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ON A GENERALIZATION OF TOTALLY BOUNDED AND COMPACT METRIC SPACES

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Abstract

We study two suitable generalizations of the notions of totally bounded and compact metric spaces, to establish comparisons and similarities with the classical case.

1 Introduction

The notions of GTB and GK space have been first introduced in [3], and used to calculate the density of the hyperspace of a

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metric space, endowed with the Hausdorff or the locally finite hypertopology. The notion of GK space is also implicit in [6], when considering metric spaces whose extent is not achieved (see, in particular, section 2).

According to an elementary topological argument, the total boundedness of an (infinite) metric space (X, d) implies a strict restriction on its density, namely $\mathbf{d}(X) = \aleph_0$. Hence, we can say that (X, d) is totally bounded if and only if $\mathbf{d}(X) = \aleph_0$ and for every $\varepsilon > 0$ there exists an ε-dense subset of X having cardinality less than $\mathbf{d}(X)$.

Now, it is natural to wonder what happens in the preceding definition if we exclude the condition: $\mathbf{d}(X) = \aleph_0$. This leads exactly to the notion of **GTB** space, which is one of the main subjects of this paper. A parallel generalization deals with compacteness, and gives rise to the **GK** metrizable spaces: X is said to be GK if every open cover of X has a subcover of cardinality less than $\mathbf{d}(X)$ (see §3).

In the following, we will investigate these two concepts and their mutual relationships, with special regards to possible extensions of results concerning the corresponding classical notions. The exposition is completed by several examples.

We note as a peculiarity the frequent use of generalized sequences indexed by singular cardinals (in fact, cardinals of cofinality $\aleph_0$); this is not common in similar frameworks, such as radial and pseudo-radial spaces, or ω_μ-metrizable spaces, where only regular cardinals are involved.

In the following, the symbol $|A|$ will denote the cardinality of the set A, while cof (ν) will be the cofinality of the cardinal ν. For other undefined symbols and notions the reader is referred to [5].

2 Generalized total boundedness

Definition: A metric space (X, d) is said **totally bounded in the generalized sense** or simply **GTB** iff for every $\varepsilon > 0$ there exists an ε-dense subset N of X with $|N| < \mathbf{d}(X)$.

A totally bounded metric space is a GTB space, as it is known that such a space is separable (i.e. has density $\aleph_0$).

An equivalent definition of generalized total boundedness is given by the following theorem. In practice, it says that uniformly discrete subsets of a GTB space cannot achieve the highest cardinality (we recall that a subset D of a metric space (X, d) is said *uniformly discrete* if there exists $\varepsilon > 0$ such that $d(x, y) \geq \varepsilon$ for distinct $x, y \in D$).

Theorem 1 *A metric space X is GTB iff every uniformly discrete subset of X has cardinality less than $\mathbf{d}(X)$.*

Proof: See [3, Theorem 4]. □

The fact that a metric space (X, d) is GTB implies a condition on the density of X. On the other hand, every cardinal number satisfying such a condition is the density of a suitable GTB metric space.

Theorem 2 *If (X, d) is a GTB metric space, then $\mathrm{cof}(\mathbf{d}(X)) = \aleph_0$. For every cardinal ξ such that $\mathrm{cof}\,(\xi) = \aleph_0$ there exists a GTB metric space (X, d) such that $\mathbf{d}(X) = \xi$.*

Proof: See [3, Theorem 5 and Example 6]. □

We demonstrate here a natural technique for constructing GTB metric spaces of density greater than $\aleph_0$.

Theorem 3 *If $(\nu_n)_{n\in\mathbb{N}}$ is a sequence of infinite cardinals such that $\nu = \sup_{n\in\mathbb{N}} \nu_n$ is greater than each of the ν_n, and for every $n \in \mathbb{N}$, (X_n, d_n) is a metric space of density ν_n, with $d_n \leq 1$, then, fixing any sequence $(r_n)_{n\in\mathbb{N}}$ of positive real numbers such that $\lim_{n\in\mathbb{N}} r_n = 0$, the metric space (X, d), where $X = \prod_{n\in\mathbb{N}} X_n$ and*

$$d\left((a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}}\right) = \sup_{n\in\mathbb{N}} r_n \cdot d_n(a_n, b_n),$$

is a GTB space of density ν.

Proof: Clearly, d is in fact a compatible metric on $X = \prod_{n\in\mathbb{N}} X_n$. Now, let us prove that, given any $\varepsilon > 0$, there exists an ε-dense subset D of X having cardinality less than ν.

Indeed, take $n \in \mathbb{N}$ such that $r_i \leq \frac{\varepsilon}{2}$ for $i > n$. For every $i \in \mathbb{N}$ with $i \leq n$, take a dense subset D_i of X_i with $|D_i| = \nu_i$, while, for $i > n$, let $D_i = \{x_i\}$ be a fixed one-element subset of X_i (which is nonempty by hypothesis). Then, putting $D = \prod_{i\in\mathbb{N}} D_i$, we have that $|D| = \nu_1 \cdot \ldots \cdot \nu_n = \max\{\nu_1, \ldots, \nu_n\} < \nu$; moreover, if $(a_i)_{i\in\mathbb{N}}$ is an arbitrary point of X, then selecting for every $i \leq n$ an $x_i \in D_i$ with $d_n(a_i, x_i) < \frac{\varepsilon}{r_i}$, it is easily seen that $d\left((x_i)_{i\in\mathbb{N}}, (a_i)_{i\in\mathbb{N}}\right) = \sup_{i\in\mathbb{N}} r_i \cdot d_i(x_i, a_i) < \varepsilon$.

Thus, all we have to show is that $\mathbf{d}(X) = \nu$. On the one hand, it is clear that $\mathbf{d}(X) \geq \nu$. On the other hand, as each factor in the definition of X has density less than (or equal to) ν, and trivially $2^\nu \geq \aleph_0$, we have that $\mathbf{d}(X) \leq \nu$ by the Hewitt-Marczewski-Pondiczery theorem (see [5, Theorem 2.3.15]). ▫

Corollary 4 *If the cardinals ν_n and ν are as in the preceding theorem and, for every $n \in \mathbb{N}$, X_n is a metrizable space of density ν_n, then the space $X = \prod_{n\in\mathbb{N}} X_n$ has density ν and can be endowed with a compatible GTB metric.*

Several properties of classical totally bounded spaces can be carried out in a suitable form to GTB spaces. In particular, let us observe that if M is a subspace of a GTB space (X, d), such that $\mathbf{d}(M) = \mathbf{d}(X)$, then M also is a GTB space.

