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RECENT RESULTS ON INDUCED MAPPINGS BETWEEN HYPERSPACES OF CONTINUA

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Abstract

For a given mapping f between continua we consider the induced mappings between corresponding hyperspaces of closed subsets, 2^f, or of subcontinua, $C(f)$. Recent results concerning interrelations between the three conditions: $f \in \mathfrak{M}$, $2^f \in \mathfrak{M}$, and $C(f) \in \mathfrak{M}$ for various classes $\mathfrak{M}$ of mappings are collected, and some questions related to the topic are asked.

1 Introduction

For a metric continuum X with a metric d we denote by 2^X, $C(X)$ and $F_1(X)$ the hyperspaces of all nonempty closed, of

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all nonempty closed connected subsets of X, and of all singletons in X, respectively, equipped with the Hausdorff metric H defined by

$$H(A, B) = \max\{\sup\{d(a, B) : a \in A\},\ \sup\{d(b, A) : b \in B\}\}.$$

The reader is referred to Nadler's book [18] for needed information on the structure of hyperspaces.

Given a mapping $f : X \to Y$ between continua X and Y, we let $2^f : 2^X \to 2^Y$ and $C(f) : C(X) \to C(Y)$ to denote the corresponding *induced mappings* defined by

$$2^f(A) = f(A) \text{ for every } A \in 2^X \quad \text{and}$$

$$C(f)(A) = f(A) \text{ for every } A \in C(X).$$

Let $\mathfrak{M}_i$, where $i \in \{1, 2, 3\}$, be some three classes of mappings between continua. A general problem which is related to a given mapping and to the two induced mappings is to find all interrelations between the following three statements:

(1.1) $f \in \mathfrak{M}_1$;

(1.2) $C(f) \in \mathfrak{M}_2$;

(1.3) $2^f \in \mathfrak{M}_3$.

In particular, if $\mathfrak{M}_1 = \mathfrak{M}_2 = \mathfrak{M}_3$ mean the class of homeomorphisms, then all three conditions (1.1), (1.2) and (1.3) are equivalent (see e.g. [18, Theorem (0.52), p. 29]). The same is true for the class of monotone mappings (see [10, Lemma 2.3, p. 2]; compare [9, Theorem 3.3, p. 4] and [18, (1.212.2), p. 204]), while not for open ones. There are some papers in which particular results concerning this problem are shown for various classes $\mathfrak{M}_i$ of mappings like open, monotone, confluent and some others (see the References). Interrelations between various classes of mappings of compact metric spaces are summarized in [17, Table II, p. 28]. In the present paper we gather recent results (some of which are not published yet) concerning induced mappings between hyperspaces, and recall some open questions and problems in the area.

2 Inducible mappings

In connection with the concept of the induced mappings one can ask under what conditions an arbitrary mapping between either the hyperspaces 2^X and 2^Y or the hyperspaces $C(X)$ and $C(Y)$ is an induced one. An answer to this question is given in [5]. To formulate it we recall an auxiliary notation.

Given two mappings between hyperspaces $g_1, g_2 : 2^X \to 2^Y$ (or $g_1, g_2 : C(X) \to C(Y)$), we will write $g_1 \prec g_2$ provided that $g_1(A) \subset g_2(A)$ for each $A \in 2^X$ (for each $A \in C(X)$, respectively). The relation $\prec$ is an order on the set of all mappings between hyperspaces (either 2^X and 2^Y, or $C(X)$ and $C(Y)$).

Let X and Y be continua. A mapping between hyperspaces, $g : 2^X \to 2^Y$ (or $g : C(X) \to C(Y)$), is said to be *inducible* provided that there exists a mapping $f : X \to Y$ such that $g = 2^f$ (or $g = C(f)$, respectively). We have the following characterization of inducible mappings (see [5, Theorem 2.2, p.7]).

2.1. Theorem *Let continua X and Y be given. A mapping between hyperspaces, $g : 2^X \to 2^Y$ (or $g : C(X) \to C(Y)$), is inducible if and only if each of the following three conditions is satisfied:*

(1) $g(F_1(X)) \subset F_1(Y)$;

(2) *$A \subset B$ implies $g(A) \subset g(B)$ for every $A, B \in 2^X$ (for every $A, B \in C(X)$, respectively);*

(3) *g is minimal with respect to the order $\prec$, i.e., if a mapping $g_0 : 2^X \to 2^Y$ (or $g_0 : C(X) \to C(Y)$) satisfies* (2),

and $g_0 \prec g$, *then* $g = g_0$.

