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Characterizing Dendrites By Deformation Retractions

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Abstract

In this paper we prove:

Theorem *Let X be a continuum, then the following assertions are equivalent:*

(a) X is a dendrite,

(b) Every subcontinuum of X is a deformation retract of X,

(c) Every subcontinuum of X is a strong deformation retract of X.

INTRODUCTION

A *continuum* is a compact, connected metric space. A *map* is a continuous function. A *monotone map* is a map with connected fibers (i.e., inverse images of one-point sets are connected). X will denote always a continuum with metric d. X is said to be *retractible* (resp. *d-retractible*, *sd-retractible, monotone retractible*) provided that for each subcontinuum Y of X there exists a retraction (resp. deformation retraction, strong deformation retraction, monotone retraction) from X onto Y. X

is said to be *hereditarily unicoherent* provided that the intersection of any two subcontinua of X is connected. A *dendroid* is a hereditarily unicoherent, arcwise connected continuum. A *dendrite* is a locally connected dendroid.

The concept of retractible continuum was studied by J. J. Charatonik in [1]. Among other characterizations of dendrites, in [3], G. R. Gordh, Jr. and L. Lum proved that A continuum is monotone retractible if and only if X is a dendrite (this is a positive answer to problem 12 in [1]). In this paper we prove:

Theorem *Let X be a continuum, then the following assertions are equivalent:*

(a) X is a dendrite,

(b) X is d-retractible and,

(c) X is sd-retractible.

1 AUXILIARY RESULTS

SOME CONVENTIONS. The unit closed interval in the real line is denoted by I. The set {1,2,...} is denoted by $\mathbb{N}$. The hyperspace of all subcontinua of X is denoted by $C(X)$, $C(X)$ is considered with the Hausdorff metric H. A *Whitney map* is a map $\mu : C(X) \to I$ such that $\mu(\{x\}) = 0$ for each $x \in X, \mu(X) = 1$ and if $A, B \in C(X)$ and $A \subset B \neq A$, then $\mu(A) < \mu(B)$. Whitney maps exist for every continuum X as was shown by H. Whitney in [7]. If $\epsilon > 0$, $p \in X$ and $A \in C(X)$ define $B(\epsilon, p) = \{q \in X : d(p,q) < \epsilon\}$ and $N(\epsilon, A) = \bigcup\{B(\epsilon, x) : x \in A\}$. Given an arc α in X and points p, q in α, define $\alpha(pq) =$ the arc from p to q contained in α, if $p \neq q$ and $\alpha(pq) = \{p\}$ if $p = q$. If there is no possibility of confusion, we simply write pq instead of $\alpha(pq)$.

A pseudocomb is a subcontinuum Y of X such that Y is of the form $Y = A \cup (\bigcup\{a_n b_n : n \in \mathbb{N}\})$, where $A \in C(X)$; for each $n \in \mathbb{N}$, $a_n b_n$ is an arc from the point a_n to the point b_n and $a_n b_n \cap A = \{b_n\}$; the arcs $a_1 b_1, a_2 b_2, \ldots$ are pairwise disjoint; the sequence $(a_n b_n)_{n \in \mathbb{N}}$ converges in $C(X)$ to a subcontinuum A_0 of A; the sequence $(a_n)_{n \in \mathbb{N}}$ (resp. $(b_n)_{n \in \mathbb{N}}$) converges to a point a_0 in A_0 (resp. b_0 in A_0) and $a_0 \neq b_0$.

Lemma 1 *Suppose that X is hereditarily arcwise connected. Let $\mu : C(X) \to I$ be a Whitney map. Then for each pair of points $p \neq q$ in X, there exists an arc α in X such that p and q are the end points of α and $\mu(\alpha)$ =min$\{\mu(\beta) : \beta$ is an arc from p to q $\}$.*

Proof: Let s_0 =inf$\{\mu(\beta) : \beta$ is an arc from p to $q\}$. Let $(\beta_n)_{n \in \mathbb{N}}$ be a sequence of arcs in X such that p and q are the end points of β_n for each $n \in \mathbb{N}$ and $\mu(\beta_n) \to s_0$. Since $C(X)$ is compact (see [6, 4.17]), we may assume that $\beta_n \to A$ for some $A \in C(X)$. Then $p,\ q \in A$. By hypothesis, there exists an arc α in A such that the end points of α are p and q. Then $s_0 \leq \mu(\alpha) \leq \mu(A) = s_0$. Since $\alpha \subset A$ and $\mu(\alpha) = \mu(A)$, then $\alpha = A$. This completes the proof of the lemma. □