A very natural question in this vein is to establish if the first statement of Theorem 2 can be inverted, in which case the preceding corollary would be obtained as a particular consequence; moreover, such a result would generalize the fact that a metrizable space is separable if and only if it admits a totally bounded compatible metric.

The answer is positive, as we are going to show below. The proof of this theorem is due to J. Pelant, to which the authors are very grateful. He improved on a preceding proof in a preliminary version of this paper, which needed the Generalized Continuum Hypothesis.

Theorem 5 *If (X, d) is a metric space with* $\mathrm{cof}(\mathbf{d}(X)) = \aleph_0$, *then there exists a metric ρ equivalent to d such that (X, ρ) is a GTB space.*

Proof: The case $\mathbf{d}(X) = \aleph_0$ is well-known; thus, we can suppose $\nu > \aleph_0$.

Let $\mathcal{A} = \bigcup_{n \in \mathbb{N}} \mathcal{A}_n$ be a σ-uniformly discrete base for X; that is, for every $n \in \mathbb{N}$ there exists $\varepsilon_n > 0$ such that

$$\forall A, B \in \mathcal{A}_n \colon (A \neq B \Longrightarrow D_d(A, B) \geq 2\varepsilon_n),$$

where $D_d(A, B) = \inf \{d(x, y) \mid x \in A, y \in B\}$. Observe that each $\mathcal{A}_n$, as a collection of pairwise disjoint open subsets of X has cardinality $\leq \nu$. Furthemore, splitting up some of the collections $\mathcal{A}_n$ into countably many pairwise disjoint subcollections, each with cardinality less than ν, we can suppose $|\mathcal{A}_n| = \nu_n < \nu$ for every $n \in \mathbb{N}$.

Given $n \in \mathbb{N}$, let $H_{\nu_n} = \left(\bigcup_{\alpha \in \nu_n} ([0, 1[\times \{\alpha\})\right) \cup \{\underline{1}\}$ be the *hedgehog* of spininess ν_n, which can be obtained, as it is well-known, by adding to the disjoint union $\bigcup_{\alpha \in \nu_n} ([0, 1[\times \{\alpha\})$ of ν_n many copies of the segment $[0, 1[$, the point $\underline{1}$ having as a fundamental system of neighbourhoods the family $\{V_m \mid m \in \mathbb{N}\}$, where

$$V_m = \{\underline{1}\} \cup \left(\bigcup_{\alpha \in \nu_n} \left(\left] 1 - \frac{1}{m}, 1 \right[\times \{\alpha\} \right) \right)$$

for every $m \in \mathbb{N}$ (in the literature, it is more frequent to use copies of the segment $]0,1]$ and the added point $\underline{0}$). Each H_{ν_n} is a metrizable space of density ν_n, and hence by Corollary 4 the product space $H = \prod_{n \in \mathbb{N}} H_{\nu_n}$ has density ν and can be endowed with a compatible GTB metric h.

For every $n \in \mathbb{N}$, write $\mathcal{A}_n = \{A_{n,\alpha} \mid \alpha \in \nu_n\}$, where $\alpha \mapsto A_{n,\alpha}$ is injective, and define $\varphi_n : X \to H_{\nu_n}$ by $\varphi_n(x) = \left(\frac{d(x, A_{n,\alpha})}{\varepsilon_n}, \alpha \right)$ if $d(x, A_{n,\alpha}) < \varepsilon_n$ for a (necessarily unique) $\alpha \in \nu_n$, and $\varphi_n(x) = \underline{1}$ otherwise. It is easily checked that each φ_n is continuous; if we can prove that the family $\{\varphi_n \mid n \in \mathbb{N}\}$ separates points and closed sets, then the map $\varphi = \Delta_{n \in \mathbb{N}} \varphi_n$ would be a topological embedding of X into H, and hence $\rho(x,y) = h(\varphi(x), \varphi(y))$ would define a compatible GTB metric on X.

Now, let C be a closed set in X and $x \in X \setminus C$. Put $r = d(x, C) > 0$: as $\mathcal{A}$ is a base for X, there exist $n \in \mathbb{N}$ and $\alpha \in \nu_n$ such that $x \in A_{n,\alpha} \subseteq S_d\left(x, \frac{r}{2}\right)$. This implies, by the triangular inequality, that $d(y,z) \geq \frac{r}{2}$ for every $y \in A_{n,\alpha}$ and $z \in C$; in particular, $\varphi_n(C) \cap ([0, s[\times \{\alpha\}) = \emptyset$, where $s = \min\left\{ \frac{r}{2\varepsilon_n}, 1 \right\}$. As $[0, s[\times \{\alpha\}$ is open in H_{ν_n}, we also have that $\mathrm{Cl}(\varphi_n(C)) \cap ([0, s[\times \{\alpha\}) = \emptyset$, and hence $\varphi_n(x) = (0, \alpha) \notin \mathrm{Cl}(\varphi_n(C))$. □

3 Generalized compactness

The notion of GTB space leads us in a natural way to introduce a generalization of compact spaces. We will say that a topological space X is **compact in the generalized sense** (briefly,

GK) if for every open cover $\mathcal{U}$ of X, there exists a subcover $\mathcal{V}$ such that $|\mathcal{V}| < \mathbf{d}(X)$. We will consider only metrizable GK spaces.

The two next characterizations generalize the well-known results that a metrizable space is compact iff it has no closed and discrete subset of cardinality $\aleph_0$ and iff every compatible metric on it is totally bounded.

Theorem 6 *A metrizable space X is GK if and only if it has no closed and discrete subset of cardinality equal to* $\mathbf{d}(X)$.

Proof: See [3, Theorem 7]. □

Theorem 7 *A metrizable space X is GK if and only if every compatible metric on X is GTB.*

Proof: See [3, Theorem 8]. □

Another characterization of generalised compactness is possible by using a suitable cardinal function.

For every topological space X, let

$$\mathrm{L}(X) = \min\{\xi \text{ cardinal} \mid \forall \text{ open cover } \mathcal{U} \text{ of } X : \\ \exists \text{ cover } \mathcal{V} \subseteq \mathcal{U} : |\mathcal{V}| \leq \xi\}$$

and

$$\mathrm{L}'(X) = \min\{\xi \text{ cardinal} \mid \forall \text{ open cover } \mathcal{U} \text{ of } X : \\ \exists \text{ cover } \mathcal{V} \subseteq \mathcal{U} : |\mathcal{V}| < \xi\}.$$

Then $\mathrm{L}(X)$ is the well-known Lindelöf number, and the preceding definitions clearly show that

(∗) $\mathrm{L}(X) \leq \mathrm{L}'(X) \leq \mathrm{L}(X)^{+}$.

