Notice that f^L sends a_2 to $f^{r_n}(0)$, a_1 to 0, I_n^L to I_n, and according to Fact 3.3 its distortion is bounded. By Fact 3.4 the ratio of $|f^{r_n}(0)|$ to I_n is bounded from below, thus

$$\frac{|a_2 - a_1|}{|I_n^L|}$$

is bounded from below by a constant. Since $\Delta = |a_2 - a_1|$, in order to finish the proof of Lemma 3.1, we need to bound

$$\frac{|I_n^L|}{|J_n|}$$

from below by a positive constant. Note that I_{n-1} contains both intervals. Indeed, it must contain r_n/r_{n-1} closest returns of the orbit of 0 to itself. Now the needed bound follows from Fact 3.3.

□

The configuration involving $J_n^{r_n-L}$ is important to understand. The foot end of this interval is at 0.

Lemma 3.2 *Choose an integer* $0 < p < L$. *Choose* c *to be* $f^p(0)$ *and so that* $J_n^{r_n-L}$ *lies between* 0 *and* c *and* f^{-p} *extends as a homeomorphism to the bigger interval while still mapping* $J_n^{r_n-L}$ *onto* $J_n^{r_n+p-L}$. *Then for every* $M > 0$ *there is* $M' > 0$ *so that if*

$$\frac{|J_n^{r_n+p-L}|}{dist(J_n^{r_n+p-L}, 0)} \leq M'$$

then

$$\frac{|J_n^{r_n-L}|}{|c|} \leq M .$$

Proof: Choose n_1 to be the smallest positive integer with the property that some $I_{n_1}^{k_1}$ is contained in the convex hull of $J_n^{r_n+p-L}$ and 0. Choose n_2 the largest integer with the property that some $I_{n_2}^{k_2}$ contains $J_n^{r_n+p-L}$. Observe that for every $k > 0$ if M' is chosen to be small enough, then $n_2 - n_1 \geq k$. This is a straightforward consequence of Fact 3.3. By mapping this configuration forward by f^p, we see that $[0, c]$ contains $I_{n_1}^{k_1-p}$ while $J_n^{r_n-L}$ is contained in $I_{n_2}^{k_2-p}$. The claim follows from Fact 3.3.

□

3.3 A change of coordinates

The idea of our change of coordinates is implicit in [16] and was expressly considered in [11].

Choose a number p between 0 and $r_n - 2$. Let f_p^{-1} be the inverse branch of f defined on $\mathbf{C} \setminus [f(0), \infty]$ chosen so that $f_p^{-1}(J_n^p) = J_n^{p+1}$. Let A_p be the affine map which sends the foot end of I_n^p to 1 and the head end of I_n^p to 0. Define ζ as the Möbius map

$$\zeta(z) = \frac{z}{1-z} .$$

The mapping ζ sends the interval $(0, 1)$ to the positive half-line while ∞ goes to -1. Take

$$T_p := \zeta \circ A_{p+1} \circ f_p^{-1} \circ A_p^{-1} \circ \zeta^{-1} .$$

The map T_p is symmetric with respect to the real line, so its enough to understand it in the upper half-plane. Let us first look at

$$A_{p+1} \circ f_p^{-1} \circ A_p^{-1} .$$

This is a univalent map of the upper half-plane into a quadrant with vertex at $A_{p+1}(0)$. The points 0, 1 and ∞ are fixed. As a consequence, T_p maps the upper half-plane into the upper half-plane minus the closed disk $\overline{\mathcal{D}(\frac{\pi}{2}, < -1, \zeta(A_{p+1}(0)) >)}$ where

$< x, y >$ means the open interval bounded by x and y without deciding which is the right or left end-point. The exception is when $\zeta(A_{p+1}(0)) = \infty$, which happens exactly for $p = r_n - L - 1$, when T_p maps the upper half-plane onto the quadrant $\{x + iy : x > -1, y > 0\}$. At the same time T_p fixes -1, 0 and ∞. It follows that T_p is related to Joukovsky's map by an affine change of coordinates. Recall that Joukovsky's map $G(z)$ is given by

$$G(z) = z + \sqrt{z^2 - 1} .$$

If B_p is the affine map which fixes -1 and sends $\zeta(A_p(f(0)))$ to 1, and C_p also fixes -1 and sends $\zeta(A_p(0))$ to 1, then

$$T_p = C_{p+1}^{-1} \circ G \circ B_p . \tag{6}$$

The action of a map T_p can be visualized as follows: it is the composition of two maps, Joukovsky's map rescaled in such a way that it "explodes" the interval $< -1, \zeta(A_p(f(0))) >$ to the circle $\partial\mathcal{D}(\frac{\pi}{2}, < -1, \zeta(A_{p+1}(0)) >)$ and the affine map which fixes -1, preserves the orientation and sends $\zeta(A_p(f(0)))$ to $\zeta(A_{p+1}(0)))$.