Theorem 2 *If X is hereditarily arcwise connected and X is not locally connected, then X contains a pseudocomb.*

Proof: Since X is not locally connected, then there exist: an open subset U of X, a component C of U and a point $p \in C-$int(C). Then every neighborhood of p intersects components of U different from C. Thus it is possible to construct a sequence of points $(p_n)_{n \in \mathbb{N}}$ of X and a sequence of components $(C_n)_{n \in \mathbb{N}}$ of U such that $p_n \to p$, $p_n \in C_n$ for every $n \in \mathbb{N}$ and $C, C_1, C_2, \ldots$ are pairwise different.

Fix a Whitney map $\mu : C(X) \to I$. Define $f : X \to I$ by $f(p) = 0$ and if $p \neq q$, $f(q)$ =min$\{\mu(\beta) : \beta$ is an arc from p to $q\}$. From Lemma 1, f is well defined. Notice that, since X is hereditarily arcwise connected, then $f(q)$ =min$\{\mu(B) : B$ is a sucontinuum of X and $p,\ q \in B\}$. For each $n \in \mathbb{N}$, fix an arc α_n in X such that p_n and p are the end points of α_n and $\mu(\alpha_n) = f(p_n)$. Taking subsequences if necessary, we may assume that $(\alpha_n)_{n\in\mathbb{N}}$ converges to an element $B \in C(X)$ and the sequences $(f(p_n))_{n\in\mathbb{N}}$ converges to a number $s_0 \in I$. Then $\mu(B) = s_0$ and $p \in B$.

If $s_0 = 0$, then $B = \{p\}$. Thus $\alpha_n \to \{p\}$. This implies that there exists $n \in \mathbb{N}$ such that $\alpha_n \subset U$. This is absurd since $\alpha_n \cap C_n \neq 0$, $\alpha_n \cap C \neq 0$ and $C \neq C$. This contradiction proves that $s_0 > 0$.

In particular, $\lim f(p_n) \neq f(p)$. Then f is not continuous. However, we will prove that:

A. *If E is a locally connected subcontinuum of X, then the restriction $f|E$ is continuous.*

In order to prove A, let $x \in E$ and let $\epsilon > 0$. Let $\delta > 0$ be such that if $H(F, G) < 2\delta$, then $|\mu(F) - \mu(G)| < \epsilon$. Since E is locally connected, there exists an arcwise connected open subset U of E such that $x \in U \subset B(\delta, x) \cap E$. Given a point $y \in U$, let α be an arc in U such that the end points of α are x and y. Let β be an arc in X such that the end points of β are p and y and $f(y) = \mu(\beta)$. Then $f(x) \leq \mu(\alpha \cup \beta)$. Since $\alpha \cup \beta \subset N(2\delta, \beta)$, we have that $H(\beta, \alpha \cup \beta) < 2\delta$. Thus $\mu(\alpha \cup \beta) < \mu(\beta) + \epsilon$. Hence $f(x) - f(y) < \epsilon$. Similarly, $f(y) - f(x) < \epsilon$. Therefore $f|E$ is continuous. This proves A.

Now we will prove that:

B. *For each $a \in I, f^{-1}[0, a]$ is a subcontinuum of X.*

Given $x \in f^{-1}[0, a]$, let α be an arc in X such that the end points of α are p and x and $f(x) = \mu(\alpha)$. For each $y \in \alpha, f(y) \leq \mu(\alpha)$, then $\alpha \subset f^{-1}[0, a]$. This proves that $f^{-1}[0, a]$ is connected.

Now, let $(x_n)_{n\in\mathbb{N}}$ be a sequence of elements of $f^{-1}[0, a]$ such that $x_n \to x$ for some $x \in X$. For each $n \in \mathbb{N}$, let β_n be an arc in X such that the end points of β_n are p and x_n and $f(x_n) = \mu(\beta_n)$. We may assume that $\beta_n \to D$ for some $D \in C(X)$. Then $p, x \in D$. Thus $f(x) \leq \mu(D) = \lim \mu(\beta_n)$. This implies that $f(x) \leq a$. Hence $f^{-1}[0, a]$ is closed in X. This completes the proof of B.