Examples are presented in [5] showing that conditions (1), (2) and (3) of Theorem 2.1 are independent in the sense that no one of them is implied by the two others.

Given two continua X and Y, let us denote by Y^X the space of all mappings from X into Y equipped with the well-known *supremum metric* ρ, that is, if d_Y stands for the metric in Y, then

$$\rho(f,g) = \sup\{d_Y(f(x), g(x)) : x \in X\} \quad \text{for all} \quad f, g \in Y^X.$$

Further, we denote by $I(X,Y)$ the space of all induced mappings between the hyperspaces 2^X and 2^Y, i.e.,

$$I(X,Y) = \{2^f : 2^X \to 2^Y : f \in Y^X\} \subset (2^Y)^{2^X},$$

and by $I_C(X,Y)$ the space of all induced mappings between the hyperspaces $C(X)$ and $C(Y)$, i.e.,

$$I_C(X,Y) = \{C(f) : C(X) \to C(Y) : f \in Y^X\} \subset C(Y)^{C(X)}.$$

Note that $I_C(X,Y) = \{f|C(X) : f \in I(X,Y)\}$. The following result is proved in [6, Theorem 3.6].

2.2. Theorem *For every two continua* X *and* Y *the function spaces* Y^X, $I(X,Y)$ *and* $I_C(X,Y)$ *are isometric.*

3 Homeomorphisms. Hereditarily weakly confluent induced mappings

It is known that if $\mathfrak{M}_1 = \mathfrak{M}_2 = \mathfrak{M}_3$ mean the class of homeomorphisms, then all three conditions (1.1), (1.2) and (1.3) are equivalent (see e.g. [18, Theorem (0.52), p. 29]). However stronger results are known in the sense that, to obtain

(1.1) one can assume much weaker conditions than that the induced mappings are homeomorphisms. To formulate them, recall some definitions.

Given a class $\mathfrak{M}$ of mappings between continua, a mapping $f : X \to Y$ between continua X and Y is said to be *hereditarily* $\mathfrak{M}$ provided that for each subcontinuum K of X the partial mapping $f|K : K \to f(K) \subset Y$ also is in $\mathfrak{M}$.

Recall that a mapping $f : X \to Y$ between continua X and Y is said to be:
– *atomic* provided that, for each subcontinuum K of X, either $f(K)$ is degenerate, or $f^{-1}(f(K)) = K$;
– *monotone* if the inverse image of each point of Y is connected;
– *confluent* if for each subcontinuum Q of Y and for each component K of $f^{-1}(Q)$ the equality $f(K) = Q$ holds;
– *semi-confluent* if for each subcontinuum Q of Y and for every two components K_1 and K_2 of $f^{-1}(Q)$ either $f(K_1) \subset f(K_2)$ or $f(K_2) \subset f(K_1)$;
– *weakly confluent* if for each subcontinuum Q of Y there is a component K of $f^{-1}(Q)$ for which the equality $f(K) = Q$ holds;
– *joining* if for each subcontinuum Q of Y and for every two components K_1 and K_2 of $f^{-1}(Q)$ we have $f(K_1) \cap f(K_2) \neq \emptyset$;
– *atriodic* if for each subcontinuum Q of Y there are two components K_1 and K_2 of $f^{-1}(Q)$ such that $f(K_1) \cup f(K_2) = Q$ and for each component K of $f^{-1}(Q)$ either $f(K) = Q$, or $f(K) \subset f(K_1)$, or $f(K) \subset f(K_2)$.

The following inclusions show relations between these classes of mappings (of continua) - see [17, (3.1), (3.7), (3.2), (3.3), (3.8), (3.4) and (3.5), p. 12 and 13].

{homeomorphisms} ⊂ {atomic} ⊂ {monotone} ⊂
{confluent} ⊂ {semi-confluent} ⊂ {weakly confluent},
{semi-confluent} ⊂ {joining},
{weakly confluent} ⊂ {atriodic}.

It is shown in [4, Theorem 3.4, p. 198, and Corollary 3.8, p. 199] that if either of the two induced mappings between the hyperspaces is hereditarily weakly confluent, then the mapping between the continua is a homeomorphism. As a consequence, the following result is obtained in Theorem 3.11 of [4, p. 199].

