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Paranormal Spaces In The Constructible Universe

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Abstract

In this note we present the construction of a first countable paranormal space with a closed discrete set that is not a regular G_δ from the combinatorial principle $\diamondsuit^{\#}$. This is a partial solution to the question whether first countable countably paracompact spaces are strongly collectionwise Hausdorff under **V=L**.

1 Introduction.

The Normal Moore Space Conjecture (NMSC) asserts that normal Moore spaces are collectionwise normal. P. Nyikos proved the consistency of NMSC by showing that it is implied by the Product Measure Extension Axiom (PMEA). Earlier, Gödel's constructible universe **L** was considered as a possible model for NMSC, and while we now know that it is not, it does have a similar influence on the structure of closed discrete sets in normal first countable spaces. In 1974, W.G. Fleissner proved

[F1] **V=L** implies that normal spaces of character $\leq 2^{\aleph_0}$ are collectionwise Hausdorff (henceforth abbreviated cwH).

Nyikos's established that PMEA implies NMSC by proving

[N1] PMEA implies that normal spaces of character $< 2^{\aleph_0}$ are collectionwise normal.

S. Watson and D.K. Burke proved analagous results for the class of regular, countably paracompact spaces.

[W] **V=L** implies that countably paracompact regular spaces of character $< 2^{\aleph_0}$ are cwH.

[B] PMEA implies that countably paracompact regular spaces of character $< 2^{\aleph_0}$ are strongly cwH.

In the same paper Burke also proved the interesting result that under PMEA, in countably metacompact T_1 spaces of character $< 2^{\aleph_0}$ closed discrete sets are G_δ, and recently Nyikos proved that under **V=L** closed discrete sets are G_δ in *locally countable* countably metacompact spaces. Note that in all these results the PMEA assumption seems to give slightly stronger conclusions. In the case of normal spaces we know that **V=L** does not imply that first countable normal spaces are collectionwise normal. In fact, the large cardinal inherent in the PMEA assumption is needed to prove the consistency of NMSC (see [F2]). For countably metacompact spaces we also know that the PMEA result is stronger as the author has recently constructed a counterexample to Burke's result from the principle $\diamondsuit^*$. However, for the class of first countable, countably paracompact spaces, the analogous question is open[1].

Definition 1.1 *A space is countably paracompact if every countable open cover has a locally finite refinement.*

Definition 1.2 *A space is paranormal if for every countable discrete collection of closed sets there is a locally finite open expansion.*

[1] For both countably metacompact and countably paracompact spaces, whether the large cardinal inherent in the PMEA assumptions is needed to get the stronger results is also not known.

It is easy to show that both countably paracompact spaces and normal spaces are paranormal. Although Burke and Watson state their theorems for the class of countably paracompact spaces, like many other results concerning separation properties of closed discrete sets, only certain open covers need be considered in the proofs (so called canonical open covers of partitions of the closed discrete sets). In both of their results the weaker assumption of paranormality suffices.

Theorem 1.3 (Burke) PMEA *implies that first countable, paranormal spaces are strongly cwH (hence closed discrete sets are regular* G_δ*).*

Theorem 1.4 (Watson) V=L *implies that first countable paranormal spaces are collectionwise Hausdorff (hence closed discrete sets are* G_δ*).*

In this note we construct from $\diamondsuit^\#$ an example of a first countable paranormal space with a closed discrete set that is not a regular G_δ . Therefore we establish that Watson's result cannot be improved to the class of paranormal spaces. Unfortunately the example is not countably paracompact. Nyikos's question remains open.

Question 1.5 *Does* **V=L** *imply that closed discrete sets are regular* G_δ *in first countable, countably paracompact spaces?*

2 A first countable, paranormal space from $\diamondsuit^\#$

The construction is very similar to the construction in [S]. However, it is complicated by the fact that we need to recursively construct locally finite open covers as we build the space and preserve that they are locally finite as we extend the topology.

Example 2.1 $\diamondsuit^{\#}$ *implies the existence of a first countable, paranormal space with a closed discrete set that is not a regular* G_δ.

Clearly any such example cannot be normal. Our space will have three pieces, $X = A \cup B \cup \omega_1$. All points of ω_1 will be isolated, A will be closed discrete and we will build in a closed set B of points to witness that A is not a regular G_δ. To help us prove that our space is paranormal we will make sure that B can't be partitioned into two disjoint uncountable closed sets.

We were unable to make the space countably paracompact. The space being paranormal guarantees that there will be lots of countable open covers that we would need to take care of to make the space countably paracompact: for any partition of the closed discrete set $A = \bigcup A_n$, paranormality implies there is a locally finite open expansion $\mathcal{W}$ of the partition. For each $x \in X$ let $ord(x, \mathcal{W})$ be the minimal n such that there exists an open neighborhood of x intersecting at most n elements of $\mathcal{W}$. Letting $U_n = \{x \in X : ord(x, \mathcal{W}) \leq n\}$ defines a countable open cover of the space which may or may not have a locally finite refinement.

