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COMPARISON OF COMPACTIFICATIONS VIA FUNCTION SPACES

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Dedicated to Professor Richard Chandler
on his sixtieth birthday

Abstract

For the function spaces $C_\alpha(X)$ and $C_\gamma(X)$ associated with compactifications αX and γX of a Tikhonov space X such that $\alpha X \leq \gamma X$, we investigate the smallest cardinal number which is the cardinality of a set $F \subseteq C_\gamma(X)$ such that $C_\gamma(X)$ is the uniform closure of the algebra generated by $C_\alpha(X) \cup F$. We apply this

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number to the comparison of some topological properties of αX and γX. Furthermore, we show that, for every cardinal number κ, there exist a locally compact Hausdorff space X and its compactifications $\alpha X \leq \gamma X$ such that, for some $f \in C^*(X)$, the algebra $C_\gamma(X)$ is generated by $C_\alpha(X) \cup \{f\}$ but the collection of all those fibres of the natural quotient map $\pi_{\gamma\alpha} : \gamma X \to \alpha X$ which are not singletons is of cardinality κ. This answers a question posed by G. D. Faulkner.

Introduction

All the spaces considered below are assumed to be completely regular and Hausdorff, i.e. Tikhonov.

For a space X, denote by $C(X)$ the algebra of all continuous real functions defined on X, and by $C^*(X)$ the subalgebra of $C(X)$ consisting of all bounded functions $f \in C(X)$.

Let $\mathcal{E}(X)$ be the collection of those sets $F \subseteq C^*(X)$ for which the diagonal map $e_F = \triangle_{f \in F} f$ is a homeomorphic embedding. If $F \in \mathcal{E}(X)$, then the closure of $e_F(X)$ in $\mathbb{R}^F$ is a compactification of X which is said to be generated by F and which is denoted by $e_F X$. The compactification $e_F X$ is the minimal compactification of X over which each function $f \in F$ is continuously extendable (cf. e.g. [1], [3], [4], [14] and [15]).

For a compactification αX of X, denote by $C_\alpha(X)$ the collection of all those functions $f \in C^*(X)$ which are continuously extendable over αX. For $f \in C_\alpha(X)$, let f^α be the continuous extension of f over αX and, for $F \subseteq C_\alpha(X)$, let $F^\alpha = \{f^\alpha : f \in F\}$. Clearly, $C_\beta(X) = C^*(X)$ where βX is the Čech-Stone compactification of X.

If $F \subseteq C^*(X)$, let the symbol $\langle F \rangle$ stand for the smallest subalgebra of $C^*(X)$ which contains F. Denote by $\overline{F}$ the closure of F in $C^*(X)$ equipped with the topology of uniform convergence.

It is well known that, for every compactification αX of X, the algebra $C_\alpha(X)$ belongs to $\mathcal{E}(X)$ and $e_{C_\alpha(X)}X = \alpha X$. Furthermore, if $F \in \mathcal{E}(X)$, then $C_{e_F}(X)$ is the smallest subalgebra of $C^*(X)$ which is closed under uniform convergence, contains F and all constant functions (cf. [3; Thm. 3.1]). Those observations give efficient tools to the comparison of compactifications. Namely, for sets $F, G \in \mathcal{E}(X)$, we have $e_F X \leq e_G X$ if and only if $C_{e_F}(X) \subseteq C_{e_G}(X)$ (cf. [5; Thm. 2.10]). Accordingly, for every compactification αX of X and for each $F \subseteq C^*(X)$, the set $G_{\alpha,F} = C_\alpha(X) \cup F$ lies in $\mathcal{E}(X)$ and generates a compactification γX such that $\alpha X \leq \gamma X$ and $C_\gamma(X) = \overline{\langle C_\alpha(X) \cup F \rangle}$. Of course, if αX and γX are any compactifications of X such that $\alpha X \leq \gamma X$, then there exists a set $F \subseteq C^*(X)$ with $C_\gamma(X) = \overline{\langle C_\alpha(X) \cup F \rangle}$. Therefore, for every pair of compactifications αX and γX of X with $\alpha X \leq \gamma X$, it seems natural to introduce the cardinal number $\varepsilon_{\gamma\alpha}(X)$ which is the smallest cardinal number κ for which there exists a set $F \subseteq C^*(X)$ of cardinality κ such that $C_\gamma(X) = \overline{\langle C_\alpha(X) \cup F \rangle}$. This cardinal number must have an essential influence on possible differences between some topological properties of αX and γX. Our purpose is to investigate $\varepsilon_{\gamma\alpha}(X)$.