Choose a strictly increasing sequence $(s_n)_{n \in \mathbb{N}}$ in $[0, s_0)$ such that $s_n \to s_0$. For each $n \in \mathbb{N}$, define $G_n = f^{-1}[0, s_n] \cap B$ and define $G_0 = f^{-1}(s_0) \cap B$. Since $p \in B, f(x) \leq \mu(B) = s_0$ for every $x \in B$. Then $B = \cup\{G_n : n \geq 0\}$. From the Baire Category Theorem, there exists $n_0 \geq 0$ such that $\text{int}_B(G_{n_0}) \neq \emptyset$.

If $G_0 \neq \emptyset$, let $x \in B$ be such that $f(x) = s_0$, let α be an arc in B such that the end points of α are p and x, then $s_0 = f(x) \leq \mu(\alpha) \leq \mu(B) = s_0$. This implies that $\alpha = B$ and $G_0 = \{x\}$. Since $p \neq x$, then $\text{int}_B(G_0) = \emptyset$. This proves that $n_0 \geq 1$.

Define $G = G_{n_0}$ and $b = s_{n_0}$. Fix a point $z \in \text{int}_B(G)$ and fix $\epsilon_0 > 0$ such that $B(\epsilon_0, z) \cap B \subset G$. Fix a number c such that $b < c < s_0$.

Since $f(p_n) \to s_0$, there exists $N \in \mathbb{N}$ such that $f(p_n) > c$ for every $n \geq N$. Given $n \geq N$, let q_n be the first point in α_n, walking from p_n to p, such that $q_n \in f^{-1}[0, c]$. We analyze two cases:

Case 1. $z \in \lim_{n \geq N} \sup p_n q_n$.

Then there exists a sequence $n_1 < n_2 < \ldots$ in $\mathbb{N}$ and there exists a sequence $(z_k)_{k \in \mathbb{N}}$ such that $z_k \in p_{n_k} q_{n_k}$ for each $k \in \mathbb{N}$ and $z_k \to z$.

Fix a number d such that $b < d < c$. Let $\delta > 0$ be such that $\delta < \epsilon_0/2$ and if $H(D, E) < 2\delta$, then $|\mu(D) - \mu(E)| < c - d$. We may assume that $z_k \in B(\delta/2, z)$ for every $k \in \mathbb{N}$.

Given $k \in \mathbb{N}$, the continuity of $f|\alpha_{n_k}$ implies that $f(x) \geq c$ for every $x \in p_{n_k}q_{n_k}$. In particular $f(z_k) \geq c$. Let y_k be the first point in α_{n_k} , walking from z_k to p, such that $y_k \in f^{-1}[0, d]$.

We assert that, for every $k \in \mathbb{N}$, the arc $z_k y_k$ is not contained in $B(\delta, z)$. Suppose on the contrary that $z_k y_k \subset B(\delta, z)$ for some $k \in \mathbb{N}$. Since $y_k \in f^{-1}[0, d]$, there exists an arc α from p to y_k such that $\mu(\alpha) = f(y_k) \leq d$. Notice that $\alpha \cup z_k y_k \in C(X)$ and $\alpha \cup z_k y_k \subset N(2\delta, \alpha)$. Then $H(\alpha \cup z_k y_k, \alpha) < 2\delta$ and $\mu(\alpha \cup z_k y_k) < \mu(\alpha) + c - d \leq c$. This implies that $f(z_k) < c$. This contradiction proves the assertion.

For each $k \in \mathbb{N}$, let x_k be the first point, walking from z_k to y_k, in the arc $z_k y_k$, such that $x_k \in X - B(\delta, z)$.

Let $k \in \mathbb{N}$, we will show that $z_k y_k$ intersects only a finite number of arcs $z_m x_m$. Suppose on the contrary that $z_k y_k$ intersects infinitely many arcs $z_m x_m$, This implies that there exists a point $w \in z_k y_k \cap \lim \sup z_m x_m \subset z_k y_k \cap \mathrm{Cl}(B(\delta, z)) \cap \lim \sup \alpha_n \subset z_k y_k \cap B(\epsilon_0, z) \cap B \subset z_k y_k \cap G$. Since $f|\alpha_{n_k}$ is continuous and $w \in z_k y_k$, we have that $f(w) \geq d$. Since $w \in G, f(w) \leq b < d$. This contradiction proves that $z_k y_k$ intersect only a finite number of sets $z_m x_m$. Similarly, it can be proved that $z_k x_k \cap B = \emptyset$ and, for each $k \in \mathbb{N}$, the arc $z_k x_k$ intersects only a finite number of arcs $z_m y_m$.