$\diamondsuit^{\#}$ is the strongest version of $\diamondsuit$ in the literature. In particular it implies $\diamondsuit^+$. A typical $\diamondsuit^+$ sequence consists of countable families of subsets of each $\alpha < \omega_1$. A $\diamondsuit^{\#}$ sequence $\{\mathcal{N}_\alpha : \alpha < \omega_1\}$ will be such that $\{\mathcal{N}_\alpha \cap P(\alpha) : \alpha < \omega_1\}$ forms a $\diamondsuit^+$ sequence and each $\mathcal{N}_\alpha$ will be a countable transitive model of a suitable portion of ZFC. Not only will $\alpha = (\omega_1)^{\mathcal{N}_\alpha}$ for stationarily many α but in addition the sequence will be Π^1_2 reflecting.

Definition 2.2 *We say that* $\{\mathcal{N}_\alpha : \alpha < \omega_1\}$ *is* Π^1_2 *reflecting if whenever* Φ *is a* Π^1_2 *sentence which is true in a structure* $\langle \omega_1, \in, (A_i)_{i<\omega} \rangle$, *where each* A_i *is a finitary predicate on* ω_1, *then there exists an* $\alpha < \omega_1$ *such that*

$$\mathcal{N}_\alpha \models \text{“}\Phi \textit{ is valid in } \langle \alpha, \in, (A_i \mid \alpha)_{i<\omega} \rangle.\text{”}$$

We state now the definition of $\diamondsuit^{\#}$, and a theorem summarizing the main results proven in [D2].

Definition 2.3 $\diamondsuit^{\#}$: *There is a sequence* $\langle \mathcal{N}_\alpha : \alpha < \omega_1 \rangle$ *such that*

(i) $\mathcal{N}_\alpha$ *is a countable, transitive primative recursive closed set containing* α.

(ii) If $X \subseteq \omega_1$, *there is a club* $C \subseteq \omega_1$ *such that* $\forall \alpha \in C$, $C \cap \alpha$ *and* $X \cap \alpha \in \mathcal{N}_\alpha$.

(iii) $\langle \mathcal{N}_\alpha : \alpha < \omega_1 \rangle$ *is* Π^1_2 *reflecting.*

The primative recursive (p.r.) set functions are defined in [D1]. A primative recursive closed set is one that is closed under the p.r. set functions. For our purposes it suffices to note that a transitive primative recursive closed set is a model of a large fragment of Set Theory, enough to carry out simple recursive constructions including the constructible hierarchy and in particular the construction of 2.1.

Theorem 2.4 $\diamondsuit^{\#}$ *holds in L. Furthermore, assuming* $\diamondsuit^{\#}$ *holds, then there is a* $\diamondsuit^{\#}$ *sequence* $\langle \mathcal{N}_\alpha : \alpha < \omega_1 \rangle$ *with the following additional properties*

(iv) $D = \{\alpha < \omega_1 : \alpha = (\omega_1)^{\mathcal{N}_\alpha}\}$ *is stationary.*

(v) For each Π^1_2 *sentence* Φ *valid in some structure* $\langle \omega_1, \in, (A_i)_{i<\omega} \rangle$ *the set of* $\alpha \in D$ *such that*

$$[\mathcal{N}_\alpha \models \text{“}\Phi \textit{ is valid in } \langle \alpha, \in, (A_i \mid \alpha)_{i<\omega} \rangle.\text{”}]$$

is stationary in ω_1.

(vi) For each α $\langle \mathcal{N}_\gamma : \gamma < \alpha \rangle \in \mathcal{N}_\alpha$.

We are interested in reflecting certain Π^1_2 properties of ω_1 over a structure of the form $\langle \omega_1, \in, \mathcal{A} \rangle$. We first fix $\mathcal{A} \subseteq \omega_1 \times \omega_1 \times \omega$ coding that each $\alpha < \omega_1$ is countable. To do this fix for each $\alpha < \omega_1$ an enumeration

$$f_\alpha : \omega \to \alpha$$

and let $\mathcal{A} = \{\langle \alpha, f_\alpha(n), n \rangle : \alpha < \omega_1, n < \omega\}$. Devlin proves theorem 2.4(v) by showing that there is a Π^1_1 statement Ψ such that $\langle \alpha, \in, \mathcal{A} \mid \alpha \rangle \models \Psi$ iff $\alpha = \omega_1$. Therefore

$$\mathcal{N}_\alpha \models \text{“}\Psi \textit{ is valid in } \langle \alpha, \in, \mathcal{A} \mid \alpha \rangle\text{” iff } \alpha = (\omega_1)^{\mathcal{N}_\alpha}.$$

We are interested in reflecting the following Π^1_1 statement that says that no unbounded (i.e., uncountable) subset of ω_1 is the union of countably many bounded subsets.

