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Mail: Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA
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On Two Questions of Arhangel'skii and Collins Regarding Submaximal Spaces

Ronnie Levy and Jack R. Porter

Abstract

An uncountable, crowded, separable Tychonoff submaximal space is constructed, and the existence of a submaximal space which is not the countable union of discrete spaces is shown to be equivalent to the existence of a crowded Baire Space which is not the union of disjoint dense sets. Both of these results are in response to questions posed in a 1995 paper by Arhangel'skii and Collins.

A topological space X is *submaximal* if every dense subset of X is open in X. Obviously, every discrete space is submaximal, and it is not hard to see that every space having only one non-isolated point is submaximal. More generally, any space X whose set of non-isolated points is discrete is submaximal. What is less clear is that there are *crowded* spaces, that is, spaces having no isolated points, that are submaximal. Examples of such spaces are *maximal* spaces, which are crowded topological spaces which have no strictly stronger crowded topologies. Maximal spaces were first studied by Hewitt in [H]. In [AC], Arhangel'skii and Collins ask if there is an uncountable, crowded, separable Hausdorff (or Tychonoff) submaximal space. In this note, we show that there is such a space. Arhangel'skii and Collins also ask if every [crowded] submaximal space is [strongly] σ-discrete, where a space is [*strongly*] *σ-discrete* if it is the union of a countable family of [closed] discrete subsets. We show that the existence of a submaximal space which is not σ-discrete is equivalent to the existence of a crowded Baire space which is not the union of disjoint dense subsets.

1 Some known results

One of our goals is to construct a Tychonoff space X which is uncountable, separable, crowded, and submaximal. As is pointed out in [AC], the complement of every dense subset of X is discrete and closed in X, so X must be the union of a dense, open, countable subset, and an uncountable closed, discrete subset. If we were not looking for a crowded space, we could settle for a Mrowka-Isbell Ψ-space. To get our example, we simply replace each isolated point of such a space with a crowded maximal space. Of course, we must do this in a way that we keep the space Tychonoff and submaximal. We will need some building blocks, and some basic facts.

Proposition 1.1 (van Douwen[vD]) *There is a countable regular (hence, normal) maximal space. Furthermore, given any such space, no point is an accumulation point of disjoint subsets.*

Van Douwen pointed out in [vD] that Proposition 1.1 says more than that there is a regular crowded space with no stronger regular crowded topology. Such spaces are easily constructed using Zorn's Lemma. The point of Proposition 1.1 is that there is a regular crowded space with no stronger crowded topology, regular or not. Constructing such a thing is quite non-trivial.

The necessary conditions of the following result were apparently first observed by Hewitt in [H] and the sufficiency of the conditions was established by Katětov in [K].

Proposition 1.2 *A crowded T_1 space is maximal if and only if it is extremally disconnected and submaximal.*

2 Uncountable submaximal separable spaces

As mentioned above, if we were merely interested in an uncountable separable submaximal Hausdorff space, a Mrowka-Isbell space would work. If we add the rquirement that the space be crowded, the situation becomes only a little trickier. We only sketch a construction because we will give a Tychonoff example below.

Pre-Example 2.1 *There is an uncountable separable submaximal crowded Hausdorff space.*

Outline of construction. Let $\mathcal{E}$ be an uncountable almost-disjoint family on ω and let Y be a countable Hausdorff maximal space (such as the space in Proposition 1.1). As a set, let $X = (\omega \times Y) \cup \mathcal{E}$. Declare $\omega \times Y$ to be open and have the product topology. For $E \in \mathcal{E}$, let a neighborhood of E be $\{E\} \cup \bigcup_{n \in (E \setminus F)}(\{n\} \times U_n)$ where U_n is dense in Y and F is a finite subset of E. The fact that X with this topology is Hausdorff follows from the almost-disjointness of $\mathcal{E}$, and the submaximality of X follows from the submaximality of Y.

$\square$

The space in 2.1 is not regular because if $p \in Y$, the subset $K = \omega \times \{p\}$ is closed in X but the closure of any neighborhood of K contains $\mathcal{E}$. Therefore, if $E \in \mathcal{E}$, there are not disjoint open subsets of X containing E and K respectively. The problem is obviously that neighborhoods of elements of $\mathcal{E}$ are too large. We circumvent this problem by taking neighborhoods of elements of $\mathcal{E}$ to be elements of a free clopen ultrafilter. This is done after establishing some notation and observing some preliminaries.