Then it is possible to construct a sequence $k_1 < k_2 < \ldots$ in $\mathbb{N}$ such that $z_{k_m} x_{k_m} \cap z_{k_r} y_{k_r} = \emptyset$ if $m \neq r$. We may assume that $(z_{k_m} x_{k_m})_{m \in \mathbb{N}}$ converges to A_0 for some $A_0 \in C(X)$; $(z_{k_m} y_{k_m})_{m \in \mathbb{N}}$ converges to B_0 for some $B_0 \in C(X)$ and $(x_{k_m})_{m \in \mathbb{N}}$ converges to an element $x_0 \in X - B(\delta, z)$.

Define $A = B \cup f^{-1}[0, d] \cup (\bigcup \{x_{k_m} y_{k_m} : m \in \mathbb{N}\})$ and $Y = A \cup (\bigcup \{z_{k_m} x_{k_m} : m \in \mathbb{N}\})$. Since $A_0, B_0 \subset B$, then A and Y are subcontinua of X. It is easy to check that Y is a pseudocomb of X.

This completes the analysis of Case 1.

Case 2. $z \notin \lim_{n \geq N} \sup p_n q_n$.

Let $(p_{n_k} q_{n_k})_{k \in \mathbb{N}}$ be a subsequence of $(p_n q_n)_{n \in \mathbb{N}}$ such that $p_{n_k} q_{n_k} \to B_1$ for some $B_1 \in C(X)$. Then $B_1 \subset B$ and $z \in B - B_1$. Thus $\mu(B_1) < \mu(B) = s_0$. Define $s^* = \mu(B_1)$. Since $p \in B_1$, $B_1 \subset f^{-1}[0, s^*]$.

Fix a number e such that c, $s^* < e < s_0$. Since $f(p_{n_k}) \to s_0$, there exists $K \in \mathbb{N}$ such that $f(p_{n_k}) > \epsilon$ for every $k \geq K$. Given $k \geq K$, let w_k be the first point, walking from p_{n_k} to p, in the arc α_{n_k} such that $w_k \in f^{-1}[0, e]$. Then $f(x) \geq e$ for every point x in the arc $p_{n_k} w_k$. Thus $p_{n_k} w_k \cap B_1 = \emptyset$ and $q_{n_k} \notin p_{n_k} w_k$. Hence $p_{n_k} w_k \subset p_{n_k} q_{n_k}$. Since $p_{n_m} q_{n_m} \to B_1, p_{n_k} w_k$ intersects only a finite number of arcs $p_{n_m} q_{n_m}$.

Then a subsequence $(p_{n_{k_m}} w_{k_m})_{m \in \mathbb{N}}$ of $(p_{n_k} w_k)_{k \in \mathbb{N}}$ can be constructed in such a way that $p_{n_{k_1}} w_{k_1}, p_{n_{k_2}} w_{k_2}, \ldots$ are pairwise disjoint. We may assume that $p_{n_{k_m}} w_{k_m} \to A_1$ for some $A_1 \in C(X)$ and $w_{k_m} \to w_0$ for some $w_0 \in X$. Notice that $A_1 \subset B$ and $w_0 \in f^{-1}[0, e]$.

We will show that A_1 is not a one-point set. Suppose on the contrary that $A_1 = \{w_0\}$. Let $m \in \mathbb{N}$, since $w_{k_m} \in f^{-1}[0, e]$, there exists an arc β_m in X such that the end points of β_m are p and w_{k_m} and $\mu(\beta_m) = f(w_{k_m}) \leq e$. Taking a subsequence if necessary, we may assume that $\beta_m \to B^*$ for some $B^* \in C(X)$. Since $f(p_{n_{k_m}}) \leq \mu(\beta_m \cup p_{n_{k_m}} w_{k_m}) \to \mu(B^* \cup A_1) = \mu(B^*)$, we have that $s_0 \leq \mu(B^*) \leq e$. This contradicts the choice of e and proves that $A_1 \neq \{w_0\}$.

Fix a point $v_0 \in A_1 - \{w_0\}$. Let $(v_m)_{m \in \mathbb{N}}$ be a sequence of points of X such that $v_m \to v_0$ and $v_m \in p_{n_{k_m}} w_{k_m} - \{w_{k_m}\}$ for every $m \in \mathbb{N}$. Taking a subsequence if necessary we may assume that $v_m w_{k_m} \to A_2$ for some $A_2 \in C(X)$. Notice that $A_2 \subset B$.