Easy properties of maps T_p.

Lemma 3.3 *For $0 \le p \le r_n - 2$, if $0 \ge \zeta(A_p(f(0))) > -1$, then*

$$\frac{1}{2} < \frac{\zeta(A_{p+1}(0)) + 1}{\zeta(A_p(f(0))) + 1} \le 1 .$$

Proof: Recall the representation (6) and let $x = B_p(0)$. Since T_p fixes 0

$$G(x) = C_{p+1}(0) = \frac{1 - \zeta(A_{p+1}(0))}{1 - \zeta(A_{p+1}(0))}$$

and Lemma 3.3 is equivalent to

$$1 \le \frac{G(x) + 1}{x + 1} < 2 .$$

But

$$G(x) + 1 = x + 1 + \sqrt{(x+1)(x-1)}$$

so this estimate is obvious.

□

Another Lemma deals with Joukovsky's map and its action as an "exploding airbag".

Lemma 3.4 *For every Q and $\delta > 0$ there is a $\gamma > 0$ so that if $r \geq 1+\delta$, $z = x+iy$ satisfies $-1 \leq x \leq 1$, $y \leq \gamma(r-x)$ and $y \leq \gamma$ then*

$$\frac{\Im G(z)}{G(r) - \Re G(z)} > Q\frac{y}{r-x} \; .$$

Proof: To make the domain of the problem compact, allow $r = \infty$. Then the claim becomes $\Im G(z) > 2Qy$. Assuming that the Lemma fails, take convergent subsequences $z_n \to t \in [-1, 1]$ so that

$$\Im G(z_n) \leq Q\frac{y_n(G(r) - \Re G(z_n))}{r - x_n} \; .$$

Because of the "explosive" behavior of G near the interval $[-1, 1]$ this is impossible.

□

Geodesic neighborhoods. By convention, let the value of $\arg z$ belong to $(-\pi, \pi]$. Let $c > 0$. Define

$$\mathcal{G}(c) := \{x + iy \in \mathbf{C} : \; x \geq 0 \;\text{ or }\; |y|/|x| > c\,\}.$$

Observe that if $0 < \alpha < \frac{\pi}{2}$, then

$$\zeta \circ A_p(\mathcal{D}(\pi - \alpha, I_n^p)) = \mathcal{G}(\tan\alpha) \; .$$

Lemma 3.5 *Let* $0 \le p \le r_n - 1$. *Choose* x *on the real so that* $x \ge \max(-1, \zeta(A_p(f(0))))$. *Let* z *be a complex number, not in* $(-\infty, x)$. *Then*

$$|\arg(T_p(z) - T_p(x))| \le |\arg(z - x)| \,.$$

Proof: Observe that T_p is a univalent map of $\mathbf{C} \setminus [-\infty, x]$ onto $\mathbf{C} \setminus [-\infty, T_p(x)]$. Hence it shrinks the respective Poincaré metrics. Note that in the region $\{z : z \notin [-\infty, 0]\}$, lines $|\arg z| = const$ are loci of points of fixed hyperbolic distance from the line $\arg z = 0$.

$\square$

Lemma 3.6 *We claim that for every* K *there are positive numbers* c_0 *and* η_1 *so that for every* $0 < c \le c_0$ *the set*

$$T_{r_n - L - 1} \circ \cdots \circ T_0(\mathcal{G}(c))$$

is contained in the intersection of the three sets specified below:

- $\mathcal{G}(c)$,
- $\{x + iy : x > -1\}$,
- $\{x + iy : x \ge -1 + \eta_1 \text{ or } \frac{|y|}{|x|} > Kc\}$.