We shall make a frequent use of the following theorem proved in [3]:

0. Theorem *For every compactification αX of X and for every $F \subseteq C^*(X)$, we have $F \in \mathcal{E}(X)$ and $e_F X = \alpha X$ if and only if $F \subseteq C_\alpha(X)$ and F^α separates points of αX.*

The results

For a compactification αX of a space X, B. J. Ball and Shoji Yokura introduced in [1] the following cardinal number which was further investigated, e. g. in [2], [4] and [14]:

$$\varepsilon(\alpha X) = \min\{|F| : F \in \mathcal{E}(X) \quad \text{and} \quad e_F X = \alpha X\}.$$

It is known that $\varepsilon(\alpha X)$ is the smallest cardinal number κ such that αX is embeddable in the cube I^κ; furthermore, if $\varepsilon(\alpha X)$ is infinite, then it is equal to the weight $w(\alpha X)$ of αX (cf. [1], [4], [14]). The cardinal number $\varepsilon(\alpha X)$ depends only on the space αX; therefore, since every compact space is a compactification of any of its dense subspaces, given a compact space Y, we have defined the cardinal number $\varepsilon(Y)$.

According to our notation, for compactifications αX and γX of X such that $\alpha X \leq \gamma X$, we have

$$\begin{aligned}\varepsilon_{\gamma\alpha}(X) &= \min\{|F| : F \subseteq C^*(X) \text{ and } C_\gamma(X) = \overline{\langle C_\alpha(X) \cup F \rangle}\} \\ &= \min\{|F| : F \subseteq C^*(X) \text{ and } e_G X = \gamma X \\ &\quad \text{where} \quad G = C_\alpha(X) \cup F\}.\end{aligned}$$

The following proposition is an immediate consequence of Theorem 0 and [1; Thm. 4.3]:

1. Proposition *For every pair $\alpha X, \gamma X$ of compactifications of X such that $\alpha X \leq \gamma X$, the inequalities*

$$\varepsilon_{\gamma\alpha}(X) \leq \varepsilon(\gamma X) \leq \varepsilon(\alpha X) + \varepsilon_{\gamma\alpha}(X)$$

hold. Moreover, if $\varepsilon(\gamma X)$ is infinite, then

$$\varepsilon(\gamma X) = \varepsilon(\alpha X) + \varepsilon_{\gamma\alpha}(X).$$

Clearly, the assumption that $\varepsilon(\gamma X) \geq \omega$ cannot be omitted in the second part of Proposition 1 even when $\varepsilon(\alpha X)$ is finite.

Example For the open interval $X = (0; 1)$, let γX be the unit interval $I = [0; 1]$. If αX is the one-point compactification of X, then $\alpha X \leq \gamma X$, while $\varepsilon(\alpha X) = 2, \varepsilon(\gamma X) = 1$ and $\varepsilon_{\gamma\alpha}(X) = 1$; hence $\varepsilon(\gamma X) < \varepsilon(\alpha X) + \varepsilon_{\gamma\alpha}(X)$.

For compactifications αX and γX of X such that $\alpha X \leq \gamma X$, put

$$P_{\gamma\alpha} = \{y \in \alpha X : |\pi_{\gamma\alpha}^{-1}(y)| > 1\}$$

where $\pi_{\gamma\alpha} : \gamma X \to \alpha X$ is the natural quotient map witnessing that $\alpha X \leq \gamma X$.