Define $Y = f^{-1}[0,e] \cup (\bigcup \{v_m w_{k_m} : m \in \mathbb{N}\})$. Since $A_2 \subset B \subset f^{-1}[0,e]$, we have that Y is a subcontinuum of X. It is easy to check that Y is a pseudocomb.

This completes the analysis of Case 2 and the proof of the theorem.

Lemma 3 *The property of being d-retractible is hereditary.*

Proof: Suppose that X is d-retractible. Let Y be a subcontinuum of X and let Z be a subcontinuum of Y. Then there exist: a retraction $R : X \to Y$; a retraction $r : X \to Z$ and a map $F : X \times I \to X$ such that $F(x,0) = x$ and $F(x,1) = r(x)$ for every $x \in X$.

Consider the map $R \circ F|Y \times I : Y \times I \to Y$, then $R \circ F(y,0) = y$ and $R \circ F(y,1) = r|Y\ (y)$ for every $y \in Y$.

Hence Y is d-retractible.

The following lemma is easy to prove:

Lemma 4 *Suppose that* $Y = A \cup (\bigcup\{a_n b_n : n \in \mathbb{N}\})$ *is a pseudocomb. Let* $n \in \mathbb{N}$ *and* $c \in a_n b_n - \{a_n\}$. *Then* $Y - \{c\} = (a_n c - \{c\}) \cup (Y - a_n c)$ *is a separation of* $Y - \{c\}$.

Theorem 5 *If X is d-retractible, then X does not contain pseudocombs.*

Proof: Suppose, on the contrary, that X contains a pseudocomb $Y = A \cup (\bigcup\{a_n b_n : n \in \mathbb{N}\})$. For each $n \in \mathbb{N}$, we consider an order in the arc $a_n b_n$ in such a way that $a_n < b_n$.

Fix $\epsilon > 0$ such that $\epsilon < d(a_0, b_0)$ ($a_0 = \lim a_n$ and $b_0 = \lim b_n$). Given $t \in (0, \epsilon)$, a sequence of points $(c_n)_{n \in \mathbb{N}}$ of X is said to be a *t-sequence* if $c_n \in a_n b_n$ for each $n \in \mathbb{N}$ and there exists $N \in \mathbb{N}$ such that $c_n \in B(t, a_0)$ for every $n \geq N$. We will prove the following assertion:

A. *If $t \in (0, \epsilon)$ and $(c_n)_{n \in \mathbb{N}}$ is a t-sequence then, for each $s \in (t, \epsilon)$, there exist sequences $(d_n)_{n \in \mathbb{N}}$ and $(e_n)_{n \in \mathbb{N}}$ such that: $(e_n)_{n \in \mathbb{N}}$ is an s-sequence; for each $n \in \mathbb{N}$, $d_n \in a_n b_n$ and there exists $M \in \mathbb{N}$ such that $a_n \leq c_n < d_n < e_n < b_n$ and $d_n \notin B(\epsilon, a_0)$ for every $n \geq M$.*

In order to prove A, let $N \in \mathbb{N}$ be such that, for each $n \geq \mathbb{N}, c_n \in B(t, a_0)$ and $d(b_n, b_0) < (d(a_0, b_0) - \epsilon)/3$, in particular, $b_n \neq B(\epsilon, a_0)$. Given $n \geq N$, let d_n be the first point, walking from c_n to b_n, in the arc $c_n b_n$ such that $d_n \notin B(\epsilon, A_0)$. Complete the sequence $(d_n)_{n \in \mathbb{N}}$ by choosing points $d_1 \in a_1 b_1, \ldots, d_{N-1} \in a_{N-1} b_{N-1}$ such that $a_1 < d_1 < b_1, \ldots, a_{N-1} < d_{N-1} < b_{N-1}$.

Define $Z = A \cup (\bigcup\{d_n b_n : n \in \mathbb{N}\})$. Since $\lim\sup d_n b_n \subset A_0 \subset A$, we have that Z is a subcontinuum of X. By hypothesis, there exists a retraction $r : X \to Z$. Let $\delta > 0$ be such that $\delta < (d(a_0, b_0) - \epsilon)/3$ and if $u, x \in X$ and $d(u, x) < \delta$, then $d(r(u), r(x)) < (d(a_0, b_0) - \epsilon)/3$.