Preliminaries 2.2 Let Z be a regular extension of a countable discrete subspace denoted as ω such that $Z\backslash\omega$ is discrete. the topology on Z is denoted as $\tau(Z)$. Here are some observations that are easy to verify:

(a) the space Z is zero-dimensional.

(b) The space Z is normal if and only if disjoint sets of $Z\backslash\omega$ are contained in disjoint open sets of Z.

(c) If Z is normal and σ is a finer topology on the underlying set of Z, then (Z,σ) is also normal.

(d) For each point $q \in Z\backslash\omega$, let $\mathcal{U}_q$ be an ultrafilter on ω containing $\{U\cap\omega : q \in U \in \tau(Z)\}$. Let σ be the finer topology on the underlying set of Z defined by $T \in \sigma$ whenever r $\in T\backslash\omega$ implies $T\cap\omega \in \mathcal{U}_r$. The space (Z,σ) is also a zero-dimensional extension of ω and is normal if Z is normal. Moreover, every subspace of (Z,σ) is extremally disconnected.

Example 2.3 *There is an uncountable separable Tychonoff maximal space.*

Construction. Let $\tilde{Y}$ be the space in 1.1, $p \in \tilde{Y}$ and $Y = \tilde{Y}\backslash\{p\}$. Note that a discrete subset of Y is also discrete and, therefore, a closed subset of $\tilde{Y}$ by 1.2 in [AC]. In particular, if D is a dense subset of Y, there is a clopen subset U of Y such that $U \subseteq D$ and $p \in\mathrm{cl}_{\tilde{Y}}U$.

Let Z be a regular extension of a countable discrete subspace denoted as ω such that $Z\backslash\omega$ is discrete. As noted in 2.2, we can assume that Z is an extremally disconnected space, i.e., for each $q \in Z\backslash\omega$, $\mathcal{U}_q = \{U \cap \omega : q \in U \in \tau(Z)\}$ is an ultrafilter on ω.

As a set, let $X = (Z\backslash\omega) \cup (\omega \times Y)$. Let $\omega \times Y$ be open in X where ω has the discrete topology and $\omega \times Y$ has the product topology. A basic neighborhood of $q \in Z\backslash\omega$ is $\{q\} \cup \bigcup\{\{n\} \times U_n : n \in A\}$ where $A \in \mathcal{U}_q$ and $U_n = \tilde{U}_n \backslash \{p\}$ where $\tilde{U}_n$ is a clopen neighborhood of $p \in \tilde{Y}$, i.e., U_n is a clopen set in Y such that $p \in \mathrm{cl}_{\tilde{Y}} U_n$. It is immediate that X is Hausdorff and $\{n\} \times Y$ is clopen in X for each $n \in \omega$. Note that X is uncountable as $Z\backslash\omega$ is uncountable and X is separable and crowded as Y is separable and crowded. We will use this easily proven result: if $M \subseteq \omega \times Y$ and $p(M) = \{n \in \omega : (n,p) \in \mathrm{cl}_{\{n\}\times\tilde{Y}}(M \cap (\{n\} \times Y))\}$, then, for $q \in Z\backslash\omega$,

$$q \in \mathrm{cl}_X M \text{ if and only if } p(M) \in \mathcal{U}_q. \qquad (*)$$

Also, for $M \subseteq X$ and $n \in \omega$, let $M_n = M \cap (\{n\} \times \omega)$.

Since Y is zero-dimensional, each point in $\omega \times Y$ has a neighborhood base of sets clopen in X. Also, a basic neighborhood of $q \in Z\backslash\omega$ is clopen in X. Thus, X is zero-dimensional. To show that X is submaximal, let D be a dense subset of X. Then for each $n \in \omega$, $D \cap (\{n\} \times Y)$ is dense in $\{n\} \times Y$. It follows that $D \cap (\omega \times Y)$ is open in X. Suppose $q \in (Z\backslash\omega) \cap D$. Then $p(D \cap (\omega \times Y)) \in \mathcal{U}_q$. For $n \in p(D \cap (\omega \times Y))$, there is a clopen set U_n in Y such that $p \in \mathrm{cl}_{\tilde{Y}} U_n$ and $\{n\} \times U_n \subseteq D$. Thus, $\{q\} \cup \bigcup\{\{n\} \times U_n : n \in p(D \cap (\omega \times Y))\} \subseteq D$. This completes the proof that D is open in X and that X is submaximal.