3.1. Theorem *Let a surjective mapping $f : X \to Y$ between continua X and Y be given. Then the following conditions are equivalent.*

(1) f *is a homeomorphism;*

(2) 2^f *is a homeomorphism;*

(3) 2^f *is atomic;*

(4) 2^f *is hereditarily monotone;*

(5) 2^f *is hereditarily confluent;*

(6) 2^f *is hereditarily semi-confluent;*

(7) 2^f *is hereditarily weakly confluent;*

(8) $C(f)$ *is a homeomorphism;*

(9) $C(f)$ *is atomic;*

(10) $C(f)$ *is hereditarily monotone;*

(11) $C(f)$ *is hereditarily confluent;*

(12) $C(f)$ *is hereditarily semi-confluent;*

(13) $C(f)$ *is hereditarily weakly confluent.*

Examples are constructed in [4] showing that neither hereditarily joining nor hereditarily atriodic induced mappings can be included to the list in the above theorem. Note that the implication from (9) to (1) has earlier been proved in [9, p. 2].

A mapping $f : X \to Y$ between continua X and Y is said to be *open* if f maps each open set in X onto an open set in Y. The class of open mappings is intermediate between homeomorphisms and confluent ones. It is known that if $\mathfrak{M}_1 = \mathfrak{M}_2 = \mathfrak{M}_3$ mean the class of open mappings, then conditions (1.1) and (1.3) are equivalent, and each of them is implied by (1.2); however, (1.1) does not imply (1.2) even for locally connected continua (see [12, Section 4, Theorem 4.3, p. 243 and the example following it]). H. Hosokawa asked in [12, Section 8, Problem 1, p. 249] whether there is an open mapping f between continua such that $C(f)$ is open but $C^2(f) = C(C(f))$ is not open. A. Illanes has recently shown [14] the following result.

3.2. Theorem *Let f be a surjective mapping between nondegenerate continua. Then $C^2(f)$ is open if and only if f is a homeomorphism.*

Using this result he has given in [14] an affirmative answer to Hosokawa's question: the natural projection f of the unit square $[0,1] \times [0,1]$ onto the first factor $[0,1]$ is open, its induced mapping $C(f)$ is also open, while $C^2(f)$ is not open by Theorem 3.2.

In connection with Theorem 3.1 recall that a class $\mathcal{C}$ of continua is said to be *C-determined* provided that for every two elements X and Y of $\mathcal{C}$, if the hyperspaces $C(X)$ and $C(Y)$ are homeomorphic, then the continua X and Y are homeomorphic, too. In [18, (0.62), p. 33] three such classes are mentioned: finite graphs different from an arc and a simple closed curve, hereditarily indecomposable continua, and smooth fans. A further progress has very recently been made by the following two results. First, S. Macias [16] has added to this list the class of indecomposable continua all of nondegenerate proper subcontinua of which are arcs. Second, A. Illanes [15] has constructed two nonhomeomorphic chainable hereditarily decomposable continua X and Y whose hyperspaces $C(X)$ and $C(Y)$ are homeomorphic. Thus the class of chainable continua is not C-determined, which answers in the negative a question of S. B. Nadler, Jr. from his book [18, (0.62), p. 33]. Note that the homeomorphism between the hyperspaces in the Illanes example is not inducible.

4 Light mappings

A mapping $f : X \to Y$ between spaces X and Y is said to be *light* provided that for each point $y \in Y$ the set $f^{-1}(y)$ has one-point components (equivalently, if $f^{-1}(f(x))$ is totally disconnected for each $x \in X$; note that if the inverse images of points are compact, this condition is equivalent to the property that they are zero-dimensional).

Lightness of a mapping f between continua can be characterized in terms of induced mappings as follows (see [3, Theorem 3.1, p. 182]).

4.1. Theorem *Let a mapping $f : X \to Y$ between continua X and Y be given. Then the following conditions are equivalent:*

(4.2) *f is light;*

(4.3) $(C(f))^{-1}(F_1(Y)) \subset F_1(X)$;

(4.4) $F_1(X) = (C(f))^{-1}(F_1(Y))$;

(4.5) $F_1(X)$ *is a component of* $(C(f))^{-1}(F_1(Y))$.