Theorem 8 *For every metrizable space X the following conditions are equivalent:*
1) X is GK;
2) $\mathrm{L}'(X) \leq \mathbf{d}(X)$*;*
3) $\mathrm{L}'(X) = \mathbf{d}(X)$*;*
4) $\mathrm{cof}\,(\mathrm{L}'(X)) = \aleph_0$*.*

Proof: Equivalence between 1) and 2) follows immediately from the definitions, whereas 2)$\Longleftrightarrow$3) is a consequence of $(*)$ and of the equality $\mathbf{d}(X) = \mathrm{L}\,(X)$ for metric spaces [7, Theorem 8.1.(c)].

If X is GK, then $\mathrm{cof}(\mathbf{d}(X)) = \aleph_0$ (by Theorems 2 and 7), and $\mathbf{d}(X) = \mathrm{L}'(X)$ by 3): thus 4) holds. Conversely, if $\mathrm{cof}\,(\mathrm{L}'(X)) = \aleph_0$, then $\mathrm{L}'(X) \neq \mathrm{L}\,(X)^+$ (as successor cardinals are regular): hence by $(*)$ we have that $\mathrm{L}\,(X) = \mathrm{L}'(X)$, and 3) holds. □

Analogously to what happens for GTB metric spaces, it is possible to prove that for every cardinal ν with $\mathrm{cof}\,(\nu) = \aleph_0$ there exists a GK metrizable space X with $\mathbf{d}(X) = \nu$.

Example 9 *Let ν be any cardinal number with* $\mathrm{cof}\,(\nu) = \aleph_0$ *and let $(\nu_n)_{n\in\mathbb{N}}$ be a strictly increasing sequence of cardinals with $\sup_{n\in\mathbb{N}} \nu_n = \nu$. Consider a set $\tilde{X} = \left(\bigcup_{n\in\mathbb{N}} X_n\right) \cup \{\infty\}$, where each X_n has cardinality ν_n, the sets X_n are pairwise disjoint and the point ∞ does not belong to any of them. Endow X with a topology τ by assuming that every point of $\bigcup_{n\in\mathbb{N}} X_n$ is discrete, and giving ∞ the fundamental system of neighbourhoods $\{V_n \mid n \in \mathbb{N}\}$, where $V_n = \left(\bigcup_{n'\geq n} X_{n'}\right) \cup \{\infty\}$ for every $n \in \mathbb{N}$. Then $\tilde{X}$ is a GK metrizable space.*

Proof: See [3, Example 9]. □

In the following, we will use the above defined space $\tilde{X}$ for further applications, and we will also consider the compatible metric $\tilde{d}$ on it, defined by:

$$\tilde{d}(x,y) = \begin{cases} 0 & \text{if } x = y \\ \frac{1}{\min\{m,n\}} & \text{if } x \in X_m,\, y \in X_n \text{ and } x \neq y \\ \frac{1}{n} & \text{if } x \in X_n \text{ and } y = \infty, \text{ or vice-versa} \end{cases}$$

Note also that by a slight modification, the above example can be chosen to be connected (see also [5, Exc 4.1.H(b)]).

Observe that, in the space X of the example, there is only one point where the local density equals the global density of X (the local density of a point is the minimum of the densities of its neighbourhoods). This is a special case of a general result which states that in a GK metrizable space, the set of points where the local density equals the global density is always compact and nonempty (see [3, Lemmas 10 and 11] or [6, Lemma 1]).

Now, let us deal with relationships between generalized compactness and sequences. If X is a metrizable space with $\mathbf{d}(X) = \xi$, is it true that X is GK if and only if every ξ-sequence in X possesses a convergent ξ-subsequence?

First of all, we say that $\left(a_{\alpha_\beta}\right)_{\beta\in\xi}$ is a ξ-subsequence of $(a_\alpha)_{\alpha\in\xi}$ if $\beta \mapsto \alpha_\beta$ is a nondecreasing function from ξ to ξ and the set $\{\alpha_\beta \mid \beta \in \xi\}$ is cofinal to ξ. Such a definition seems in accordance with the general notion of subnet of a net [5, §1.6], and in the case $\xi = \aleph_0$ it works the same way of the classical one (although it is less restrictive).

Then the above question has a negative answer for $\xi > \aleph_0$; as a matter of fact, it results too strong to impose the condition of the convergent subsequence on all the ξ-sequences of X. Consider, for instance, the space $\tilde{X}$ defined in Example 9: let $\{a_n \mid n \in \mathbb{N}\}$ be a countable subset of X_1, where $n \mapsto a_n$ is one-to-one. Define $(x_\alpha)_{\alpha\in\nu}$ by putting $x_\alpha = a_n$ for $\alpha \in A_n = \{\alpha \in \nu \mid \nu_{n-1} \leq \alpha < \nu_n\}$. It is apparent that no ν-subsequence of $(x_\alpha)_{\alpha\in\nu}$ can converge to a point of $\tilde{X}$.

Thus, to avoid degenerations, we must prevent the ν-

sequences in question from becoming, in fact, ζ-sequences with $\zeta < \nu$. To this end, we will only consider injective ν-sequences. We have a result whose proof uses the Generalized Continuum Hypothesis.

Theorem 10 (ZFC+GCH) *Let (X, d) be a GK metric space with $\mathbf{d}(X) = \nu$. Then every injective ν-sequence possesses a convergent ν-subsequence.*

Proof: The case $\nu = \aleph_0$ is well-known. For $\nu > \aleph_0$, put, as usual, $\nu = \sup_{n\in\omega} \nu_n$, where $n \mapsto \nu_n$ is strictly increasing and $\nu_0 = 0$ (this is possible because $\mathrm{cof}(\nu) = \aleph_0$). Also, we can choose the cardinals ν_n in such a way that there exist corresponding *regular* cardinals $\mu_n > \aleph_0$ with:

$$\mu_n^+ = \nu_n < \mu_{n+1} < \mu_{n+1}^+ = \nu_{n+1}$$

for every $n \in \mathbb{N}$ (in particular, each ν_n is in turn a successor cardinal).