Proof: Of the three sets participating in the intersection, the first one is clearly there by Lemma 3.5 and the second is there because $T_{r_n - L - 1}$ is a root mapping the upper half-plane into the quadrant. So only the last one requires a new estimate. Without loss of generality, $K > 1$ or the third set contains the first one. Set $c_0 = \eta_1 = \frac{1}{10K^2}$. If z is in the difference between the first and third sets, then $|z + 1| < \frac{1}{5K}$ and $|z + 1| < 2Kc$. Since $T_{r_n - L - 1}$ has the form

$$z \to \sqrt{z + 1} - 1 \,,$$

we have

$$|T^{-1}_{r_n-L-1}(z)+1| < \frac{1}{5K}|z+1| < c/2$$

which means that $T^{-1}_{r_n-L-1}(z)$ is not in $\mathcal{G}(c)$. By Lemma 3.5 this implies that

$$z \notin T_{r_n-L-1}\circ\cdots\circ T_0(\mathcal{G}(c)) .$$

□

Orbits under mappings T_m.

Lemma 3.7 *Let $\Re z > -1$ but suppose that*

$$\Re\,[T_{r_n-2}\circ\cdots\circ T_{r_n-L}(z)] \leq -1 .$$

We claim that there is an integer $r_n - L \leq q \leq r_n - 2$ and for every $\eta > 0$ there is an $\eta_2 > 0$ with these properties:

- $-1 < \Re\,[T_{q-1}\circ\cdots\circ T_{r_n-L}(z)] < \zeta(A_{q-1}(f(0)))$,
- *if* $\Re z \geq -1+\eta$, *then* $\zeta(A_{q-1}(f(0))) \geq -1+\eta_2$.

Proof: Because of the symmetry of the problem, we may assume that z is in the upper half-plane. Let us find the smallest q' so that

$$\Re\,[T_{q'}\circ\cdots\circ T_{r_n-L}(z)] \leq -1 .$$

Proceed to define a sequence a_m for $r_n - L \leq m \leq q'$ by reverse induction. Take $a_{q'} := \zeta(A_{q'}(f(0)))$. Then assuming that a_{m+1} was defined, set $a_m := T_m^{-1}(a_{m+1})$ if a_{m+1} is in the range of T_m, or let $a_m := \zeta(A_m(f(0)))$ otherwise.

Our first observation is that for every m between 0 and q' we have

$$\arg(T_{m-1}\circ\cdots\circ T_{r_n-L}(z) - a_m) > \frac{\pi}{2} . \tag{7}$$

This is proved by reverse induction. For $m = q'$ we check first that the critical point $a_{q'}$ of $T_{q'}$ must be positive. To this end we will show that Joukowski's map curves the line $\Re z = -1$ leftward, i.e. the image of $\Re z = -1$ by G lies in the half-plane $\{\Re z \leq -1\}$. By symmetry, $G(z) = -G(-z)$ and $\overline{G(z)} = G(\overline{z})$. Hence we will look at the image of the half-line $\Re z = 1, \Im z \geq 0$ instead. As in (4) we calculate

$$\frac{dG}{dy} = \frac{iG(y)}{\sqrt{-y^2 + 2iy}} .$$

Here, the appropriate branch of the square root is the one which maps into the upper half-plane. Hence, for positive y

$$\arg(\frac{\frac{dG}{dy}}{G(y)}) = \frac{\pi}{2} - \frac{1}{2}\cot^{-1} y .$$

varies between 0 and $\pi/4$. This means that as y grows, the trajectory $G(y)$ "curves rightward" staying all the time in the quadrant $\{\Re z \geq 1\}$. After conjugation with symmetries the needed inclusion for the image of $\Re z = -1$ follows. Now, recall that $T_{q'} = C_{q'+1} \circ G \circ B_q$ is a composition of Joukowsky's map with two affine maps which fix -1 and simultanously reverse or preserve orientation in dependence on whether $a_{q'}$ is greater than -1 or not. Therefore, if $a_{q'}$ were smaller than -1 than the image of the line $\Re z = -1$ by $T_{q'}$ would be contained in the half-plane $\Re z \geq -1$ and $T_{q'}$ would not map the point $T_{q'-1} \circ \cdots \circ T_{r_n - L}(z)$ to the other side of -1.

From what was said above we see immediately that if $a_{q'}$ is positive then the image of the line $\Re z = a_{q'}$ lies in the half plane $\Re \geq a_{q'}$. Again if $\Re T_{m-1} \circ \cdots \circ T_{r_n - L}(z) \geq a_{q'}$ then this point would not be maped to the other side of -1.

For an inductive step, if a_{m+1} has a preimage by T_m, then by Lemma 3.5 the inductive step follows. Otherwise, $a_{m+1} < \zeta(A_m(0))$. By (7) and the induction all already considered a_m are positive. Hence

$$\arg(T_m \circ \cdots \circ T_{r_n-L}(z) - \zeta(A_m(0))) >$$

$$\arg(T_m \circ \cdots \circ T_{r_n-L}(z) - a_{m+1}) > \frac{\pi}{2}$$

and the inductive step is completed by invoking Fact 3.5 to deal with the preimage by T_m.