2. Proposition *For compactifications αX and γX of X such that $\alpha X \leq \gamma X$, the following inequalities hold:*

$$\sup\{\varepsilon(\pi_{\gamma\alpha}^{-1}(y)) : y \in P_{\gamma\alpha}\} \leq \varepsilon_{\gamma\alpha}(X) \leq$$

$$|P_{\gamma\alpha}| \cdot \sup\{\varepsilon(\pi_{\gamma\alpha}^{-1}(y)) : y \in P_{\gamma\alpha}\}.$$

Furthermore, if the set $P_{\gamma\alpha}$ is finite, then

$$\varepsilon_{\gamma\alpha}(X) = \sup\{\varepsilon(\pi_{\gamma\alpha}^{-1}(y)) : y \in P_{\gamma\alpha}\}.$$

Proof: Put $\kappa = \sup\{\varepsilon(\pi_{\gamma\alpha}^{-1}(y)) : y \in P_{\gamma\alpha}\}$. Suppose first that $C_\alpha(X) \cup F$ generates γX. Since, in view of Theorem 0, the set $(C_\alpha(X) \cup F)^\gamma$ separates points of γX and, moreover, each function from $C_\alpha(X)^\gamma$ is constant on the fibres of $\pi_{\gamma\alpha}$, we have that F^γ separates points of $\pi_{\gamma\alpha}^{-1}(y)$ for any $y \in P_{\gamma\alpha}$. This, together with Theorem 0, implies that $\{f^\gamma \restriction \pi_{\gamma\alpha}^{-1}(y) : f \in F\} \in \mathcal{E}(\pi_{\gamma\alpha}^{-1}(y))$ for any $y \in P_{\gamma\alpha}$. Hence $\kappa \leq |F|$ and, in consequence, $\kappa \leq \varepsilon_{\gamma\alpha}(X)$.

For each $y \in P_{\gamma\alpha}$, let us choose a set $F_y \in \mathcal{E}(\pi_{\gamma\alpha}^{-1}(y))$ such that $|F_y| \leq \kappa$. Extend each function $f \in F_y$ to a function $\tilde{f} \in C(\gamma X)$ and put $F = \bigcup_{y \in P_{\gamma\alpha}} \{\tilde{f} \restriction X : f \in F_y\}$. Then, by Theorem 0, the set $C_\alpha(X) \cup F$ generates γX. Since $|F| \leq |P_{\gamma\alpha}| \cdot \kappa$, we have that $\varepsilon_{\gamma\alpha}(X) \leq |P_{\gamma\alpha}| \cdot \kappa$.

Now, assume that the set $P_{\gamma\alpha}$ is finite. If κ is infinite, then the set G of all possible combinations $\nabla_{y \in P_{\gamma\alpha}} f_y$ of functions $f_y \in F_y$ with $y \in P_{\gamma\alpha}$ is of cardinality $\leq \kappa$. All the functions from G are continuous on the compact set $\pi_{\gamma\alpha}^{-1}(P_{\gamma\alpha})$. If we extend each function $g \in G$ to a function $\tilde{g} \in C(\gamma X)$, then we obtain a set $F = \{\tilde{g} \restriction X : g \in G\}$ of cardinality $\leq \kappa$ such that $C_\alpha(X) \cup F$ generates γX; hence $\varepsilon_{\gamma\alpha}(X) \leq \kappa$.