Next, to show that X is extremally disconnected, let U and V be disjoint open sets in X. Since $\mathrm{cl}_X U = \mathrm{cl}_X(U \cap (\omega \times Y))$, we can assume that $U \cup V \subseteq \omega \times Y$. As $\omega \times \tilde{Y}$ is extremally disconnected, it follows that $\mathrm{cl}_{\omega\times\tilde{Y}} U \cap \mathrm{cl}_{\omega\times\tilde{Y}} V = \emptyset$. Using this statement and (*), it follows that $\mathrm{cl}_X U \cap \mathrm{cl}_X V = \emptyset$.. This completes the proof that X is extermally disconnected. It follows by 1.2 that X is maximal space.

□

Corollary 2.4 *There is an uncountable, separable normal maximal space if and only if* $2^\omega = 2^{\omega_1}$.

Proof: Suppose that E is an uncountable separable normal maximal space with D being the countable dense subset. By submaximality, D is open and $E\backslash D$ is an uncountable closed discrete subset of E. By Jones Lemma, $|\mathcal{P}(\mathcal{E}\backslash\mathcal{D})| \leq |\mathcal{P}(\mathcal{D})|$. As $E\backslash D$ is uncountable, then $2^{\omega_1} \leq 2^{|E\backslash D|} \leq 2^\omega$. It follows that $2^\omega = 2^{\omega_1}$.

The set theoretic axiom $2^\omega = 2^{\omega_1}$ guarantees that the Stone-Čech compactification $\beta\omega_1$ (where ω_1 has the discrete topology) of ω_1 can be embedded in $\beta\omega\backslash\omega$. The subspace $Z = \omega_1 \cup \omega$ of $\beta\omega$ is normal as ω_1 is C^*-embedded in $\beta\omega_1$ and therefore in $\beta\omega$. By 2.2, the extension Z of ω is extremally disconnected. Starting with a separable normal maximal space Y, construct X as in the proof of 2.3. By 2.3, X is an uncountable, separable Tychonoff maximal space.

It remains to show that X is normal. Let $F : X \to Z$ be the map which fixes each element of $X\backslash(\omega \times Y)$ and takes (n, x) to n for each $(n, x) \in \omega \times Y$. Let H and K be disjoint closed subsets of X. Since Z is normal and strongly zero-dimensional, there exists a clopen subset A of Z such that $F[H]\backslash\omega \subseteq A$ and $F[K]\backslash\omega \subseteq Z\backslash A$. Let $X_H = F^{\leftarrow}[A]$ and $X_K = F^{\leftarrow}[Z\backslash A]$. Then X_H and X_K partition X into clopen sets with the property that $X_H \cap (X\backslash(\omega \times Y)) \cap K = \emptyset$ and $X_K \cap (X\backslash(\omega \times Y)) \cap H = \emptyset$. If we can find disjoint open subsets U_H and V_H of X_H containing $H \cap X_H$ and $K \cap X_H$, and $K \cap X_K$, we will have shown that H and K are contained in disjoint open subsets of X. We show how to find the sets U_H and V_H, the construction of U_K and V_K being analogous. For each $n \in \omega$ such that $F(n) \in A$ and $(n, p) \notin \mathrm{cl}_{\{n\}\times\tilde{Y}}(K \cap (\{n\} \times Y))$, let $\tilde{W}_n$ be a clopen subset of $\{n\} \times \tilde{Y}$ such that $p \in \tilde{W}_n$ and $\tilde{W}_n \cap K = \emptyset$, and let $W_n = \tilde{W}_n\backslash\{(n, p)\}$. For all other $n \in \omega$ such that $F(n) \in A$, let $W_n = \emptyset$. Since no element of $X_H\backslash(\omega \times Y)$ is an element of the closed set K, and since $X_H \cap (\omega \times Y)$ is dense in X_H, every element of $X_H\backslash(\omega \times Y)$ is an element of the closure in X_H of $H \cup \bigcup_{n\in\omega} W_n$. Since $\omega \times Y$ is normal, there exists a continuous function $f : X_H \cap (\omega \times Y) \to [0,1]$ such that $f(x) = 0$ if $x \in H \cup \bigcup_{n\in\omega} W_n$ and $f(x) = 1$ if $x \in K$. Since X_H is extremally disconnected, and every dense subset of an extremally disconnected space is C^*-embedded, there is a continuous extension $f^* : X_H \to [0,1]$ of f. Let $U_H = (f^*)^{\leftarrow}[(-\infty, \frac{1}{2})]$ and $V_H = (f^*)^{\leftarrow}[(\frac{1}{2}, \infty)]$. Then, since every element of $X_H\backslash(\omega \times Y)$ is in the closure of a set on which f^* is identically 0, $(X_H\backslash(\omega \times Y)) \cup H \subseteq U_H$, and $X_H \cap K \subseteq V_H$ by the definition of f. This completes the proof that X is normal.