Now choose q to be the smallest m so that $a_m = \zeta(A_m(f(0)))$. With this choice of q, the first claim of Lemma 3.7 is satisfied by the choice of q' and estimate (7). To prove the second claim, note that since $(T_{q-1} \circ \cdots \circ T_{r_n-L})^{-1}(a_q)$ is well-defined, it follows that $f(0)$ and J_n^q are preimages of $f^{q-r_n+L}(f(0))$ and $J_n^{r_n-L}$, respectively, by the same inverse branch of f^{q-r_n+L}. Hence, we are in the setting of Lemma 3.2 with $p := q - r_n + L + 1$. Since

$$a_{r_n-L} = \zeta(A_{r_n-L}(f^{q-r_n+L+1}(0)))$$

and $\arg(z - a_{r_n-L}) > \frac{\pi}{2}$, we have that $\Re a_{r_n-L} > \Re z$. The map $\zeta^{-1} = \frac{z}{1+z}$ sends the half line $\{x : x \geq -1+\eta\}$ into the half line $\{x : x \geq -M(\eta)\}$, where $M(\eta)$ is a positive constant. Hence

$$\frac{|J_n^{r_n-L}|}{|f^{q+r_n-L+1}(0)|} = \frac{1}{|A_{r_n-L}(f^{q-r_n+L+1}(0))|} \geq \frac{1}{M(\eta)} .$$

Lemma 3.1 now provides a constant $M' > 0$ depending only on $M(\eta)$ so that

$$\frac{|J_n^{q+1}|}{\mathrm{dist}(J_n^{q+1}, 0)} \geq M' .$$

The points $\zeta \circ A_q(f(0))$ and $\zeta \circ A_{q+1}(0)$ are on the same side of -1, so by the choice of q, they are both larger than -1. Therefore, there exists η_2 which depends on M' only so that $\zeta(A_{q+1}(0)) \geq -1 + \eta_2$. From Lemma 3.3, $\zeta(A_q(f(0))) \geq \zeta(A_{q+1}(0))$.

□

The main estimate.

Proposition 6 *For every K there is a $c_0 > 0$ so that for every $0 < c \leq c_0$ the set*

$$T_{r_n-2} \circ \cdots \circ T_0(\mathcal{G}(c)) \subset \mathcal{G}(Kc) \ .$$

The boundaries of the region $T_{r_n-L-1} \circ \cdots \circ T_0(\mathcal{G}(c))$ are described by Lemma 3.6. Pick a point

$$w \in T_{r_n-2} \circ \cdots \circ T_0(\mathcal{G}(c))$$

and find z such that

$$w = T_{r_n-2} \circ \cdots \circ T_{r_n-L}(z) \ .$$

If we choose the same K in that Lemma as given in Proposition 6, we get an $\eta_1 > 0$ and if $\Re z < -1 + \eta_1$, then $z \in \mathcal{G}(Kc)$, hence $w \in \mathcal{G}(Kc)$ by Lemma 3.5. So let us assume without loss of generality that $\Re z + 1 \geq \eta_1$. Then substitute $\eta := \eta_1$ in Lemma 3.7, choose the q from that Lemma, and focus on

$$z' := B_q \circ T_{q-1} \circ \cdots \circ T_{r_n-L}(z)$$

(recall B_q from Formula (6)). As a consequence of Lemma 3.7, B_q is affine with derivative bounded from above by $\frac{2}{\eta_2}$. Remember that η_2 depends only on η_1, hence ultimately on K. From the first claim of Lemma 3.7, $-1 \leq \Re z' \leq 1$. Also by our assumptions, z' belongs to $\mathcal{G}(c)$ translated by the vector $B_q(0)$. Now apply Lemma 3.4 with $r := B_q(0)$ and $Q := K$. Depending on γ specified by that Lemma, if we make $c_0 \leq \eta_2\gamma/2$, we obtain

$$\Im G(z') > Kc(G(r) - \Re G(z)) \ ,$$

thus $C_{q+1}^{-1}(z') \in \mathcal{G}(Kc)$. A reference to Lemma 3.5 ends the proof.