Let $K \in \mathbb{N}$ be such that $K \geq N$ and, for every $n \geq K$, $H(a_n b_n, A_0) < \delta$.

Let $n \geq K$, we will show that $r(c_n d_n) \subset d_n b_n - \{b_n\}$. Take $x \in c_n d_n \subset a_n b_n \subset N(\delta, A_0)$, then there exists $y \in A_0 \subset Z$ such that $d(x, y) < \delta$, thus $d(r(x), y) < (d(a_0, b_0) - \epsilon)/3$. From the choice of $d_n, d(x, a_0) \leq \epsilon$. Then $d(a_0, r(x)) \leq d(a_0, x) + d(x, y) + d(y, r(x)) < 2d(a_0, b_0)/3 + \epsilon/3$. Thus $d(a_0, r(x)) < 2d(a_0, b_0)/3 + \epsilon/3$. On the other hand, $d(b_0, b_n) < (d(a_0, b_0) - \epsilon)/3$. This implies that $r(x) \neq b_n$. Hence $r(c_n d_n) \subset Y - \{b_n\}$. Since $r(d_n) = d_n \in (a_n b_n - \{b_n\})$, the connectedness of $r(c_n d_n)$ and Lemma 4 imply that $r(c_n d_n) \subset a_n b_n - \{b_n\}$. Since $r(c_n d_n) \subset Z$, we conclude that $r(c_n d_n) \subset d_n b_n - \{b_n\}$.

Define, for each $n \geq K$, $e_n = r(c_n)$. To complete the sequence $(e_n)_{n \in \mathbb{N}}$, choose points $e_1 \in a_1 b_1, \ldots, e_{K-1} \in a_{K-1} b_{K-1}$ such that $d_1 < e_1 < b_1, \ldots, d_{K-1} < e_{K-1} < b_{K-1}$.

We will prove that $(e_n)_{n \in \mathbb{N}}$ is an s-sequence. Suppose on the contrary that for each $n_0 \in \mathbb{N}$, there exists $n \geq n_0$ such that $e_n \notin B(s, a_0)$. Then a sequence $n_1 < n_2 < \ldots$ in N can be constructed such that $e_{n_k} \notin B(s, a_0)$ for every $k \in \mathbb{N}$. We may assume that $c_{n_k} \to c_0$ for some $c_0 \in X$. Notice that $c_0 \in Z$. Then $r(c_{n_k}) \to c_0$. Thus $e_{n_k} \to c_0$. Hence $c_0 \notin B(s, a_0)$. But $c_{n_k} \in B(t, a_0)$ for almost every k, then $d(a_0, c_0) \leq t < s$. This contradiction proves that $(e_n)_{n \in \mathbb{N}}$ is an s-sequence.

Then there exists $M \in \mathbb{N}$ such that $M \geq K$ and $e_n \in B(s, a_0)$ for every $n \geq M$. Given $n \geq M$, from the choice of $d_n, d(a_0, d_n) = \epsilon$, then $c_n < d_n < e_n < b_n$. This completes the proof of assertion A.

From Lemma 3, there exists a retraction $R : Y \to A$ and there exists a map $F : Y \times I \to Y$ such that $F(y, 0) = y$ and $F(y, 1) = R(y)$ for every $y \in Y$. Let $\eta > 0$ be such that if (v, s), $(y, t) \in Y \times I$ and $d(v, y)$, $|s - t| < \eta$, then $d(F(v, s), F(y, t)) < \epsilon/2$. Fix a number $L \in \mathbb{N}$ such that $2L\eta > 1$.

By successive applications of assertion A, it is possible to construct two families of sequences $(c_n^{(0)})_{n \in \mathbb{N}}, (c_n^{(1)})_{n \in \mathbb{N}}, \ldots, (c_n^{(L)})_{n \in \mathbb{N}}$ and $(d_n^{(1)})_{n \in \mathbb{N}}, \ldots, (d_n^{(L)})_{n \in \mathbb{N}}$ such that:
(a) $c_n^{(0)} = a_n$ for every $n \in \mathbb{N}$,
(b) For each $i \in \{0, 1, \ldots, L\}$, $(c_n^{(i)})_{n \in \mathbb{N}}$, is a $\epsilon(i+1)/2(L+1)$-sequence and $d_n^{(i)} \in a_n b_n$ for every $i \in \{0, 1, \ldots, L\}$ and every $n \in \mathbb{N}$,
(c) There exists $M_0 \in \mathbb{N}$ such that, for each $n \geq M_0$, $a_n \leq c_n^{(0)} < d_n^{(1)} < c_n^{(1)} < d_n^{(2)} < c_n^{(2)} < \ldots < d_N^{(L)} < c_n^{(L)} < b_n$ and,
(d) For every $i \in \{1, \ldots, L\}$ and every $n \geq M, d_n^{(i)} \notin B(\epsilon, a_0)$.