Suppose that there exists an injective ν-sequence $(x_\alpha)_{\alpha\in\nu}$ in X having no convergent ν-subsequence. For every $n \in \mathbb{N}$, let $S_n = \{x_\alpha \mid \alpha < \nu_n\}$. Then it is possible to find for every $n \in \mathbb{N}$, $n \geq 2$, a subset $\tilde{A}_n$ of $A_n = \{\alpha \in \nu \mid \nu_{n-1} \leq \alpha < \nu_n\}$ such that

$$|\tilde{A}_n| = \nu_n \tag{1}$$

and

$$\left(\mathrm{Cl}\left\{x_\alpha \mid \alpha \in \tilde{A}_n\right\}\right) \cap \mathrm{Cl}\,(S_{n-1}) = \emptyset \tag{2}$$

(the definition of the sets $\tilde{A}_n$ will be completed by putting $\tilde{A}_1 = A_1 = \nu_1$).

Indeed, let $\tilde{n} \in \mathbb{N}$ with $\tilde{n} \geq 2$. Then the relation $|S_{\tilde{n}-1}| = \nu_{\tilde{n}-1}$ implies that $|\mathrm{Cl}\,(S_{\tilde{n}-1})| \leq {\nu_{\tilde{n}-1}}^{\aleph_0} \leq {\nu_{\tilde{n}-1}}^{\nu_{\tilde{n}-1}} = 2^{\nu_{\tilde{n}-1}} = \nu_{\tilde{n}-1}^+ < \mu_{\tilde{n}}^+ = \nu_{\tilde{n}}$ (by GCH and the initial assumptions on the cardinals ν_n): hence, $|\{\alpha \in A_{\tilde{n}} \mid x_\alpha \notin \mathrm{Cl}\,(S_{\tilde{n}-1})\}| = \nu_{\tilde{n}}$. By perfect normality of X, there exists a sequence $(W_m)_{m\in\mathbb{N}}$ of open subsets of X such that $\bigcap_{m\in\mathbb{N}} W_m = \mathrm{Cl}\,(S_{\tilde{n}-1})$; since

$$|\bigcup_{m\in\mathbb{N}} \{\alpha \in A_{\tilde{n}} \,|\, x_\alpha \notin W_m\}| = |\{\alpha \in A_{\tilde{n}} \,|\, x_\alpha \notin \mathrm{Cl}\,(S_{\tilde{n}-1})\}| = \nu_{\tilde{n}}$$

and $\nu_{\tilde{n}}$ is regular, there exists $\tilde{m} \in \mathbb{N}$ such that

$$|\{\alpha \in A_{\tilde{n}} \,|\, x_\alpha \notin W_{\tilde{m}}\}| = \nu_{\tilde{n}}.$$

Putting $\tilde{A}_{\tilde{n}} = \{\alpha \in A_{\tilde{n}} \,|\, x_\alpha \notin W_{\tilde{m}}\}$, we have that $\mathrm{Cl}\Big\{x_\alpha \,|\, \alpha \in \tilde{A}_{\tilde{n}}\Big\} \subseteq X \setminus W_{\tilde{m}}$, and hence $\mathrm{Cl}\Big\{x_\alpha \,|\, \alpha \in \tilde{A}_{\tilde{n}}\Big\} \cap \mathrm{Cl}\,(S_{\tilde{n}-1}) = \emptyset$.

Now let us define, for every $n \in \mathbb{N}$:

$$C_n = \mathrm{Cl}\Big\{x_\alpha \,|\, \alpha \in \tilde{A}_n\Big\};$$

then we have by (2) and the definition of the S_n that, for $n' < n''$: $C_{n'} \cap C_{n''} \subseteq \mathrm{Cl}\,(S_{n'}) \cap C_{n''} \subseteq \mathrm{Cl}\,(S_{n''-1}) \cap C_{n''} = \emptyset$. Thus the sets C_n are pairwise disjoint. We claim that the family $\{C_n \,|\, n \in \mathbb{N}\}$ is locally finite.

By contradiction, suppose $\bar{x} \in X$ be such that every neighbourhood of $\bar{x}$ intersects infinitely many sets C_n. Let $\{U_n \,|\, n \in \mathbb{N}\}$ be a fundamental system of open neighbourhoods for $\bar{x}$ with $U_1 \supseteq U_2 \supseteq U_3 \supseteq \ldots$: we will construct an increasing sequence $(\alpha'_n)_{n\in\omega}$ of ordinals in ν such that $\{\alpha'_n \,|\, n \in \omega\}$ is cofinal to ν. Put $\alpha'_0 = 0$; if α'_n is defined, and $m_n \in \mathbb{N}$ is such that $\alpha'_n \in A_{m_n}$, then, as the set U_n is an open neighbourhood of $\bar{x}$, there exists $m_{n+1} > m_n$ such that $U_n \cap C_{m_{n+1}} \neq \emptyset$; U_n open implies further that there exists $\alpha'_{n+1} \in \tilde{A}_{m_{n+1}} \subseteq A_{m_{n+1}}$ such that

$$x_{\alpha'_{n+1}} \in U_n. \tag{3}$$

Thus it is clear that $\alpha'_{n+1} > \alpha'_n$ (as $\alpha'_{n+1} \in A_{m_{n+1}}$ and $\alpha'_n \in A_{m_n}$) and that the set $\{\alpha'_n \,|\, n \in \omega\}$ is cofinal to ν; also, the decreasing character of the neighbourhoods U_n implies by (3) that $\lim_{n\to\infty} x_{\alpha'_n} = \bar{x}$. Putting, for every $\beta \in \nu$, $\alpha_\beta = \alpha'_{n_\beta}$ — where $n_\beta = \min\{n \in \omega \mid \alpha'_n \geq \beta\}$ —, we have that $\beta \mapsto \alpha_\beta$ is non-decreasing, the set $\{\alpha_\beta \mid \beta \in \nu\} = \{\alpha'_n \mid n \in \omega\}$ is cofinal to ν and $\left(x_{\alpha_\beta}\right)_{\beta\in\nu}$ converges to $\bar{x}$, which is impossible. Thus the family $\{C_n \mid n \in \mathbb{N}\}$ is locally finite.