Φ “$(\forall X)(\forall f : X \to \omega)[(\forall n < \omega \exists \gamma f^{-1}(n) \subseteq \gamma) \to (\exists \gamma, X \subseteq \gamma)]$.”

It will suffice our purposes that if E is the stationary set on which

$$\mathcal{N}_\alpha \models \Phi \textit{ is valid in } \langle \alpha, \in, \mathcal{A} \mid \alpha \rangle$$

then $\diamondsuit(E)$ holds.

Before we present the construction we need one more technical definition reflecting the fact that $\{\mathcal{N}_\alpha \cap P(\alpha) : \alpha < \omega_1\}$ forms a $\diamondsuit^+$ sequence.

Definition 2.5 *For each* α *and each* $C \in \mathcal{N}_\alpha \cap P(\alpha)$ *and* $H \in \mathcal{N}_\alpha \cap {}^\alpha\omega$ *such that* C *is closed, we say that the pair* (C, H) *is good if for each* $\beta \in C$ *both* $C \cap \beta$ *and* $H \mid \beta$ *are contained in* $\mathcal{N}_\beta$.

The Construction

The point set for the space is $X = A \cup B \cup \omega_1$ where

$$A = \{a_\alpha : \alpha < \omega_1\}, \quad B = \{b_\alpha : \alpha < \omega_1\}$$

are disjoint copies of ω_1. A first countable, zero-dimensional topology on X will be constructed so that

- A will be closed discrete, separated (hence a G_δ), but not a regular G_δ,
- ω_1 will consist of isolated points, and
- B will be closed unseparated from A such that for every $C,\ D \in [B]^{\omega_1}$, $\overline{C} \cap \overline{D} \neq \emptyset$.

The subspace topology on $A \cup \omega_1$, τ, will be coded by a function $\mathcal{U} : \omega_1^2 \to \omega$ (see below) and neighborhoods will look up, while σ, the subspace topology on $B \cup \omega_1$ will be locally countable, locally compact and neighborhoods will look down.
Let $\{\mathcal{N}_\alpha : \alpha < \omega_1\}$ be a $\diamondsuit^\#$ sequence, and let $\{g_\alpha : \alpha \in E\}$ be a $\diamondsuit(E)$ sequence on ${}^{\omega_1 \times \omega}\omega$. Where $E = \{\alpha < \omega_1 : \mathcal{N}_\alpha \models \Phi \textit{ is valid in } \langle \alpha, \in, \mathcal{A} \mid \alpha \rangle\}$.
Given a function $\mathcal{U} : \omega_1^2 \to \omega$ let

$$U_n(a_\alpha) = \{a_\alpha\} \cup \{\beta : \alpha < \beta < \omega_1,\ \mathcal{U}(\alpha, \beta) \geq n\}.$$

Definition 2.6 *We say that $\mathcal{U} : \omega_1^2 \to \omega$ codes the topology τ on $A \cup \omega_1$ if τ is the topology generated by isolating ω_1 and by taking $\{U_n(a_\alpha) : n < \omega, \alpha < \omega_1\}$ as a base for the points of A.*

We will define by recursion on $\alpha < \omega_1$ topologies σ_α on $(B \mid (\alpha+\omega)) \cup (\alpha+\omega)$ and τ_α on $(A \mid \alpha) \cup (\alpha+\omega)$. We let $A_\alpha = A \mid \alpha$ and $B_\alpha = B \mid (\alpha + \omega)$ and let $X_\alpha = A_\alpha \cup B_\alpha \cup (\alpha + \omega)$. For $\beta < \alpha$ we will demand that σ_α be a conservative extension of σ_β while τ_α will be coded in the sense of 2.6 by a function $\mathcal{U}_\alpha : \alpha \times (\alpha+\omega)$ extending the previous $\mathcal{U}_\beta$'s. In the end we will let σ be the topology on $B \cup \omega_1$ generated by $\bigcup_{\alpha<\omega_1} \sigma_\alpha$, and we will let τ be the topology on $A \cup \omega_1$ coded by $\mathcal{U} = \bigcup_{\alpha<\omega_1} \mathcal{U}_\alpha$. We glue the topologies together by letting v be the topology generated by $\tau \cup \sigma$. The $\diamondsuit$ sequence $\{g_\alpha : \alpha < \omega_1\}$ is used to assure that A will not be a regular G_δ. This is made precise in the following lemma.