The following theorem gives characterizations of lightness of induced mappings between hyperspaces (see [3, Theorems 3.6 and 3.7, p. 183]; for $C(f)$ see [18, (1.212), p. 204]).

4.6. Theorem *Let a mapping $f : X \to Y$ between continua X and Y be given. Then:*

(4.7) $2^f : 2^X \to 2^Y$ *is light if and only if for every $A, B \in 2^X$ the conditions $A \subsetneq B$ and each component of B intersects A imply the condition $f(A) \subsetneq f(B)$.*

(4.8) $C(f) : C(X) \to C(Y)$ *is light if and only if for every $A, B \in C(X)$ the condition $A \subsetneq B$ implies the condition $f(A) \subsetneq f(B)$.*

Interrelations between lightness of a mapping f and lightness of the two induced mappings are given in the next result [3, Theorem 3.10, p. 184, and Corollary 5.5, p. 190].

4.9. Theorem *Let a mapping $f : X \to Y$ between continua X and Y be given. Consider the following conditions:*

(a) *f is light;*

(b) *$C(f) : C(X) \to C(Y)$ is light;*

(c) *for every two continua $P, Q \in C(X) \setminus F_1(X)$ with $P \cap Q = \emptyset$ the inequality $f(P) \setminus f(Q) \neq \emptyset$ holds;*

(d) *$2^f : 2^X \to 2^Y$ is light.*

Then the implications (d) $\Rightarrow$ (c) $\Rightarrow$ (b) $\Rightarrow$ (a) *hold, and none of them can be reversed.*

The known continua showing that the conditions (1.1), (1.2) and (1.3) are not equivalent if $\mathfrak{M}_1 = \mathfrak{M}_2 = \mathfrak{M}_3$ mean the class of light mappings are not locally connected. They are not arcwise connected even. Thus the following question is natural (see [3, Question 5.1, p, 188]).

4.10. Question Are lightness of the induced mappings 2^f and $C(f)$ equivalent conditions for a mapping f between arcwise connected (in particular, locally connected) continua?

It is known [3, Corollary 5.7 and Remark 5.8, p. 191] that for mappings f with an arcwise connected (in particular with a locally connected) domain X conditions (b) and (c) of Theorem 4.9 are equivalent, while (b) does not imply (a) even for mappings f between locally connected continua. The question on the implication from (d) to (c) under this additional assumption is another form of Question 4.10.

An important class of light mappings between continua form local homeomorphisms. Recall that a mapping $f : X \to Y$ is said to be:
– *of a constant degree* if there is an $n \in \mathbb{N}$ such that card $f^{-1}(y) = n$ for each $y \in Y$ (in some papers these mappings are called n-to-1 maps);
– a *local homeomorphism* provided that every point $x \in X$ has an open neighborhood U such that $f(U)$ is an open subset of Y and the partial mapping $f|U : U \to f(U)$ is a homeomorphism.

It is known that a mapping $f : X \to Y$ of a compact space X onto a connected space Y is a local homeomorphism if and only if it is open and of a constant degree ([17, (4.27), p. 20]. Since each mapping of a constant degree is obviously light, we see that any local homeomorphism between continua is light.

Denote by $\mathbb{C}$ the set of complex numbers and put $S^1 = \{z \in \mathbb{C} : |z| = 1\}$. The mapping $f : S^1 \to S^1$ defined by $f(z) = z^2$ is a local homeomorphism, while $C(f)$ and 2^f are not light even. The following result (see [3, Theorem 4.2 and Corollary 4.3, p. 186]) concerns opposite implications.

4.11. Theorem *Let $f : X \to Y$ be a mapping between continua X and Y. If, for some $n \in \mathbb{N}$, the induced mapping either 2^f or $C(f)$ is of the constant degree n (in particular, if it is a local homeomorphism), then $n = 1$ and f, 2^f and $C(f)$ are homeomorphisms.*

5 Monotone mappings and near-homeomorphisms

As it has been recalled in the introduction, if $\mathfrak{M}_1 = \mathfrak{M}_2 = \mathfrak{M}_3$ mean the class of monotone mappings, then all three conditions (1.1), (1.2) and (1.3) are equivalent. However, monotoneity of f implies even stronger conditions related to induced mappings, namely their cell-likeness (see [19, Lemma 2.1, p. 750]). Recall that a continuum is said to have *trivial shape* if it is the intersection of a decreasing sequence of compact absolute retracts. A mapping $f : X \to Y$ between continua X and Y is called *cell-like* if, for each point $y \in Y$, the preimage $f^{-1}(y)$ is a continuum of trivial shape. In particular, cell-like mappings are monotone. Combining the above quoted results one gets the following theorem.