If both $P_{\gamma\alpha}$ and κ are finite, let $F_y = \{f_{1,y}, \dots, f_{\kappa,y}\}$ for $y \in P_{\gamma\alpha}$. For $i = 1, \dots, \kappa$, denote by f_i the combination $\bigtriangledown_{y \in P_{\gamma\alpha}} f_{i,y}$ of the functions $f_{i,y}$ with $y \in P_{\gamma\alpha}$. Extend each function $f_i \in C^*(\pi_{\gamma\alpha}^{-1}(P_{\gamma\alpha}))$ to a function $\tilde{f}_i \in C(\gamma X)$ and put $F = \{\tilde{f}_i \restriction X : i = 1, \dots, \kappa\}$. Then $C_\alpha(X) \cup F$ generates γX, which implies that $\varepsilon_{\gamma\alpha}(X) \leq \kappa$. This completes the proof. $\square$

The following example shows that the assumption that the set $P_{\gamma\alpha}$ be finite, cannot be omitted in the second part of Proposition 2:

Example Let Z be the well-known double arrow space, i.e. the interval $[-1; 1)$ equipped with the topology having as a base for open sets the collection of all the sets of the form:

$$[-b; -a) \cup [a; b)$$

where $0 \leq a \leq b \leq 1$. Since Z is a compact perfectly normal space, there exists a compactification $\gamma\mathbb{N}$ of the space $\mathbb{N}$ of positive integers such that $Z = \gamma\mathbb{N} \setminus \mathbb{N}$ (cf. [11]). The function $f : Z \to I$ defined by $f(x) = |x|$ for $x \in Z$ is continuous. It follows from Magill's theorem (cf. [5; Thm. 7.2]) that there exists a compactification $\alpha\mathbb{N}$ of $\mathbb{N}$ such that $\alpha\mathbb{N} \leq \gamma\mathbb{N}$, $\alpha\mathbb{N} \setminus \mathbb{N} = I$ and $\pi_{\gamma\alpha} \restriction Z = f$. Then

$$\sup\{\varepsilon(\pi_{\gamma\alpha}^{-1}(y)) : y \in P_{\gamma\alpha}\} = 1.$$

Since $\gamma\mathbb{N}$ is of weight 2^ω, we have $\varepsilon(\gamma\mathbb{N}) = 2^\omega$. The compactification $\alpha\mathbb{N}$ of $\mathbb{N}$ is metrizable; thus $\varepsilon(\alpha\mathbb{N}) + \omega = \omega$. All this taken together with Proposition 1 implies that $\varepsilon_{\gamma\alpha}(\mathbb{N}) = 2^\omega$.

3. Theorem *For any compactifications αX and γX of X such that $\alpha X \leq \gamma X$, the following equality holds:*

$$w(\gamma X) = w(\alpha X) + \varepsilon_{\gamma\alpha}(X).$$

Proof: If $\varepsilon(\gamma X)$ is infinite, then $w(\gamma X) = \varepsilon(\gamma X)$; hence, by Proposition 1, $w(\gamma X) = \varepsilon(\alpha X) + \varepsilon_{\gamma\alpha}(X) \leq w(\alpha X) + \varepsilon_{\gamma\alpha}(X)$. If

$\varepsilon(\gamma X)$ is finite, then $w(\gamma X) = \omega$, which implies that $w(\gamma X) \leq w(\alpha X) + \varepsilon_{\gamma\alpha}(X)$. On the other hand, since αX is a continuous image of γX, we have $w(\alpha X) \leq w(\gamma X)$; moreover, the inequalities $\varepsilon_{\gamma\alpha}(X) \leq \varepsilon(\gamma X) \leq w(\gamma X)$ always hold. In consequence, $w(\alpha X) + \varepsilon_{\gamma\alpha}(X) \leq w(\gamma X)$. □

Mimicking the proof of Theorem 4 of [15], we can get the following lemma:

4. Lemma *For a set $F \subseteq C^*(X)$, denote by $\mathcal{B}_F$ the collection of all sets of the form*

$$\bigcap_{i=1}^{n} f_i^{-1}((a_i; b_i))$$

where $a_i < b_i$ are rational numbers and $f_i \in F$ for $i = 1, \ldots, n$. Then $F \in \mathcal{E}(X)$ if and only if the collection $\mathcal{B}_F$ is an open base for X.