Lemma 2.7 *Let v be the topology on $X = A \cup B \cup \omega_1$ defined by τ and σ as above. Suppose that for each $g \in {}^{\omega_1 \times \omega}\omega$ $\exists \alpha < \omega_1$ such that $g \mid \alpha \times \omega = g_\alpha$ and such that there is a sequence $\{\alpha(n) : n < \omega\}$ cofinal in α satisfying*

(a) $\mathcal{U}(\alpha(n), \alpha + n) = g_\alpha(\alpha(n), n)$.

(b) $\{\alpha + n : n < \omega\}$ converges to b_α.

Then A is not a regular G_δ in X.

Proof: Fix a sequence $\{W(n) : n < \omega\}$ of open neighborhoods of A. By shrinking the $W(n)$'s a little we may assume that $\{W(n) : n < \omega\}$ is decreasing. Define $g : \omega_1 \times \omega \to \omega$ by letting $g(\alpha, n)$ be the minimal i such that $U_i(a_\alpha)$ is contained in $W(n)$. If there is an α and a sequence $\{\alpha(n) : n < \omega\}$ satisfying both (a) and (b), then for each n, $\alpha + n \in W(n)$. The sequence $\{\alpha + n : n < \omega\}$ converging to b_α implies that $b_\alpha \in \bigcap_{n<\omega} \overline{W(n)}$. Therefore A is not a regular G_δ.

To make the space paranormal we need to keep control of what kind of countable discrete collections of closed sets can appear. We will do this by assuring that B cannot contain disjoint uncountable closed subsets and that A is a separated closed discrete set. This is summarized in the following lemma whose proof is routine.

Lemma 2.8 *Let v be a topology on $X = A \cup B \cup \omega_1$. Suppose that A is a separated closed discrete collection and that B cannot be partitioned into uncountable disjoint closed sets. Then X is paranormal iff*

(†) *For every partition $H : A \to \omega$ there exists an open expansion $\mathcal{W} = \{W_n : n < \omega\}$ which is locally finite at each point of B.*

To make sure that (†) will hold we only need $\diamondsuit^*$ (as in section 1) however, to preserve (†) in our construction while guaranteeing that B contains no disjoint uncountable closed subsets we will need $\diamondsuit^\#$ (in particular we will need to reflect Φ). We are now ready to state our inductive hypotheses:
For each $\beta \leq \alpha$ we have $\mathcal{U}_\beta : \beta \times (\beta + \omega) \to \omega$ and topologies σ_β satisfying

(1) For all $\eta < \beta \leq \alpha$, $\mathcal{U}_\eta \subseteq \mathcal{U}_\beta$.

(2) For all $\eta < \beta \leq \alpha$, σ_β is a conservative extension of σ_η (i.e., $\sigma_\eta \subseteq \sigma_\beta$ and for all $u \in \sigma_\beta$, $u \cap X_\eta \in \sigma_\eta$).

In (3) and (4) below $U_n^\alpha(a_\beta)$ is defined from $\mathcal{U}_\alpha$ analogously to the definition of the U_n's from $\mathcal{U}$ given before 2.6.

(3) For all $\beta \neq \eta < \alpha$,

$$U_1^\alpha(a_\beta) \cap U_1^\alpha(a_\eta) = \emptyset$$

(4) For all $\beta < \alpha$, $U_1^\alpha(a_\beta)$ is clopen in X_α with respect to $\sigma_\alpha \cup \tau_\alpha$.

(5) For all $\beta < \alpha + \omega$ and $n < \omega$, $V_n(b_\beta)$ is clopen in X_α with respect to $\sigma_\alpha \cup \tau_\alpha$.

As well, for each $\beta \leq \alpha$ and each good pair $(C, H) \in \mathcal{N}_\beta$ we have

$$F^{C,H,\beta} : \beta \to \omega$$

satisfying

(6) The partition (coded by H) of A_β has an open in X_α, expansion $\mathcal{W}^{C,H,\beta}$, coded by $F^{C,H,\beta}$, and $\mathcal{W}^{C,H,\beta}$ is locally finite at each point of B_α.

(7) For all $\eta < \beta \leq \alpha$, for all good pairs $(C, H) \in \mathcal{N}_\eta$ and $(C', H') \in \mathcal{N}_\beta$, if $\eta \in C'$, $C' \cap \eta = C$ and $H' \mid \eta = H$, then $F^{C,H,\eta} \subseteq F^{C',H',\beta}$.

(8) $F^{C,H,\beta} \geq H$ for each $\beta \leq \alpha$ and each good pair $(C, H) \in \mathcal{N}_\beta$.