3.4 The last pull-back

In this way we come to the last application of f^{-1}. Let $J_n^{r_n}$ be the preimage of $J_n^{r_n-1}$ by f and denote by A the orientation-preserving affine mapping from $J_n^{r_n}$ onto the interval $(0,1)$. Then consider

$$T := \zeta \circ A \circ f^{-1} \circ A_{r_n-1}^{-1} \circ \zeta^{-1} .$$

Here f^{-1} denotes the inverse branch of f defined on the complex plane except the half-line $(f(0), \infty)$ onto the right half-plane.

Remember that A_{r_n-1}, by convention, sends the head end of $J_n^{r_n-1}$ to 0. By looking at the range of T and its action at three convenient points, we see that T is closely related to Joukovsky's map by the formula

$$T(z) = G \circ B(z)$$

where B is the affine map defined by the requirement that it fixes -1 and sends $\zeta(A_{r_n-1}(f(0)))$ to 1.

The action of T.

Lemma 3.8 *For every δ there are $c_0 > 0$ and K_0 so that for every $0 < c \leq c_0$ if $1 \leq \zeta(A_{r_n-1}(f(0))) \leq \delta$, then*

$$T(\mathcal{G}(K_0 c)) \subset \mathcal{G}(c) .$$

Proof: To facilitate the notation, let $b := \zeta(A_{r_n-1}(f(0)))$. Assume in addition that $1 \leq b$. For every $\delta > 0$ there is an $\epsilon > 0$ so that $B(0) \geq 1 - \epsilon$. Excluding easy cases assume that $\epsilon < 1/2$.

The region $B(\mathcal{G}(Kc))$ is a translation of $\mathcal{G}(Kc)$ by the vector $B(0)$. We will show that the image of $B(\mathcal{G})(Kc)$ by Joukovsky's maps is contained in $\mathcal{G}(c)$ for a suitable choice of c and K.

The crucial observation is that the image of every half-line $z(y) = 1 - \epsilon + (-1 + Kci)y$, $y \geq 0$, by Joukovski's map first slides down but then recovers its altitude constantly curving up until it begins asymptotically approaching the direction $\alpha = \arg dz(y)/dy$.

$$\frac{d}{dy} \arg G(z(y)) = \frac{d}{dy} \Im \left(\log G(z(y))\right) = \Im \left(\frac{\frac{dz(y)}{dy}}{\sqrt{(z(y))^2 - 1}} \right) .$$

The derivative of $\arg G(z(y))$ can have zeros only at these points ξ for which $\arg \sqrt{(z(\xi))^2 - 1} = \alpha$. We check the boundary values of $\arg G(z(y))$ at 0 and ∞.

$$\lim_{y \to \infty} \arg(G(z(y))) =$$

$$\lim_{y \to \infty} \left(\arg z(y) + \arg \left(1 + \sqrt{1 - \frac{1}{(z(y))^2}} \right) \right) = \alpha .$$

Denote by $e^{\theta i}$ the point of intersection of $z(y)$ with the unit circle. The argument of $G(z(0))$ is obviously θ and depends on ϵ only.

We will estimate from above

$$\arg G(z(\xi)) = \arg(\xi + e^{i\alpha} \sqrt{|\xi^2 - 1|}) .$$

Since $\pi < \arg \xi < \alpha$, the following estimate holds for $|\xi| \geq 1 - \epsilon/2$

$$\arg G(z(\xi)) \leq \max(\alpha, \arg \xi) \leq \max(\alpha, \pi - \tan^{-1}(2cK\epsilon)) .$$

If $|\xi| < 1 - \epsilon/2$ than

$$\arg G(z(\xi)) \leq \arg(-1 + \frac{1}{2}\epsilon + \frac{1}{2}e^{i\alpha}\epsilon) \leq \pi - \tan^{-1}(\frac{1}{2}cK\epsilon) .$$

If c is small enough than $\alpha \approx \pi - cK$ and $\tan^{-1}(\frac{1}{2}cK\epsilon) \approx \frac{1}{2}cK\epsilon$. Hence $G(z(t)) \subset \mathcal{G}(c)$ if only $0 < c < c_0$ and $K \geq K_0$. By the symmetry of Joukovsky's map, the lemma follows.

□

Proof of Theorem 3. We assumed that $\zeta(A_{r_n-1}(f(0))) \geq 1$ in the proof of Lemma 3.8. We could do this without loss of generality since otherwise the image of f covers more than a half of $J_n^{r_n-1}$ and consequently the preimage D_1 of $\mathcal{D}(\frac{\pi}{2}, J_n^{r_n-1})$ is contained in $\mathcal{D}(\frac{\pi}{2}, J_n^{r_n})$. Now we take D_1 as the domain of the analytic continuation of f^{r_n} demanded in Theorem 3. The bound on the modulus holds by Lemma 3.1. The bound of the diameter is also clearly satisfied.