5.1. Theorem *For any two continua X and Y and a mapping $f : X \to Y$, the following conditions are equivalent:*

(1) *f is a monotone surjection;*

(2) *2^f is a monotone surjection;*

(3) *$C(f)$ is a monotone surjection;*

(4) *2^f is a cell-like surjection;*

(5) *$C(f)$ is a cell-like surjection.*

The above equivalences were recently used to obtain some results on induced mappings for locally connected continua. To formulate them, recall that a mapping between continua X and Y is called a *near-homeomorphism* if it is the uniform limit of homeomorphisms from X onto Y. The following can easily be shown.

5.2. Theorem *If a surjective mapping $f : X \to Y$ between continua X and Y is a near-homeomorphism, then the two* *induced mappings $2^f : 2^X \to 2^Y$ and $C(f) : C(X) \to C(Y)$ are also near-homeomorphisms.*

An arc ab in a space X is said to be *free* provided that $ab \setminus \{a, b\}$ is an open subset of X. The next result is Theorem 3 in [8].

5.3. Theorem *Let continua X and Y be locally connected (be locally connected without free arcs), and let a mapping $f : X \to Y$ be monotone. Then the induced mapping 2^f (the induced mapping $C(f)$) is a near-homeomorphism between 2^X and 2^Y (between $C(X)$ and $C(Y)$, respectively), which are homeomorphic to the Hilbert cube.*

The inverse implications to these of Theorems 5.2 and 5.3 are not true in general. Namely the following examples have recently been constructed (see 8, Examples 4, 5 and 6]).

5.4. Examples a) *There are nonhomeomorphic locally connected continua X and Y and a mapping $f : X \to Y$ such that the induced mappings 2^f and $C(f)$ are near-homeomorphisms, while f is not.* b) *There are a locally connected continuum X and a mapping $f : X \to X$ such that the induced mappings 2^f and $C(f)$ are near-homeomorphisms, while f is not.* c) *There are a locally connected continuum X and a mapping $f : X \to X$ such that the induced mapping 2^f is a near-homeomorphism, while f and $C(f)$ are not.*

However, the following questions remain open [8, Questions 7].

5.5. Questions Let a mapping $f : X \to Y$ between continua X and Y be such that the induced mapping $C(f)$ is a near-homeomorphism (in particular, $C(X)$ and $C(Y)$ are homeomorphic). Does it imply that 2^f is a near-homeomorphism? The same question, if $X = Y$.

In connection with Theorems 5.1. and 5.3 recall that monotoneity of a surjective mapping onto a locally connected continuum has also other consequences that concern the two induced mappings, and that were applied in [19] to show the fixed point property for some hyperspaces. A mapping $f : X \to Y$ between spaces X and Y is said to be *universal* provided that it has a coincidence with every mapping from X into Y; more precisely, provided that for every mapping $g : X \to Y$ there is a point $p \in X$ such that $f(p) = g(p)$. The following result is proved in [19, Theorem 2.3, p. 752].

5.6. Theorem *If a mapping $f : X \to Y$ from a continuum X onto a locally connected continuum Y is monotone, then the two induced mappings 2^f and $C(f)$ are universal.*

Theorem 5.6 has been applied to show in [19, Proposition 3.1, p. 753] that if a continuum X is the inverse limit of an inverse sequence of locally connected continua X_n with bonding mappings $f_n : X_{n+1} \to X_n$ such that for each index $n \in \mathbb{N}$ there is a subcontinuum Y_{n+1} of X_{n+1} for which the restriction $f_n|Y_{n+1} : Y_{n+1} \to X_n$ is a monotone surjection, then the hyperspaces 2^X and $C(X)$ have the fixed point property. In particular, the conclusion holds if X is the inverse limit of an inverse sequence of dendrites with quasi-monotone bonding mappings [19, Theorem 3.3, p. 753].