Fix γ a limit ordinal and suppose that for each $\alpha < \gamma$ the induction hypotheses (1)-(8) are satisfied. We need to define $\mathcal{U}_\gamma$, σ_γ and $F^{C,H,\gamma}$ for each good pair $(C, H) \in \mathcal{N}_\gamma$. We will do this in the following order:

First we will define $F^{C,H,\gamma}$ for those good pairs $(C, H) \in \mathcal{N}_\gamma$ satisfying C is unbounded in γ.
Second we will extend $\bigcup_{\alpha<\gamma} \mathcal{U}_\alpha$ to $\mathcal{U}_\gamma$.
Third we will extend $\bigcup_{\alpha<\gamma} \sigma_\alpha$ to σ_γ.
Lastly we will define $F^{C,H,\gamma}$ for those good pairs $(C, H) \in \mathcal{N}_\gamma$ satisfying C is bounded in γ.

The first step is easy. If $(C, H,) \in \mathcal{N}_\gamma$ is good and C is unbounded in γ, then $(C \cap \alpha, H \mid \alpha)$ is a good pair in $\mathcal{N}_\alpha$ for each $\alpha \in C$. Therefore, by inductive hypothesis (7) we have no choice but to define $F^{C,H,\gamma} = \bigcup_{\alpha \in C} F^{C\cap\alpha, H|\alpha, \alpha}$. Note that (8) is clearly preserved when we do this.

Next we need to pick a sequence $\{\gamma(n) : n < \omega\}$ cofinal in γ satisfying 2.7. In the following definition the family $\mathcal{G}$ will be an enumeration of the pairs $(F^{C,H,\gamma}, H)$ for good pairs (C, H) such that C is unbounded in γ.

Definition 2.9 *Let $g : \gamma \times \omega \to \omega$, and let $\mathcal{G} = \{(F_i, H_i) : i < \omega\} \subseteq ({}^\gamma\omega)^2$. We say that the sequence $\{\gamma(n) : n < \omega\}$ diagonalizes $(g, \mathcal{G})$ if for each $i < \omega$, either*

$$\exists m < \omega \{\gamma(n) : n < \omega\} \subseteq^* H_i^{-1}(m)$$

or

$$\exists m < \omega \forall n > m,\ g(\gamma(n), n) < F_i(\gamma(n)).$$

Now we can extend $\bigcup_{\alpha<\gamma}\mathcal{U}_\alpha$ to $\mathcal{U}_\gamma$. Let $\mathcal{G}$ be an enumeration of the pairs $(F^{C,H,\gamma}, H)$ for good pairs (C, H) such that C is unbounded in γ.
If there exists a sequence $\{\gamma(n) : n < \omega\}$ ω-cofinal in γ diagonalizing $(g_\gamma, \mathcal{G})$ then we let

$$\mathcal{U}_\gamma(\gamma(n), \gamma + n) = g_\gamma(\gamma(n), n).$$

Restricted to $\gamma \times \gamma$, $\mathcal{U}_\gamma = \bigcup_{\alpha<\gamma}\mathcal{U}_\alpha$. And at all other points in $\gamma \times (\gamma + \omega)$ we let $\mathcal{U}_\gamma = 0$.
If there is no such sequence diagonalizing $(g_\gamma, \mathcal{G})$ restricted to $\gamma \times \gamma$, we let $\mathcal{U}_\gamma = \bigcup_{\alpha<\gamma}\mathcal{U}_\alpha$ and on $(\gamma \times (\gamma + \omega) \setminus (\gamma \times \gamma)$ we let $\mathcal{U}_\gamma = 0$.

Next we define σ_γ by defining local neighborhood bases $\{V_i(b_{\gamma+n}) : n < \omega\}$ for each $n < \omega$. To start off let $\sigma'_\gamma = \bigcup_{\alpha<\gamma}\sigma_\alpha$.
$n = 0$: Let $V_n(b_\gamma) = \{b_\gamma\} \cup \{\gamma + m : n < m < \omega\}$.
$n > 0$: List all pairs (X, Y) from $\mathcal{N}_\gamma \cap P(\gamma)$ such that

$$\mathcal{N}_\gamma \models \text{“}X \textit{ and } Y \textit{ are uncountable.”}$$

as $\{(X_n, Y_n) : n < \omega\}$.
In addition, list all open covers coded by all $F^{C,H,\alpha}$'s thus far defined as $\{\mathcal{W}_n : n < \omega\}$. An important thing to note is that each $F^{C,H,\alpha}$ thus far defined is in $\mathcal{N}_\gamma$ even in the case that $\alpha = \gamma$ and C is unbounded in γ. For each n we fix an enumeration of $\mathcal{W}_n = \{W_n(i) : i < \omega\}$. For an open set V and a countable open cover $\mathcal{W}$ enumerated by $\mathcal{W} = \{W_i : i < \omega\}$, $ord(V, \mathcal{W}) \preceq N$ means that $\{i : V \cap W_i \neq \emptyset\} \subseteq N$.