5. Theorem *For any compactifications αX and γX of X such that $\alpha X \leq \gamma X$, the following equality holds:*

$$w(\gamma X \setminus X) = w(\alpha X \setminus X) + \varepsilon_{\gamma\alpha}(X).$$

Proof: Let us take a set $F \subseteq C^*(X)$ such that $|F| = \varepsilon_{\gamma\alpha}(X)$ and $C_\gamma(X) = \overline{\langle C_\alpha(X) \cup F \rangle}$. Since cozero-sets of αX serve as an open base for αX, there exists a set $G \subseteq C_\alpha(X)$ such that $|G| \leq w(\alpha X \setminus X)$ and the collection of all sets of the form

$$(\alpha X \setminus X) \cap (g^\alpha)^{-1}((0; 1)),$$

where $g \in G$, is an open base for $\alpha X \setminus X$. Denote by $\mathcal{B}$ the collection of all sets of the form

$$(\gamma X \setminus X) \cap \bigcap_{i=1}^{n} (h_i^\gamma)^{-1}((a_i; b_i))$$

where $a_i < b_i$ are rational numbers and $h_i \in G \cup F$ for $i = 1, \dots, n$. We shall show that $\mathcal{B}$ is an open base for $\gamma X \setminus X$. To this end, consider any open set $V \subseteq \gamma X$ and any $x \in (\gamma X \setminus X) \cap V$. Let $y = \pi_{\gamma\alpha}(x)$ and suppose that $\pi_{\gamma\alpha}^{-1}(y) \subseteq V$. The mapping $\pi_{\gamma\alpha}$ being closed, there exists an open neighbourhood U of y in $\alpha X \setminus X$ such that $\pi_{\gamma\alpha}^{-1}(U) \subseteq V$. There exists a function $g \in G$ such that $y \in (\alpha X \setminus X) \cap (g^\alpha)^{-1}((0;1)) \subseteq U$. Then $x \in \pi_{\gamma\alpha}^{-1}(y) \subseteq (g^\gamma)^{-1}((0;1)) \subseteq V$.

Assume now that $\pi_{\gamma\alpha}^{-1}(y) \setminus V \neq \emptyset$. Since F^γ separates points of $\pi_{\gamma\alpha}^{-1}(y)$, it follows from Theorem 0 and Lemma 4 that there exist functions $f_j \in F$ and rational numbers $a_j < c_j < d_j < b_j$, such that

$$x \in \pi_{\gamma\alpha}^{-1}(y) \cap \bigcap_{j=1}^{m} (f_j^\gamma)^{-1}((c_j; d_j)) \subseteq \pi_{\gamma\alpha}^{-1}(y) \cap \bigcap_{j=1}^{m} (f_j^\gamma)^{-1}((a_j; b_j)) \subseteq V.$$

The set $A = (\gamma X \setminus V) \cap \bigcap_{j=1}^{m} (f_j^\gamma)^{-1}([c_j; d_j])$ is closed in γX and it does not meet $\pi_{\gamma\alpha}^{-1}(y)$. Therefore, since $\pi_{\gamma\alpha}$ is a closed map, there exists a function $g \in G$ such that

$$\pi_{\gamma\alpha}^{-1}(y) \subseteq (g^\gamma)^{-1}((0;1)) \subseteq \gamma X \setminus A.$$

Then

$$x \in (g^\gamma)^{-1}((0;1)) \cap \bigcap_{j=1}^{m} (f_j^\gamma)^{-1}((c_j; d_j)) \subseteq V.$$

All this taken together implies that $\mathcal{B}$ is an open base for $\gamma X \setminus X$. Since $|\mathcal{B}| \leq w(\alpha X \setminus X) + \varepsilon_{\gamma\alpha}(X)$, we have $w(\gamma X \setminus X) \leq w(\alpha X \setminus X) + \varepsilon_{\gamma\alpha}(X)$. The mapping $\pi_{\gamma\alpha} \restriction (\gamma X \setminus X)$ being perfect, in view of Theorem 3.7.19 of [7], the inequality $w(\alpha X \setminus X) \leq w(\gamma X \setminus X)$ always holds. According to Theorem 4.1 of [1], there exists a set $F \subseteq C_\gamma(X)$ such that F^γ separates points of $\gamma X \setminus X$ and $|F| \leq w(\gamma X \setminus X)$. This, along with Theorem 0, yields that also $\varepsilon_{\gamma\alpha}(X) \leq w(\gamma X \setminus X)$. Hence