For every $n \in \mathbb{N}$, as $\left\{x_\alpha \mid \alpha \in \tilde{A}_n\right\} \subseteq C_n$ and $|\tilde{A}_n| = \nu_n$ by (1), we have that $\mu_n^+ = \nu_n \leq |C_n| \leq \mathbf{d}(C_n)^{\aleph_0} \leq 2^{\mathbf{d}(C_n)} = \mathbf{d}(C_n)^+$ and hence $\mu_n \leq \mathbf{d}(C_n)$; on the other hand, $\mathbf{d}(C_n) \leq |\left\{x_\alpha \mid \alpha \in \tilde{\mathbf{A}}_n\right\}| = \nu_n = \mu_n^+$; this implies that $\mathbf{d}(C_n)$ is in any case a regular cardinal greater than $\aleph_0$ and hence C_n is not GK. Thus there exists a closed and discrete subset D_n of C_n with $|D_n| = \mathbf{d}(C_n)$; putting $D = \cup_{n\in\mathbb{N}} D_n$, we have that $|D| = \nu$ and, since the family $\{D_n \mid n \in \mathbb{N}\}$ is locally finite, D is closed and discrete in X. Clearly, this contradicts the fact that X is GK. □

The above theorem cannot be proved in **ZFC**, as the following example shows. Observe that the assumption $2^{\aleph_0} = \aleph_{\omega+1}$ is compatible with **ZFC**, as $\operatorname{cof}(\aleph_{\omega+1}) = \aleph_{\omega+1} > \aleph_0$ (see [4, Theorem 1]).

Example 11 (ZFC + ($2^{\aleph_0} = \aleph_{\omega+1}$)) Let $\mathbb{R}$ be the real line with the euclidean topology, and $\tilde{X}$ be the GK space defined in Example 9, with $\nu_n = \aleph_n$ for every $n \in \mathbb{N}$ (hence $\nu = \aleph_\omega$). Put $Y = \mathbb{R} \oplus \tilde{X}$, the topological disjoint sum of $\mathbb{R}$ and $\tilde{X}$. Clearly, $\mathbf{d}(Y) = \aleph_\omega$ and it is easily proved that Y is GK. Indeed, given any open cover $\mathcal{U}$ of Y, there exist $\mathcal{U}_1, \mathcal{U}_2 \subseteq \mathcal{U}$ with $|\mathcal{U}_1| \leq \aleph_0$, $\bigcup \mathcal{U}_1 \supseteq \mathbb{R}$, $|\mathcal{U}_2| < \aleph_\omega$ and $\bigcup \mathcal{U}_2 \supseteq \tilde{X}$; thus $\mathcal{U}_1 \cup \mathcal{U}_2$ is a subcover of Y of cardinality less than $\aleph_\omega$.

On the other hand, for every $n \in \mathbb{N}$ the subset $]n-1, n[$ of $\mathbb{R}$ has cardinality $2^{\aleph_0} = \aleph_{\omega+1} > \aleph_n$, and hence it is possible to find an injective function $y_n \colon A_n \to]n-1, n[$ (as in this case $|A_n| = \aleph_n$). Define $(a_\alpha)_{\alpha\in\aleph_\omega}$ by $a_\alpha = y_{n_\alpha}(\alpha)$, where n_α is the unique element of $\mathbb{N}$ such that $\alpha \in A_{n_\alpha}$. Then the $\aleph_\omega$-sequence $(a_\alpha)_{\alpha\in\aleph_\omega}$ is injective and has no convergent subsequence (for otherwise it would be possible to take a point x_n in $]n-1,n[$ for each $n \in \mathbb{N}$, in such a way that the sequence $(x_n)_{n\in\mathbb{N}}$ have an accumulation point in $\mathbb{R}$). ▫

On the contrary, it is easily seen that the converse of Theorem 10 is true in **ZFC** without need of additional assumptions.

Theorem 12 *Let X be a metrizable space with $\mathbf{d}(X) = \xi$, and suppose that every injective ξ-sequence in X has a convergent ξ-subsequence. Then the space X is GK.*

Proof: Suppose X is not GK: then there exists a closed and discrete subset $\{x_\alpha \mid \alpha \in \xi\}$ of X, where $\alpha \mapsto x_\alpha$ is one-to-one. Then the injective ξ-sequence $(x_\alpha)_{\alpha\in\xi}$ has no convergent ξ-subsequence. ▫

The attempt of characterizing GK spaces by means of generalized convergent sequences leads us to deal with another classical result for compact metric spaces, that is a metric space is compact if and only if it is complete and totally bounded.

Let (X,d) be a metric space and let ξ be an arbitrary cardinal (in particular, we are interested to the case where $\xi = \mathbf{d}(X)$). We will say that a ξ-sequence $(x_\alpha)_{\alpha\in\xi} \subseteq X$ is Cauchy if

$$\forall\varepsilon > 0\colon \exists\alpha \in \xi\colon \forall\alpha', \alpha'' \geq \alpha\colon d\left(x_{\alpha'}, x_{\alpha''}\right) \leq \varepsilon.$$

A metric space (X,d) with $\mathbf{d}(X) = \xi$ is **GC** (**complete in the generalized sense**) if every injective Cauchy ξ-sequence in X is convergent. It is not hard to prove that every metric space which is complete in the classical sense, is GC.

Note that for every cardinal number ν with $\operatorname{cof}(\nu) = \aleph_0$, the space $\left(\tilde{X}, \tilde{d}\right)$ of Example 9 is GC.

Compatible with **ZFC** is the existence of some GK metric space which is not GC. The construction is very similar to Example 11.

Example 13 (ZFC + $(2^{\aleph_0} = \aleph_{\omega+1})$) Consider the segment $]0,1[$ of the real line, and put $Y =]0,1[\oplus \tilde{X}$, where $\tilde{X}$ is the space defined in Example 9. The metric ρ on Y defined by

$$\rho(x,y) = \begin{cases} |x-y| & \text{if } x,y \in]0,1[\\ \tilde{d}(x,y) & \text{if } x,y \in \tilde{X} \\ 1 & \text{if } x \in]0,1[\text{ and } y \in \tilde{X} \text{ or vice-versa} \end{cases}$$

is compatible with the topology of the disjoint sum. The space Y is GK for the same reasons explained in Example 11.

Nevertheless, (Y,ρ) is not GC. To see this, let us construct an injective ν-sequence $(a_\alpha)_{\alpha\in\nu}$ such that, for every $n \in \mathbb{N}$: $\{a_\alpha \mid \alpha \in A_n\} \subseteq \left]\frac{1}{n+1}, \frac{1}{n}\right[$; then $(a_\alpha)_{\alpha\in\nu}$ is Cauchy, but it doesn't converge to any point of Y. □

On the other hand, it is easy to show in **ZFC+GCH** that every GK metric space is GC. Indeed, every injective Cauchy ν-sequence must have a convergent ν-subsequence by Theorem 10; and it is trivial to verify that, as for classical sequences, if the Cauchy ν-sequence $(a_\alpha)_{\alpha\in\nu}$ has a ν-subsequence which converges to a point a of X, then $(a_\alpha)_{\alpha\in\nu}$ itself converges to a.