For each $n < \omega$ pick if possible a sequence $S_n = \{\gamma_n(i) : i < \omega\} \subseteq \gamma$ satisfying

(i) $tp(S_n) = \omega$.

(ii) $S_n \subseteq X_n \cup Y_n$.

(iii) $|S_n \cap X_n| = |S_n \cap Y_n| = \aleph_0$.

(iv) For each $i < \omega$ there exists a basic open neighborhood $V(\gamma_n(i))$ of $b_{\gamma_n(i)}$ such that for each j there is an N_j such that for all $i > j$,

$$ord(V(\gamma_n(i)), \mathcal{W}_j) \preceq N_j.$$

If there is no such sequence, let $S_n = \emptyset$. Clearly if we have for each n such a sequence, we can thin them all out so that in addition they satisfy

(v) $S_n \cap S_m = \emptyset$ for each $n \neq m$.

We wish to extend the topology σ'_γ so that for each n, if $S_n \neq \emptyset$ then $\{b_{\gamma_n(i)} : i < \omega\}$ converges to $b_{\gamma+n}$. The condition (iv) guarantees that if we do this we won't destroy the local finiteness of the covers thus far defined.

In general, suppose (Y, τ_0) is a countable, zero-dimensional topological space and that $S \subseteq Y$ is closed discrete. Fix a point $x \notin Y$. Then there is a conservative zero-dimensional extension τ_1 of τ_0 on $Y' = Y \cup \{x\}$ such that in τ_1, S converges to x. This can be done by enumerating the points of Y and recursively choosing a discrete separation $\{V(s) : s \in S\}$ of S. We then let $\{x\} \cup \bigcup\{V(s) : s \in S \setminus F\}$ be a typical basic open neighborhood of x where F a finite subset of S.

We can do a bit more. Given countably many disjoint closed discrete sequences $\{S_n : n < \omega\}$ in Y. We can fix points $\{x_n : n < \omega\}$ and construct a conservative zero-dimensional extension of τ_0 so that for each n, S_n converges to x_n. For each n the local neighborhood base at x_n is as described above.

In our case let σ_γ be a conservative extension of σ'_γ such that if S_n is not empty, then it converges to $b_{\gamma+n}$. Otherwise $b_{\gamma+n}$ is an isolated point. We also demand that the discrete sequences of neighborhoods separating each S_n also refine the open sets given by clause (iv) above.

Clearly, by this choice each of the open covers $\mathcal{W}_n$ is locally finite at each point of B_γ. Therefore we need only define $F^{C,H,\gamma}$ for those good pairs $(C,H) \in \mathcal{N}_\gamma$ where C is bounded in γ. For any such pair let $\alpha_C = max(C)$. Then we know that $F^{C\cap\alpha_C,H|\alpha_C,\alpha_C}$ codes a locally finite open expansion of the partition of A_{α_C} coded by H (namely $\{\{a_\beta : \beta \in H^{-1}(n) \cap \alpha_C\} : n < \omega\}$). We know it is locally finite since it was one of the enumerated families $\mathcal{W}_n$. We need to extend this open expansion to a locally finite expansion of the full partition of A_γ coded by H. Keeping in mind the easy to preserve inductive hypothesis (8), that F dominates H, we extend the rest of the F's using the following lemma.

Lemma 2.10 *Let Y be a countable zero-dimensional topological space. Suppose that $\{A_n : n < \omega\}$ is a closed discrete collection and $\{W_n : n < \omega\}$ a precise locally finite open expansion. Furthermore suppose that $\{B_n : n < \omega\}$ is a closed discrete collection such that for each n, $A_n \subseteq B_n$. Then there exist open $V_n \supseteq B_n \setminus A_n$ such that $\{W_n \cup V_n : n < \omega\}$ is a locally finite expansion of $\{B_n : n < \omega\}$.*

Proof: Enumerate the points of Y in type ω and construct $\{V_n : n < \omega\}$ recursively.

This completes the construction of $\mathcal{U}_\gamma$ and σ_γ. It is straightforward, albeit a little trying in light of all the notation, to verify that the inductive hypotheses (1)-(5), (7) and (8) are satisfied. (6) is a little trickier. The proof splits up into cases. We wish to verify that for each $\alpha \leq \gamma$ and each good pair $(C,H) \in \mathcal{N}_\alpha$, the open family coded by $F^{C,H,\alpha}$ is locally finite at each point $\{b_{\gamma+n} : n < \omega\}$.