6 Confluent mappings

The definition of a confluent mapping has been recalled here in Section 3. We add that a mapping $f : X \to Y$ between continua X and Y is said to be *pseudo-confluent* provided that for each irreducible subcontinuum Q of Y there is a component K of $f^{-1}(Q)$ for which the equality $f(K) = Q$ holds. Thus each weakly confluent mapping is pseudo-confluent, but not invertedly (see e.g. [17, Example (4.49), p. 27]). Implications from confluence of a mapping f to confluence of the induced mappings are not true in general, unless some additional conditions are imposed concerning the structure of either continua X and Y or their hyperspaces. The following example, due to Hiroshi Hosokawa and Kazuhiro Kawamura, is known (see [11, Example 5.1], compare [7, Examples 4.1, p. 131, and 4.24, p. 143]).

6.1. Example *There are continua X and Y and a confluent mapping $f : X \to Y$ such that the two induced mappings 2^f and $C(f)$ both are neither pseudo-confluent nor joining.*

A continuum X is said to have the *arc approximation property* if every subcontinuum K of X is the limit of a sequence of arcwise connected subcontinua of X all containing a fixed point of K. In particular, each locally connected continuum has this property. The following result is shown in [7, Theorem 4.4, p. 133].

6.2. Theorem *Let a surjective mapping $f : X \to Y$ between continua X and Y be confluent. If the hyperspace either 2^Y or $C(Y)$ has the arc approximation property, then the induced mapping either 2^f or $C(f)$, respectively, is confluent.*

As a corollary one gets a known implication from confluence of f to confluence of the two induced mappings if Y is locally connected (see [11, Theorem 4.4] and [9, Theorem 2.5, p. 3]). This result has recently been strengthened to the following one (see [2, Theorem 5.2]).

6.3. Theorem *Let a continuum Y be the inverse limit of an inverse sequence of continua Y_n such that the hyperspaces $C(Y_n)$ have the arc approximation property and that the bonding mappings are confluent. Then for every continuum X confluence of a mapping $f : X \to Y$ implies confluence of the induced mapping $C(f) : C(X) \to C(Y)$.*

This result generalizes a similar one in which the members Y_n of the inverse sequence are assumed to be locally connected (see [13, Theorem 2.7, p. 775]).

Recall that a *Knaster's type continuum* (a *solenoid*) means the inverse limit of arcs (of simple closed curves, respectively) with open (equivalently: with confluent, see [1, Corollaries 6.1 and 6.2, p. 228 and 229]) bonding mappings. As an application of Theorem 6.3 one gets the following result.

6.4. Corollary *For each confluent mapping f of a continuum onto either a Knaster's type continuum or a solenoid the induced mapping $C(f)$ is confluent.*

Another property that is weaker than local connectedness is the property of Kelley. We say that a continuum X has the *property of Kelley* provided that for each point $x \in X$, for each sequence of points x_n converging to x and for each continuum K containing x there is a sequence of subcontinua K_n of X converging to K, with $x_n \in K_n$ for each index n. However, this property is not sufficient for the discussed implication, because of the next example [7, Example 4.7, p. 135].

6.5. Example *There are continua X and Y both having the property of Kelley, and a confluent surjection of X onto Y such that neither of the two induced mappings is pseudo-confluent.*

If implications from conditions (1.2) and (1.3) to (1.1) are considered for some classes $\mathfrak{M}_1$, $\mathfrak{M}_2$ and $\mathfrak{M}_3$ of mappings related to confluent ones, it turns out that much weaker assumptions than confluence of the induced mappings suffice to attain confluence of f. Namely we have the following result [7, Theorem 4.20, p. 142].

6.6. Theorem *Let a surjective mapping f between continua be given. If either of the two induced mappings is surjective and joining, then f is confluent.*

Concerning the other implications between (1.1), (1.2) and (1.3) related to the class of confluent mappings, let us mention one more example and a question (see [7, Example 4.12, p. 138, and Question 4.25, p. 144]).

6.7. Example *There are continua X and Y and a confluent mapping $f : X \to Y$ such that $C(f)$ is confluent while 2^f is not.*

6.8. Question Let a mapping f between continua be such that the induced mapping 2^f is confluent. Does it imply that the induced mapping $C(f)$ is confluent, too?

Let us mention that Chapters 5, 6 and 7 of [7] contain several results concerning interrelations between conditions (1.1), (1.2) and (1.3) for semi-confluent, joining, weakly confluent, pseudo-confluent and some related classes of mappings.