The following example, which can be obtained in **ZFC**, shows that in general GC+GTB$\not\Rightarrow$GK, and hence that the corresponding classical result cannot be generalized even by means of additional set-theoretic assumptions.

Example 14 Let $\left(\tilde{X}, \tilde{d}\right)$ be the space defined in Example 9, and put $Y = \tilde{X} \times \mathbb{N}$; define a compatible metric ρ on Y by:

$$\rho\Big((x,m'),(y,m'')\Big) = \begin{cases} \tilde{d}(x,y) & \text{if } m' = m'' \\ 1 & \text{if } m' \neq m'' \end{cases}$$

Clearly, $\mathbf{d}(Y) = \left(\operatorname{dens}(\tilde{X})\right) \cdot \aleph_0 = \nu$. For every $m \in \mathbb{N}$, it is possible to find an open partition $\mathcal{U}_m$ of $\tilde{X} \times \{m\}$ with $|\mathcal{U}_m| = \nu_m$. As $\tilde{X} \times \{m\}$ is open in Y, the collection $\mathcal{U}_m$ is still open in Y. Put $\mathcal{U} = \bigcup_{m \in \mathbb{N}} \mathcal{U}_m$: then $\mathcal{U}$ is an open partition of Y and $|\mathcal{U}| = \nu$. Thus Y is not GK.

The fact that $\left(\tilde{X}, \tilde{d}\right)$ is GTB and GC is quite clear. □

For the classical notions, we know another fundamental relationship between compactness and completeness, namely a metrizable space is compact if and only if every compatible metric on it is complete. We have already seen that the GK property for a metrizable space X implies that every compatible metric on X is GC, only under **GCH**. What about the reverse implication?

It turns out that the property in question holds, if we make sure in advance that the cofinality of $\mathbf{d}(X)$ is $\aleph_0$. The reason for this is clear from the following easy result.

Theorem 15 *If (X, d) is a metric space with $\operatorname{cof}(\mathbf{d}(X)) > \aleph_0$, then (X, d) is GC.*

Proof: Let $\zeta = \mathbf{d}(X)$: if we can show that in (X, d) there is no injective Cauchy ζ-sequence, then the result will trivially follow. Suppose $(a_\alpha)_{\alpha \in \zeta}$ to be a Cauchy ζ-sequence in X: then for every $n \in \mathbb{N}$ there exists an $\alpha_n \in \zeta$ such that

$$\forall \alpha', \alpha'' \geq \alpha_n \colon d\left(a_{\alpha'}, a_{\alpha''}\right) \leq \frac{1}{n}.$$

Let $\hat{\alpha} = \sup_{n \in \mathbb{N}} \alpha_n$: then $\hat{\alpha} \in \zeta$ as $\operatorname{cof}(\zeta) > \aleph_0$. Thus

$$\forall \alpha', \alpha'' \geq \hat{\alpha} \colon d\left(a_{\alpha'}, a_{\alpha''}\right) = 0,$$

that is $(a_\alpha)_{\alpha \in \zeta}$ is eventually constant. In particular, it is not injective. □

Therefore, we have the desired result in the following form.

Theorem 16 *If X is a metrizable space with* $\mathrm{cof}(\mathbf{d}(X)) = \aleph_0$, *such that every compatible metric on it is GC, then X is GK.*

Proof: Let $\mathbf{d}(X) = \nu = \sup_{n \in \mathbb{N}} \nu_n$: if, by contradiction, X is not GK, then there exists a closed and discrete subset D of X with $|D| = \nu$. Write $D = \cup_{n \in \mathbb{N}} D_n$, where $|D_n| = \nu_n$ for every $n \in \mathbb{N}$ and the sets D_n are pairwise disjoint, and endow D by the compatible metric ρ, defined as:

$$\rho(x,y) = \begin{cases} 0 & \text{if } x = y \\ \frac{1}{\min\{l(x), l(y)\}} & \text{if } x \neq y \end{cases}$$

where $l(x)$ is, as usual, the unique $n \in \mathbb{N}$ such that $x \in D_n$ (see also the definition of the metric $\tilde{d}$ of Example 13). Now, extend ρ by Hausdorff's theorem to a compatible metric ρ^* on X. Indexing each D_n as $D_n = \{x_\alpha \,|\, \alpha \in A_n\}$, we see that the ν-sequence $(x_\alpha)_{\alpha \in \nu}$ is Cauchy with respect to ρ^*, but it cannot converge to any point of D, neither of X. □

One more question about transfinite sequences concerns a possible generalization of the well known fact that every metric space (X, d) is totally bounded (in the classical sense) if and only if every sequence on it admits a Cauchy subsequence. Is it possible to extend this property, in a suitable way, to GTB spaces? We are going to see that only one implication may be preserved.

Theorem 17 *If (X, d) is a metric space with* $\mathbf{d}(X) = \xi$, *such that every injective ξ-sequence on it admits a Cauchy ξ-subsequence, then (X, d) is GTB.*

Proof: Fix any $\varepsilon > 0$: if, by contradiction, there existed an ε-uniformly discrete subset D of X with $|D| = \xi$, then indexing D as $\{x_\alpha \,|\, \alpha \in \xi\}$ (with $\alpha \mapsto x_\alpha$ one-to-one) gives an injective ξ-sequence in X with no Cauchy ξ-subsequence. □

The converse of the above theorem is false, as the following example shows.

Example 18 Fix a cardinal $\nu > \aleph_0$ with $\operatorname{cof}(\nu) = \aleph_0$, and a strictly increasing sequence $(\nu_n)_{n\in\mathbb{N}}$ of infinite cardinals with $\sup_{n\in\mathbb{N}} \nu_n = \nu$; then consider again the GTB metric space of [3, Example 6]. That is, fix pairwise disjoint sets X_n for $n \in \mathbb{N}$, with $|X_n| = \nu_n$ for every n, and on the set $X = \bigcup_{n\in\mathbb{N}} X_n$ introduce the metric d defined by:

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ \frac{1}{n} & \text{if } x \neq y \text{ and } l(x) = l(y) = n \\ 1 & \text{if } l(x) \neq l(y) \end{cases}$$

where, for $x \in X$, $l(x)$ is the unique $n \in \mathbb{N}$ such that $x \in X_n$.