CASE 1 $n = 0$ and $\alpha < \gamma$: Recall that

$$V_N(b_\gamma) = \{b_\gamma\} \cup \{\gamma + n : N < n < \omega\}$$

and note that the only nontrivial basic open set containing $\gamma + n$ is a $U_i^\gamma(a_{\gamma(n)})$ for some i. Therefore $\gamma + n \in W^{C,H,\alpha}(i)$

only if $\gamma(n) \in H^{-1}(i)$. But since $\alpha < \gamma$ and $\{\gamma(n) : n < \omega\}$ is ω-cofinal in γ, this can only happen for finitely many of the $\gamma(n)$'s.
CASE 2 $n > 0$ and $\alpha < \gamma$ or $\alpha = \gamma$ with C unbounded in γ: The only difficulty that can occur is if $b_{\gamma+n}$ is non-isolated. Therefore we were able to choose a sequence S_n satisfying clauses (i)-(iv) above. The open cover $\mathcal{W}^{C,H,\alpha}$ was defined at this point so it was enumerated as some $\mathcal{W}_j$. Recall $S_n = \{\gamma_n(i) : i < \omega\}$ and by clause (iv) we have for each $i < \omega$ a basic open neighborhood $V(\gamma_n(i))$ of $b_{\gamma_n(i)}$ an N_j such that for all $i > j$,

$$ord(V(\gamma_n(i)), \mathcal{W}_j) \preceq N_j.$$

Also we defined the basic open sets around $b_{\gamma+n}$ to satisfy

$$V_m(b_{\gamma+n}) \subseteq \{b_{\gamma+n}\} \cup \bigcup \{V(\gamma_n(i)) : m < i < \omega\}$$

Therefore $ord(V_j(b_{\gamma+n}), \mathcal{W}_j) \preceq N_j$.
CASE 3 $n = 0$, $\alpha = \gamma$ and C is unbounded in γ: The only difficulty that will arise is if we were able to choose the sequence $\{\gamma(n) : n < \omega\}$ diagonalizing $(g_\gamma, \mathcal{G})$. If so, then as $\mathcal{W}^{C,H,\gamma} \in \mathcal{G}$ either

$$\exists m < \omega \{\gamma(n) : n < \omega\} \subseteq^* H^{-1}(m)$$

or

$$\exists m < \omega \forall n > m,\ g(\gamma(n), n) < F(\gamma(n)).$$

Both cases easily imply that $\mathcal{W}^{C,H,\gamma}$ is locally finite at b_γ.
CASE 4 $\alpha = \gamma$ and C is bounded in γ: In this case we defined the locally finite covers coded by $F^{C,H,\gamma}$ after defining the topologies so there is nothing to prove.

This completes the construction. We let $\mathcal{U} = \bigcup_{\alpha<\omega_1} \mathcal{U}_\alpha$ and let τ be the topology coded by $\mathcal{U}$. Let $\sigma = \bigcup_{\alpha<\omega_1} \sigma_\alpha$, and finally let υ be the topology generated by $\tau \cup \sigma$.

CLAIM 1: For each C and $D \in [B]^{\omega_1}$ $\overline{C} \cap \overline{D} \neq \emptyset$.

Proof: The set of α such that $C \cap \alpha$, $D \cap \alpha$, and $v \mid (\{a_\beta : \beta < \alpha\} \cup \{b_\beta : \beta < \alpha\} \cup \alpha)$ are all in $\mathcal{N}_\alpha$ forms a club. Therefore there is a γ in this club such that $\gamma \in E$, i.e.,

$$\mathcal{N}_\gamma \models \Phi \textit{ is valid in } \langle \gamma, \in, \mathcal{A} \mid \gamma . \rangle$$

Recall when defining σ_γ we enumerated some of the locally finite open families from $\mathcal{N}_\gamma$ as $\{\mathcal{W}_i : i < \omega\}$ and that (C, D) was listed as (X_n, Y_n). If we can find a sequence S_n satisfying clauses (i)-(iv) above for (X_n, Y_n), then we will have that $b_{\gamma+n} \in \overline{C} \cap \overline{D}$.
As $\mathcal{N}_\gamma \models \Phi$ we can fix $N_0 \in \omega$ and sets $C_0 \subseteq C$, $D_0 \subseteq D$ such that

$$\mathcal{N}_\gamma \models C_0 \textit{ and } D_0 \textit{ are uncountable.}$$

and such that for each $\alpha \in C_0 \cup D_0$ there is an open $V(\alpha)$ containing b_α such that $ord(V(\alpha), \mathcal{W}_0) \preceq N_0$. Recursively we choose sets $\{C_j, D_j : j < \omega\} \subseteq \mathcal{N}_\gamma$ and integers $\{N_j : j < \omega\}$ such that for each j

$$\mathcal{N}_\gamma \models C_j \textit{ and } D_j \textit{ are uncountable.}$$

and $C_{j+1} \subseteq C_j$, $D_{j+1} \subseteq D_j$ and for each $\alpha \in C_j \cup D_j$ there is an open $V(\alpha)$ containing b_α such that $ord(V(\alpha), \mathcal{W}_j) \preceq N_j$. Now choose $S_n = \{\gamma_n(i) : i < \omega\}$ so that $supS_n = \gamma$ and so that for all $k < \omega$, $\gamma_n(2k) \in C_k$ and $\gamma_n(2k+1) \in D_k$. This can be done as all the C_i and D_i are unbounded in γ.