□

Remarks 2.5 1. Actually, we have proved more than what is stated in 2.4. The following are equivalent:

(i) There is an uncountable, separable normal maximal space in which every subset is G_δ.

(ii) There is an uncountable, separable normal submaximal space.

(iii) $2^\omega = 2^{\omega_1}$.

2. Another way of generating an uncountable, separable crowded Hausdorff submaximal space is to start with an uncountable separable crowded Hausdorff space X (e.g., $\boldsymbol{R}^{\boldsymbol{c}}$ where $\boldsymbol{c} = 2^\omega$) with S a countable dense subset. Let $\mathcal{D} = \{D \subseteq X : D$ is dense in $X\}$ and $\mathcal{F} \subseteq \mathcal{D}$ be a maximal $\mathcal{D}$-filter containing S. Then the underlying set of X with the topology τ generated by $\tau(X) \cup \mathcal{F}$ is a separable crowded Hausdorff submaximal topology on X. That S is dense in (X,σ) and that (X,σ) is submaximal and crowded follow from results of Bourbaki (see [B]). It is clear that (X,σ) is uncountable and Hausdorff. We observe that if X is a space of cardinality $2^{\boldsymbol{c}}$, then (X,σ) will

be a separable submaximal Hausdorff space having as large a cardinal as possible. In particular, (X, σ) has a closed discrete subset of cardinality $2^{\mathbf{c}}$. Since a separable Tychonoff space has weight at most $\mathbf{c}$, there can be no such example which is Tychonoff.

3. The referee has kindly supplied the following simpler construction of spaces having the properties of the spaces in 2.3 and 2.4. Let X be the space described in 1.1. Since X is Lindelöf but not compact, it is not pseudocompact, so it has a closed, discrete, infinite, countable C^*-embedded subset D. Then $\mathrm{cl}_{\beta X} D$ is homeomorphic to the Stone-Čech compactification $\beta\omega$ of ω and therefore has an uncountable relatively discrete subset S. Furthermore, if $2^{\omega_1} = 2^{\omega}$, S may be taken to be C^* in $\mathrm{cl}_{\beta X} D$. Then the space $X \cup S$ is maximal, separable, uncountable, and Tychonoff, and if S is C^*-embedded in $\mathrm{cl}_{\beta X} D$, it is also normal.

3 The question of σ-discreteness

In this section, we discuss some other questions raised in [AC]. Arhangel'skii and Collins ask whether every submaximal space is σ-discrete and whether every crowded submaximal space is strongly σ-discrete, where a space is *σ-discrete* if it is the union of countably many discrete subsets, and *strongly σ-discrete* if it is the union of countably many closed discrete subsets. We show that the consistency of the existence of a measurable cardinal implies the consistency of the existence of a crowded submaximal space which is not σ-discrete. More precisely, we show that there exists such a space if and only if there is a crowded irresolvable Baire space, where a space is *irresolvable* it if does not have disjoint dense subsets. The consistency result then follows from [KST], where it is shown that the consistency of the existence of a measurable cardinal implies the existence of a (Tychonoff) crowded irresolvable space each of whose subsets is Baire.