As the first step, we will show that for some *constant* $c > 0$ there is always

$$T \circ T_{r_n-2} \circ \cdots \circ T_0(\mathcal{G}(c)) \subset \mathcal{G}(c)\,. \tag{8}$$

This inclusion is satisfied if we pick $c \leq c_0$ from Lemma 3.8 and can somehow show that

$$T_{r_n-2} \circ \cdots \circ T_0(\mathcal{G}(c)) \subset G(K_0 c) \tag{9}$$

where K_0 is another constant obtained from Lemma 3.8. Both constants c_0 and K_0 depend on the parameter δ from the hypothesis of Lemma 3.8. We specify δ as a function of the constant K_3 from Lemma 3.1. According to Proposition 6 the inclusion (9) holds provided that c is bounded by another constant depending only on K_0. Taking c equal to the minimum of these two bounds, we have proved our claim.

Going back to the original dynamical plane, the inclusion (8) is equivalent to

$$D_1 := f^{-r_n}(\mathcal{D}(\pi - \tan^{-1}(c), J_n)) \subset \mathcal{D}(\pi - \tan^{-1}(c), J_n^{r_n})\,.$$

From here we see that D_1 can be the domain of the analytic continuation of f^{r_n} as a quadratic-like map, and the modulus estimate follows from Lemma 3.1.

References

[1] Douady, A. & Hubbard, J.H.: *On the dynamics of polynomial-like mappings*, Ann. Sci. Ec. Norm. Sup. (Paris) **18** (1985), 287-343.

[2] Graczyk, J. & Świątek, G.: *Critical circle maps near bifurcation*, Commun. Math. Phys. **176** (1996), 227-260.

[3] Graczyk, J. & Świątek, G.: *Induced expansion for quadratic polynomials*, Ann. Sc. d'ENS **29** (1996), 399-482.

[4] Graczyk, J. & Świątek, G.: *Generic Hyperbolicity in the Logistic Family*, to appear in Annals of Math.

[5] Graczyk, J. & Świątek, G.: *Polynomial-like mappings induced by real polynomials*, in World Scientific Series in Applicable Analysis, Vol. 4, 315-328, World Scientific, Singapore (1995).

[6] Guckenheimer, J. & Johnson, S.: *Distortion of S-unimodal maps*, Annals of Math. **132** (1990), 71-130.

[7] Jakobson, M.: *Absolutely continuous invariant measures for one-parameter families of one-dimensional maps*, Commun. Math. Phys. **81** (1981), 39-88.

[8] Jakobson, M. & Świątek, G.: *Metric properties of non-renormalizable S-unimodal maps*, Ergod. Th. and Dynam. Sys. **14** (1994), 721-755.

[9] Levin, G. & van Strien, S.: *Local connectivity of the Julia set of real polynomials*, to appear in Ann. of Math.

[10] Lehto, O.& Virtanen, K.: *Quasikonforme Abbildungen*, Springer-Verlag, Berlin-Heidelberg-New York (1965).

[11] de Melo, W. & van Strien, S.: *One-Dimensional Dynamics*, Springer-Verlag, New York (1993).

[12] de Melo, W. & van Strien, S.: *A structure theorem in one-dimensional dynamics*, Ann. Math. **129** (1989), 519-546.

[13] Milnor, J.: The Yoccoz theorem on local connectivity of Julia sets. A proof with pictures., class notes, Stony Brook, (1991-92).

[14] Milnor, J. & Thurston, W.: *On iterated maps of the interval I, II*, preprint (1977), in: Lecture Notes in Mathematics, Vol. 1342 (Springer, Berlin, 1988), 465.

[15] Preston, C.: *Iterates of maps on an interval*, Lecture Notes in Mathematics, Vol. 999.

[16] Sullivan, D.: *On the structure of infinitely many dynamical systems nested inside or outside a given one*, preprint of IHES no. **IHES/M/90/75**.

[17] Yoccoz, J.-C.: unpublished results, see the discussion in [13].

[18] Yampolsky, M.: *Complex bound for critical circle maps*, Stony Brook preprint (1995).

Caltech
Pasadena, CA 91125

Pennsylvania State University
University Park, PA 16802