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A Hausdorff Countably Compact Space On Which Every Continuous Real–Valued Function Is Constant

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Abstract

We construct a Hausdorff countably compact space in which no pair of distinct points have disjoint closed neighbourhoods, and hence on this space every continuous real-valued function is constant.

Spaces on which every continuous real–valued function (or, more generally, spaces on which every continuous function into a given space R) is constant, are examined in [1]-[4], [6]-[8] and [11]-[15]. None of them is countably compact. All constructions of these spaces make use of an auxiliary space T containing two points a, b such that $f(a) = f(b)$, for every continuous real–valued function f of T (or, $f(a) = f(b)$, for every continuous function f of T into the given space R) and of a condensation process. The points a, b having this property are called by van Douwen [3], twins. While it is easy to construct a regular countably compact space containing two points not separated by a continuous real–valued function–(consider the countably compact non–normal space in [5, 8L] on which every continuous real–valued function is constant on a deleted neighbourhood of the corner point, and then apply the method of [9])– none of the condensation processes (nor the c–process [10]) yield a countably compact space.

We construct a Hausdorff countably compact space having the property that no pair of distinct points have disjoint closed neighbourhoods. Obviously from this property follows that on such a space every continuous real-valued function is constant. The construction is a modification of the van Douwen's construction in [3].

Proposition *There exists a Hausdorff countably compact space on which every continuous real–valued function is constant.*

Proof: Let $[0,\Omega]$ be the closed ordinal space and let $[0,\Omega]_i,\ i \in I,\ |I| = |[0,\Omega]|$ be disjoint copies of $[0,\Omega]$. To the topological sum $X = \bigcup_{i\in I}[0,\Omega]_i$ we add the open ordinal space $[0,\Omega)$ and we consider the set $Y = X \cup [0,\Omega)$. We define the bases of open neighbourhoods of the points of $[0,\Omega)$ in Y as follows: Let $V(x)$ be an open neighbourhood of x in $[0,\Omega)$ and let $V(x_i)$ be the copy of $V(x)$ in $[0,\Omega]_i$. Then a basis of open neighbourhoods of x in Y is the collection of sets $V(x) \cup (\bigcup C)$, where C is the set consisting of all but a finite number of $V(x_i)$.

It can be easily proved that the space Y is regular and that X is an open dense subspace of Y. Observe that since $[0,\Omega)$ is countably compact, every sequence frequently in $Y \setminus \{\Omega_i : i \in I\}$ has an accumulation point (either in X or in $[0,\Omega)$). Therefore the only not accumulating sequences in Y, are those whose all but finitely many of terms belong to $L = \{\Omega_i : i \in I\}$.

Since the set L and the set D of isolated points of X have the same cardinality there exists an one-to-one mapping g of L onto D. On the quotient space

$$Z = \{x, (\Omega_i, g(\Omega_i)) : x \in (X \setminus L \cup D) \cup [0,\Omega),\ i \in I\}$$

we define a weaker topology τ as follows: Let $U(\Omega)$ be an open neighbourhood of Ω in $[0,\Omega]$ and let $U(\Omega_i)$ be the copy of

$U(\Omega)$ in $[0,\Omega]_i$. Let also π be the natural projection of Y onto Z. For every $x_i \in X \setminus L \cup D$ a basis of open neighbourhoods is the collection of open sets $O(x_i)$ for which

$$\pi^{-1}(O(x_i)) \cap X = V(x_i) \cup \bigcup U(\Omega_k),$$

where k varies through all positive integers for which, for some finite sequence of positive integers $n_1, ..., n_m$, $n_m = k$, $g(\Omega_{n_1}) \in V(x_i)$, and for all $1 \leq j < m$, $g(\Omega_{n_{j+1}}) \in U(\Omega_{n_j})$. For every $x \in [0,\Omega)$ a basis of open neighbourhoods is the collection of open sets $O(x)$ for which

$$\pi^{-1}(O(x)) = V(x) \cup \bigcup C \cup \bigcup U(\Omega_k),$$

where k again varies through all positive integers for which, for some $n_1, ..., n_m$, $n_m = k$, $g(\Omega_{n_1}) \in \bigcup C$, and for all $1 \leq j < m$, $g(\Omega_{n_{j+1}}) \in U(\Omega_{n_j})$. For every point $(\Omega_i, g(\Omega_i))$ a basis of open neighbourhoods is the collection of open sets $O\Big((\Omega_i, g(\Omega_i))\Big)$ for which

$$\pi^{-1}\Big(O\Big((\Omega_i, g(\Omega_i))\Big)\Big) \cap X = \{g(\Omega_i)\} \cup \bigcup U(\Omega_k),$$

where now for k and $n_1, ..., n_m$, $n_m = k$, $g(\Omega_{n_1}) \in U(\Omega_k)$, and for all $1 \leq j < m$, $g(\Omega_{n_{j+1}}) \in U(\Omega_{n_j})$.

We observe that the open sets in τ are just those open sets of the quotient topology each of whose inverse images under π is a (saturated) open set for which $i, V(x_i)$, and $U(\Omega)$ are fixed and k varies through the smallest set so needed. Obviously (Z,τ) is countably compact.

We prove that (Z,τ) is Hausdorff. Let $x_i \in X \setminus L \cup D$ and $B = (\Omega_i, g(\Omega_i))$. There exist open neighbourhoods $V(x_i)$, $U(\Omega_i)$ in Y of x_i and Ω_i, respectively, such that $V(x_i) \cap U(\Omega_i) = \emptyset$ and $g(\Omega_i) \notin V(x_i)$. Hence the sets

$$E_1\Big(V(x_i)\Big) = V(x_i) \cup \{\Omega_k : g(\Omega_k) \in V(x_i)\}$$

and

$$E_1\Big(U(B)\Big) = \{g(\Omega_i)\} \cup U(\Omega_i) \cup \{\Omega_l : g(\Omega_l) \in U(\Omega_i\}$$

are disjoint. Consequently, for appropriate open neighbourhoods $U(\Omega_k)$, $V(\Omega_l)$ in Y of the points Ω_k and Ω_l, respectively the sets

$$E_{n+1}\Big(V(x_i)\Big) = E_n\Big(V(x_i)\Big) \cup \bigcup_{g(\Omega_k)\in E_n(V(x_i))} U(\Omega_k)$$

and

$$E_{n+1}\Big(U(B)\Big) = E_n\Big(U(B)\Big) \cup \bigcup_{g(\Omega_l)\in E_n(U(B))} U(\Omega_l)$$

for n=1,2,..., are disjoint. Therefore the sets

$$E\Big(V(x_i)\Big) = \bigcup_{n=1}^{\infty} E_n\Big(V(x_i)\Big)$$

and

$$E\Big(U(B)\Big) = \bigcup_{n=1}^{\infty} E_n\Big(U(B)\Big)$$

are disjoint open in Y, and such that the sets $\pi\Big(E\Big(V(x_i)\Big)\Big)$, $\pi\Big(E\Big(U(B)\Big)\Big)$ are disjoint open in (Z,τ).

Similarly are proved and the other cases. Therefore (Z,τ) is Hausdorff.

That every continuous real-valued function of (Z,τ) is constant is obvious since by the definition of topology, no pair of distinct points of Z have disjoint closed neighbourhoods.

We observe that the space (Z,τ) is neither regular nor first countable nor separable.

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