A space is *hereditarily irresolvable* if every subset is irresolvable. It is known (see [vD] or [KST]) that every irresolvable space has a non-empty hereditarily irresolvable open subset. The main tool is the following Proposition.

Proposition 3.1 *The following are equivalent.*

(i) There exists a submaximal Hausdorff space which is not σ-discrete.

(ii) There exists a crowded submaximal Hausdorff space which is not σ-discrete.

(iii) There exists a maximal Hausdorff space which is not σ-discrete.

(iv) There exista a crowded submaximal Hausdorff space which is not strongly σ-discrete.

(v) There exists a maximal Hausdorff space which is not strongly σ-discrete.

(vi) There exists a crowded irresolvable Hausdorff Baire space.

Proof: The implications (ii) $\Rightarrow$ (i), (iii) $\Rightarrow$ (ii) $\Rightarrow$ (iv), and (iii) $\Rightarrow$ (v) $\Rightarrow$ (iv) are trivial.

(i) $\Rightarrow$ (ii). We first note that a scattered submaximal space has Cantor-Bendixsen height at most two, since if the set of limit points have a limit point p, then $\{x|x \text{ is isolated}\} \cup \{p\}$ would be a dense subset which is not open. Therefore, any scattered submaximal space is σ-discrete. (This observation was made in [AC].) Let Z be a submaximal Hausdorff space which is not σ-discrete. Let $X = \mathrm{cl}_Z(\bigcup\{A \subseteq Z|A \text{ is crowded}\})$. Then X is a crowded submaximal Hausdorff space, and since

$Z \backslash X$ is scattered, it is σ-discrete. Since Z is not σ-discrete, neither is X.

(iv) $\Rightarrow$ (vi). Suppose that there exists a crowded submaximal space Y which is not strongly σ-discrete. The property of being submaximal is hereditary, so it suffices to show that Y has a crowded Baire subset. Since in a submaximal space every nowhere-dense subset is closed and discrete, Y is second category in itself. Therefore, Y has a non-empty open, and therefore crowded, subset X which is Baire.

(vi) $\Rightarrow$ (iii). Let Y be a Hausdorff crowded irresolvable Baire space. Let $\widehat{H}$ be an open hereditarily irresolvable subset of Y and denote the topology on $\widehat{X}$ by $\widehat{\tau}$. Then $\widehat{X}$ is Baire, and since it is hereditarily irresolvable, every somewhere-dense subset of $\widehat{X}$ contains a non-empty open subset of $\widehat{X}$. Let τ be a maximal topology on $\widehat{X}$ which strengthens $\widehat{\tau}$, and let X be $\widehat{X}$ with the topology τ. Since $\widehat{X}$ is Hausdorff, so is X. Suppose X were σ-discrete, say $X = \cup_{k \in \omega} D_k$ where each set D_k is discrete as a subset of X. Since $\widehat{X} = \cup_{k \in \omega} D_k$, there exists $k_0 \in \omega$ such that D_{k_0} is somewhere-dense in $\widehat{X}$. Therefore, there exists a non-empty open subset U of $\widehat{X}$ such that $U \subseteq D_{k_0}$. Since $\widehat{\tau} \subseteq \tau$, U is also open in X. Therefore, D_{k_0} contains a non-empty open subset of X, and since D_{k_0} is discrete, X has an isolated point, contradicting the fact that X is crowded.

□

The significance of Proposition 3.1 lies in the fact, shown in [M], that condition (vi) in the Proposition is known to be equivalent to an affirmative answer to the old question, due to Katětov, which asks if there is a crowded Hausdorff space with the property that every real-valued function defined on the space is somewhere continuous. A consistent answer to Katětov's question was given in [KST] assuming the consistency of the existence of a measurable cardinal. Therefore, we get the following corollary.

Corollary 3.2 *If it is consistent that there is a measurable cardinal, then it is consistent that there is a crowded submaximal Hausdorff space (in fact, a crowded maximal Hausdorff space) which is not σ-discrete.*

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George Mason
Fairfax, VA 22030

University of Kansas
Lawrence, KS 66045-2142