For every $n \in \mathbb{N}$ define (as in the proof of Theorem 10):

$$A_n = \{\alpha \in \nu \mid \nu_{n-1} \leq \alpha < \nu_n\}$$

— with $\nu_0 = 0$ — and put $X_n = \{x_\alpha \mid \alpha \in A_n\}$, where $\alpha \mapsto x_\alpha$ is one-to-one on A_n. Then it is easy to see that the (injective) ν-sequence $(x_\alpha)_{\alpha\in\nu}$ has no Cauchy ν-subsequence. □

In many cases, extensions of classical properties of compact (metrizable) spaces to GK spaces are possible provided that we take care of the density of the spaces which are involved.

In general, for instance, we can claim that a closed subset of a GK space is still GK, only if it has the same density of the first space. In the same way, if f is a continuous mapping from a GK space X to a metrizable space Y, then it is possible to infer that $f(X)$ is GK only if $\mathbf{d}(X) = \mathbf{d}(f(X))$.

On the contrary, there is no satisfactory generalization of the classical property that compact subspaces are closed. Consider the space $\tilde{X}$ of Example 9 and fix a subset $\{a_m \mid m \in \mathbb{N}\}$ of X_1 (where $m \mapsto a_m$ is one-to-one). Adjoin a point x^* to X, and endow it with the fundamental system of neighbourhoods: $\{V_m\}_{m\in\mathbb{N}}$, where $V_m = \{x^*\} \cup \{a_{m'} \mid m' \geq m\}$ for every $m \in \mathbb{N}$. Then $Z = \tilde{X} \cup \{x^*\}$ is a (GK) metrizable space having density ν, but its GK subspace $\tilde{X}$ is not closed in Z.

All we can say with respect to the last subject is that, if X is a GK subspace of a metrizable space Y, with $\mathbf{d}(X) = \mathbf{d}(Y) = \nu$, and each point in $(\mathrm{Cl}X) \setminus X$ can be obtained as the limit of an injective ν-sequence of points in X, then — under **GCH** — X is closed in Y.

We deal now with product spaces, to give necessary and sufficient conditions for a countable product of metrizable spaces to be GK.

Lemma 19 *Let X and Y be two metrizable spaces: then $X \times Y$ is GK if and only if X and Y are both GK and at least one of them is compact.*

Proof: First, suppose X to be GK and Y compact. Clearly, we can suppose X infinite; thus, if $\mathbf{d}(X) = \nu$, then $\mathbf{d}(X \times Y) = \nu$, too.

If, by contradiction, there exists a closed and discrete subset D of $X \times Y$ with $|D| = \nu$, let $D_1 = \mathrm{pr}_1(D)$ (where pr_1 and pr_2 are the canonical projections from $X \times Y$ on X and Y, respectively). For every $x \in D_1$, the set $\{y \in Y \mid (x, y) \in D\}$ is finite (since otherwise it would have an accumulation point $\bar{y}$, and hence $(x, \bar{y})$ would be an accumulation point for D, which is impossible). Thus $|D_1| = |D| = \nu$.

For every $x \in D_1$, select $y_x \in Y$ such that $(x, y_x) \in D$. As X is GK with $\mathbf{d}(X) = \nu$, the set D_1 cannot be closed and discrete in X, and hence it has there an accumulation point $\bar{x}$. Thus we can find an injective sequence $(x_n)_{n\in\mathrm{N}} \subseteq D_1$ such that $\lim_{n\to\infty} x_n = \bar{x}$. Correspondently, the sequence $(y_{x_n})_{n\in\mathrm{N}}$ in Y must have a subsequence $\left(y_{x_{n_m}}\right)_{m\in\mathrm{N}}$ converging to a point $\bar{y} \in Y$. It results that $(\bar{x}, \bar{y})$ is an accumulation point for D, in contrast with the assumption that this set is closed and discrete.

Suppose now that $X \times Y$ is *GK*: we can suppose also, without loss of generality, that $\mathbf{d}(X) = \nu \geq \mathbf{d}(Y)$. Therefore

$\mathbf{d}(X \times Y) = \mathbf{d}(X) = \nu$ and $X = \mathrm{pr}_1(X \times Y)$ is GK. Thus we only have to show that Y is compact.

Suppose the contrary: then there exists a closed and discrete subset $D'' = \{y_n \mid n \in \mathbb{N}\}$ of Y, where $n \mapsto y_n$ is one-to-one. Write, as usual, $\nu = \sup_{n \in \mathbb{N}} \nu_n$ with $n \mapsto \nu_n$ strictly increasing: for every $n \in \mathbb{N}$, there exists a closed and discrete subset D'_n of X with $|D'_n| = \nu_n$. We claim that $D = \bigcup_{n \in \mathbb{N}} (D'_n \times \{y_n\})$ is a closed and discrete subset of $X \times Y$ having cardinality ν, thus contradicting the assumption that $X \times Y$ is GK.

Indeed, if $(x, y) \notin D$, then either $y \notin D''$ — in which case there exists an open neighbourhood V of y such that $V \cap D'' = \emptyset$, and hence $X \times V$ is an open neighbourhood of (x, y) not intersecting D —, or $y = y_{\bar{n}}$ for some $\bar{n} \in \mathbb{N}$ and $x \notin D'_{\bar{n}}$ — in which case there exist U, V open neighbourhoods of x, y respectively, such that $U \cap D'_{\bar{n}} = \emptyset$ and $V \cap D'' = \{y_{\bar{n}}\}$, and we have also $(U \times V) \cap D = \emptyset$. Consider now an arbitrary point (x, y_n) of D (with $x \in D'_n$): there exist again U, V open neighbourhoods of x, y respectively such that $U \cap D'_n = \{x\}$ and $V \cap D'' = \{y_n\}$, and hence $(U \times V) \cap D = \{(x, y_n)\}$. □

Theorem 20 *A countable product $X = \prod_{n \in \mathbb{N}} X_n$ of metrizable spaces is GK if and only if there exists $n \in \mathbb{N}$ such that X_n is GK and each $X_{n'}$ with $n' \neq n$ is compact.*

Proof: If the above condition is satisfied, then $\prod_{n' \neq n} X_{n'}$ is compact by the Tychonoff theorem and hence $X \approx X_n \times \left(\prod_{n' \neq n} X_{n'}\right)$ is GK by Lemma 19.