Fix $n \in \mathbb{N}$ such that $n \geq M_0, d(a_0, a_n) < \eta$ and, for every $i \in \{0, 1, \ldots, L\}$, $c_n^{(i)} \in B(\epsilon(i+1)/2(L+1), a_0)$. Define $\alpha : I \to Y$ by $\alpha(t) = F(a_n, t)$. Then $\alpha(0) = a_n, \alpha(1) \in A$ and if $s, t \in I$ and $|s - t| < \eta$, then $d(\alpha(s), \alpha(t)) < \epsilon/2$.

Since $\alpha(0) \in a_n d_n^{(1)} - \{d_n^{(1)}\}$ and $\alpha(1) \in Y - a_n d_n^{(1)}$, there exists $s_1 \in I$ such that $\alpha(s_1) = d_n^{(1)}$ (see Lemma 4). Similarly, there exists $t_1 \in [s_1, 1]$ such that $\alpha(t_1) = c_n^{(1)}$. Proceeding in this way it is possible to find numbers $s_1, \ldots, s_L$ and $t_1, \ldots, t_L$ such that $0 \leq s_1 \leq t_1 \leq s_2 \leq t_2 \leq \ldots s_L \leq t_L \leq 1$ and $\alpha(s_i) = d_n^{(i)}, \alpha(t_i) = c_n^{(i)}$ for every $i \in \{1, \ldots, L\}$.

For each $i \in \{1, \ldots, L\}, d_n^{(i)} \notin B(\epsilon, a_0)$ and $c_n^{(i)} \in B(\epsilon/2, a_0)$, then $d(c_n^{(i)}, d_n^{(i)}) > \epsilon/2$. This implies that $t_i - s_i \geq \eta$. Similarly, $s_i - t_{i-1} \geq \eta \ \ (t_0 = 0)$.

Then $s_1 - 0 \geq \eta, t_1 - s_1 \geq \eta, s_2 - t_1 \geq \eta, t_2 - s_2 \geq \eta, \ldots, s_L - t_{L-1} \geq \eta$ and $t_L - s_L \geq \eta$. Adding these inequalities, $1 \geq t_L \geq 2L\eta$. This contradicts the choice of L and completes the proof of the theorem.

2 MAIN THEOREM

Theorem *Let X be a continuum, then the following assertions are equivalent:*

(a) X is a dendrite,

(b) X is d-retractible and,

(c) X is sd-retractible.

Proof: (c) $\Rightarrow$ (b) is immediate.

(a) $\Rightarrow$ (c) Suppose that X is a dendrite. Let Y be a subcontinuum of X. From [4, Thm. 2.1], there exists a retraction $r : X \to Y$. For each $x \in X$, let $xr(x)$ denote the unique arc in X such that the end points of $xr(x)$ are x and $r(x)$, in the case that $r(x) = x$, let $xr(x) = \{x\}$.

Define $F : X \times I \to X$ by $F(x,t) =$ the unique point in $xr(x)$ such that $\mu(xF(x,t)) = t\mu(xr(x))$. Clearly, F is continuous, $F(x,0) = x$ and $F(x,1) = r(x)$. If $y \in Y, yr(y) = \{y\}$, then $F(y,t) = \{y\}$ for every $t \in I$. This proves that r is a strong deformation retraction.

(b) $\Rightarrow$ (a) From Theorem 5, X does not contain pseudo-combs. Let $p \in X$, since $\{p\}$ is a deformation retract of X, it follows that X is contractible. This implies that every map from X to the unit circle in the Euclidean plane is homotopic to a constant. Then, combining the main theorem in [5] and [2, Thm. 2 and 3], we have that X is unicoherent. From Lemma 3, X is hereditarily contractible. Thus X is hereditarily arcwise connected and hereditarily unicoherent. Then X is a dendroid and, from Theorem 2, X is locally connected. Therefore X is a dendrite.

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