7 Limit properties

A class $\mathfrak{M}$ of mappings between continua is said to be *admissible* provided that it contains all homeomorphisms, and the composition of every two mappings belonging to $\mathfrak{M}$ is in $\mathfrak{M}$. Let a class $\mathfrak{M}$ of mappings between continua be admissible. A mapping $f : X \to Y$ is said to be *near*-$\mathfrak{M}$ if f is the uniform limit of a sequence of mappings from $\mathfrak{M}$. More precisely, $f \in$ near-$\mathfrak{M}$ provided that there exists a sequence of mappings $f_n : X \to Y$ in $\mathfrak{M}$ such that $f = \lim f_n$, where the limit is taken with respect to the supremum metric. It should be stressed that, in the above definition, the terms f_n of the sequence of mappings are defined on the same domain space X and have the same range space Y as the limit mapping f.

We intend to discuss interrelations between the conditions

(7.1) $f \in$ near-$\mathfrak{M}$;

(7.2) $C(f) \in$ near-$\mathfrak{M}$;

(7.3) $2^f \in$ near-$\mathfrak{M}$.

for some particular admissible classes $\mathfrak{M}$ of mappings for which some corresponding relations between conditions (1.1), (1.2) and (1.3) (see Introduction) are assumed. We start with a general result [6, Theorem 4.4].

7.4. Theorem *Let an admissible class* $\mathfrak{M}$ *of mappings between continua be given such that*

(7.5) *the condition* $f \in \mathfrak{M}$ *implies the condition* $2^f \in \mathfrak{M}$ *(or the condition* $C(f) \in \mathfrak{M}$*).*

Then

(7.6) *the condition* $f \in$ near-$\mathfrak{M}$ *implies the condition* $2^f \in$ near-$\mathfrak{M}$ *(or the condition* $C(f) \in$ near-$\mathfrak{M}$*, respectively).*

It can be seen that for most admissible classes $\mathfrak{M}$ (as homeomorphisms, monotone, open, or some other mappings) the converse is not true, i.e., the condition that the induced mapping is in near-$\mathfrak{M}$ does not imply that f is in near-$\mathfrak{M}$ even if $2^f \in \mathfrak{M}$ (or $C(f) \in \mathfrak{M}$) implies $f \in \mathfrak{M}$. Similar assertion concerns the two implications between induced mappings. Appropriate examples have been constructed in [6].

For the class $\mathfrak{M}$ of open mappings conditions (1.1) and (1.3) are equivalent, and each of them is implied by (1.2) (see [12, Theorem 4.3, p. 243]). Thus it follows from Theorem 7.4 that

(7.7) if a surjective mapping $f : X \to Y$ between continua X and Y is near-open, then the induced mapping $2^f : 2^X \to 2^Y$ is near-open, too.

Examples are known (see [9, Example 3.2, p. 4] and [12, Section 4]) of open surjective mappings $f : X \to Y$ between continua X and Y such that the induced mappings $C(f) : C(X) \to C(Y)$ are not open. The following result is even stronger (see [6, Example 5.8]), and it shows that an analog of (7.7) is not true.

7.8. Example *There are plane dendroids* X *and* $Y \subset X$ *and an open retraction* $f : X \to Y$ *such that the induced mapping* $C(f) : C(X) \to C(Y)$ *is not near-open.*

The next two examples [6, Examples 5.12 and 5.15] show that implications from (7.2) and (7.3) to (7.1) are not true for the class $\mathfrak{M}$ of open mappings, even if local connectedness of the considered continua is assumed.

7.9. Example *There is a monotone mapping f of a dendrite onto itself, such that 2^f is a near-homeomorphism (so it is near-open) while f is not near-open.*

7.10. Example *There is a monotone mapping f of a plane locally connected continuum onto itself, such that the two induced mappings 2^f and $C(f)$ are near-homeomorphisms while f is not near-open.*

The quoted paper [6] contains also some particular results on possible implications between conditions (7.1), (7.2) and (7.3) for some related classes $\mathfrak{M}$ of mappings between continua, as monotone or compositions of open and monotone mappings, but these results are rather far from being complete. Also a discussion of interrelations between the conditions for other classes of mappings, as for example confluent ones, deserves a further study.

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