Conversely, suppose that X is GK. If all the X_n spaces are compact, then the thesis is satisfied. Thus, suppose that there exists a $\bar{n} \in \mathbb{N}$ such that $X_{\bar{n}}$ is not compact: as $X \approx X_{\bar{n}} \times \left(\prod_{n \neq \bar{n}} X_n\right)$, by the preceding lemma $X_{\bar{n}}$ is GK and the product $\prod_{n \neq \bar{n}} X_n$ is compact. Hence the thesis follows from the Tychonoff theorem. □

Another question arises from the well known fact that every totally bounded metric space (or, equivalently, every separable metrizable space) can be (topologically) embedded in a suitable compact metrizable space, namely in the product of $\aleph_0$ many copies of the interval $[0,1]$ (the "Hilbert cube"). Can such a result be extended to GTB and GK spaces, respectively? In Theorem 3 we showed that every GTB metric space of density ν can be embedded in a countable product of metrizable spaces, whose densities have ν as their supremum; nevertheless, Theorem 20 above shows that such a product fails to be GK, and hence this doesn't solve our problem.

In the following example, we give a negative answer to the above question; this emphasizes the different behaviours between the classical and the generalized case.

Example 21 *For every cardinal number ν with $\nu > \aleph_0$ and* $\operatorname{cof}(\nu) = \aleph_0$*, there exists a metrizable space X with $\mathbf{d}(X) = \nu$ such that X cannot be embedded in any GK metrizable space having density ν.*

Proof: Given ν as above, let $\tilde{X}$ be the GK metrizable space of Example 9: then $\tilde{X} = \bigcup_{n\in\mathbb{N}} X_n \cup \{\infty\}$, where the sets X_n are pairwise disjoint and $\mathbf{d}(X_n) = \nu_n$ for every $n \in \mathbb{N}$; furthermore, we can suppose that each ν_n is regular. Consider $Y = \bigcup_{\alpha\in\aleph_1} \left(\tilde{X} \times \{\alpha\}\right)$, and endow Y with the topology of the disjoint sum of $\aleph_1$ many copies of $\tilde{X}$. Then Y is a metrizable space with $\mathbf{d}(Y) = \nu \cdot \aleph_1 = \nu$. We claim that Y cannot be embedded in a GK metrizable space having the same density.

By contradiction, suppose that it is possible to envisage Y as a subspace of a GK metrizable space Z of density ν, and let d be a compatible metric on Z. Consider the subset $M = \{(\infty, \alpha) \mid \alpha \in \aleph_1\}$ of Y: then M is discrete and hence $\mathbf{d}(M) = \aleph_1$. By [**3**, Lemma 2], choosing for every $n \in \mathbb{N}$ a $M_n \in \mathcal{UD}^{\max}_{\frac{1}{n}}(M)$, we have that $\sup_{n\in\mathbb{N}} |M_n| = \mathbf{d}(M) = \aleph_1$; it follows that there exists $\tilde{n} \in \mathbb{N}$ such that $|M_{\tilde{n}}| = \aleph_1$.

Let $\tilde{A} = \{\alpha \in \aleph_1 \,|\, (\infty, \alpha) \in M_{\tilde{n}}\}$: then $|\tilde{A}| = \aleph_1$. For every $\alpha \in \aleph_1$, the ball $S_d\left((\infty, \alpha), \frac{1}{3\tilde{n}}\right)$ is an open neighbourhood of (∞, α) in Z: this implies, in particular (by the definition of the topology of $\tilde{X}$), that there exists $n_\alpha \in \mathbb{N}$ such that $\{(\infty, \alpha)\} \cup \left\{(x, \alpha) \,|\, x \in \bigcup_{n \geq n_\alpha} X_n\right\} \subseteq S_d\left((\infty, \alpha), \frac{1}{3\tilde{n}}\right)$. For every $n \in \mathbb{N}$, let $\tilde{A}_n = \left\{\alpha \in \tilde{A} \,|\, n_\alpha \leq n\right\}$: then $\tilde{A} = \bigcup_{n \in \mathbb{N}} \tilde{A}_n$ and hence there exists $n^\star \in \mathbb{N}$ such that $|\tilde{A}_{n^\star}| = \aleph_1$; fix a countable subset $A^\star = \{\alpha_m \,|\, m \geq n^\star\}$ of $\tilde{A}_{n^\star}$, where $m \mapsto \alpha_m$ is one-to-one. For every $m \geq n^\star$, the set $X_m \times \{\alpha_m\}$ is discrete, and hence $\mathbf{d}(X_m \times \{\alpha_m\}) = |X_m \times \{\alpha_m\}| = \nu_m$; again, by [**3**, Lemma 2] and regularity of ν_m, it is possible to find a $p_m \in \mathbb{N}$ and a $\frac{1}{p_m}$-uniformly discrete subset P_m of $X_m \times \{\alpha_m\}$ such that $|P_m| = \nu_m$.

Now, consider the set $D = \bigcup_{m \geq n^\star} P_m$: it is clear that $|D| = \sum_{m \geq n^\star} \nu_m = \nu$; we will obtain a contradiction with the GK character of Z by showing that D is discrete and closed in Z. Indeed, let z be an arbitrary point of Z, and consider the open ball $S_d\left(z, \frac{1}{3\tilde{n}}\right)$: if it doesn't intersect D, then we have nothing more to prove; if $S_d\left(z, \frac{1}{3\tilde{n}}\right) \cap D \neq \emptyset$, then there is a unique $\bar{m} \geq n^\star$ such that $S_d\left(z, \frac{1}{3\tilde{n}}\right) \cap P_{\bar{m}} \neq \emptyset$ (as if $m' \neq m''$ with $m', m'' \geq n^\star$ then $D_d(P_{m'}, P_{m''}) \geq D_d(X_{m'} \times \{\alpha_{m'}\}, X_{m''} \times \{\alpha_{m''}\}) \geq \frac{1}{3\tilde{n}}$, since $X_{m'} \times \{\alpha_{m'}\} \subseteq S_d\left((\infty, \alpha_{m'}), \frac{1}{3\tilde{n}}\right)$, $X_{m''} \times \{\alpha_{m''}\} \subseteq S_d\left((\infty, \alpha_{m''}), \frac{1}{3\tilde{n}}\right)$ and $d((\infty, \alpha_{m'}), (\infty, \alpha_{m''})) \geq \frac{1}{\tilde{n}}$). In this case, the open ball $S_d\left(z, \min\left\{\frac{1}{3\tilde{n}}, \frac{1}{2p_{\bar{m}}}\right\}\right)$ intersects at most one point of D, and hence it is also possible to find a suitable neighbourhood of z which either doesn't intersect D or has in common with D the only point z. Thus D is closed and discrete. □

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