CLAIM 2: *X is paranormal.*

Proof: Notice that by CLAIM 1, the hypotheses of 2.8 are satisfied. Therefore we need to check that for every partition of A into countably many pieces, we have an open expansion locally finite at each point of B. So fix a partition $H : \omega_1 \to \omega$ of A. By $\diamondsuit^\#$ there is a club C in ω_1 such that $\forall \alpha \in C$ both $C \cap \alpha$ and $H \mid \alpha$ are in $\mathcal{N}_\alpha$. In particular for each $\alpha \in C$ the pair $(C \cap \alpha, H \mid \alpha)$ is a good pair. Therefore for each $\alpha \in C$

we have defined $F_\alpha = F^{C\cap\alpha, H|\alpha, \alpha}$. Let

$$F = \bigcup_{\alpha \in C} F_\alpha.$$

Then F codes an open expansion $\mathcal{W} = \{W_n : n < \omega\}$ of the partition H by

$$W(n) = \bigcup \{U_{F(\alpha)}(a_\alpha) : \alpha \in H^{-1}(n)\}.$$

We claim that $\mathcal{W}$ is locally finite at each point in B. So fix $\beta \in \omega_1$ and let $\alpha \in C$ be greater than β. At stage γ we have that F_α codes a locally finite open expansion of the partition $H \mid \alpha$ at each point in B_α, in particular at b_β. Clearly this open expansion is given by $W(n) \cap X_\alpha$ where $X_\alpha = A_\alpha \cup B_\alpha \cup \alpha$. The fact that neighborhoods for b_β are contained in X_α implies that $\mathcal{W}$ is locally finite at b_β.

CLAIM 3: *A is not a regular G_δ.*

Proof: By 2.7 and the construction it suffices to fix $g : \omega_1 \times \omega \to \omega$ and find an α such that $g \mid \alpha \times \omega = g_\alpha$ and such that there exists a diagonalizing sequence $\{\alpha(n) : n < \omega\}$ in the sense of 2.9. Fix α such that $g \mid \alpha \times \omega = g_\alpha$ and such that $g_\alpha \in \mathcal{N}_\alpha \models \alpha = \omega_1$. Recall that in the construction at stage α we fixed $\mathcal{G} = \{F_i : i < \omega\}$ all the $F^{C,H,\alpha}$'s for good pairs (C, H) such that C was unbounded in α. Note that $\mathcal{G} \subseteq \mathcal{N}_\alpha$. We wish to construct a sequence diagonalizing $(g_\alpha, \mathcal{G})$.
Fix n_0 such that if $S_0 = \{\beta : g_\alpha(\beta, 0) \leq n_0\}$ then

$$\mathcal{N}_\alpha \models S_0 \textit{ is uncountable.}$$

Recursively choose $n_{i+1} \geq n_i$ such that if $S_{i+1} = \{\beta \in S_i : g_\alpha(\beta, i+1) \leq n_{i+1}\}$ then

$$\mathcal{N}_\alpha \models S_{i+1} \textit{ is uncountable.}$$

Having done this, for each i fix $\beta(i) \in S_i$ such that $T = \{\beta(i) : i < \omega\}$ is an increasing sequence ω-cofinal with α.

Note that if $\{\alpha(i) : i < \omega\}$ is any subsequence of T then $g_\alpha(\alpha(i), i) \leq g_\alpha(\beta(i), i)$.
Now we pick a subsequence of T that will diagonalize $(g_\alpha, \mathcal{G})$. First fix $T_0 \subseteq T$ such that for each i either $\exists m < \omega$, $T_0 \subseteq H_i^{-1}(m)$ or $\forall m < \omega$, $T_0 \cap H_i^{-1}(m)$ is finite (i.e., H_i is either almost constant or finite to one on T_0).
Next, choose for each $i < \omega$ $\alpha(i) \in T_0$ as follows. Having $\alpha(i)$ choose $\alpha(i+1) \in T_0$ above $\alpha(i)$ such that for all $j \leq i+1$ if H_j is finite to one on T_0 then $F_j(\alpha(i+1)) > n_{i+1}$. Clearly we can do this since inductive hypothesis (8) guarantees that if H_j is finite to one on T_0 then so is F_j. Clearly the sequence $\{\alpha(i) : i < \omega\}$ diagonalizes $(g_\alpha, \mathcal{G})$.

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