$w(\alpha X \setminus X) + \varepsilon_{\gamma\alpha}(X) \leq w(\gamma X \setminus X)$, which concludes the proof. $\square$

Recall that $\mathfrak{s}$ is the smallest among those cardinal numbers which are the cardinalities of splitting families in $[\omega]^\omega$ (cf. [6; p. 115]).

6. Theorem *Let αX and γX be compactifications of X such that $\alpha X \leq \gamma X$. If αX is sequentially compact and $\varepsilon_{\gamma\alpha}(X) < \mathfrak{s}$, then γX is sequentially compact.*

Proof: Let $\langle p_n \rangle$ be a sequence of points of γX. Since the space αX is sequentially compact, the sequence $\langle \pi_{\gamma\alpha}(p_n) \rangle$ contains a convergent in αX subsequence. For simplicity, assume that $\langle \pi_{\gamma\alpha}(p_n) \rangle$ converges in αX to a point p. It is easily seen that the set $P = \{p_n : n \in \mathbb{N}\} \cup \pi_{\gamma\alpha}^{-1}(p)$ is compact. Since $\varepsilon_{\gamma\alpha}(X) < \mathfrak{s}$, the space $\pi_{\gamma\alpha}^{-1}(p)$ is embeddable in the Tikhonov cube $I^\varkappa$ for some $\varkappa < \mathfrak{s}$. This implies that the space P is of weight $< \mathfrak{s}$. As every compact space of weight $< \mathfrak{s}$ is sequentially compact (cf. [6; Thm. 6.1] or [13; Thm. 5.12]), the sequence $\langle p_n \rangle$ contains a convergent in P subsequence $\langle p_{n_k} \rangle$. Then the sequence $\langle p_{n_k} \rangle$ is convergent in γX. $\square$

Using similar arguments, we can prove the following theorem:

7. Theorem *Let αX and γX be compactifications of X such that $\alpha X \leq \gamma X$. If $\alpha X \setminus X$ is sequentially compact and $\varepsilon_{\gamma\alpha}(X) < \mathfrak{s}$, then $\gamma X \setminus X$ is sequentially compact.*

Remark. The inequality $\varepsilon_{\gamma\alpha}(X) < \mathfrak{s}$ cannot be replaced by $\varepsilon_{\gamma\alpha}(X) \leq \mathfrak{s}$ in Theorems 6 and 7. Indeed, since $\mathfrak{s} \leq 2^\omega$, we can consider a compact space K of weight $\mathfrak{s}$ which is not sequentially compact but which is the remainder of a compactification $\gamma\mathbb{N}$ of $\mathbb{N}$. Now, for the one-point compactification $\omega\mathbb{N}$ of $\mathbb{N}$, we have $\varepsilon_{\gamma\omega}(\mathbb{N}) = \mathfrak{s}$, the space $\omega\mathbb{N}$ is sequentially compact, while both the spaces $\gamma\mathbb{N}$ and $\gamma\mathbb{N} \setminus \mathbb{N}$ are not sequentially compact.

For a set $F \subseteq C^*(X)$ and a positive integer n, let $M^n(F)$ be the collection of all functions of the form $\phi \circ \triangle_{i=1}^n f_i$ where

$\phi \in C^*(\mathbb{R}^n)$ and $f_i \in F$ for $i = 1, \ldots, n$. Then, if $F \subseteq C^*(X)$ is non-void, the collection $M(F) = \bigcup_{n=1}^{\infty} M^n(F)$ is a subalgebra of $C^*(X)$ which contains F and all the constants (cf. [2] and [14]). Furthermore, $F \subseteq C^*(X)$ generates a compactification αX of X if and only if the algebra $M(F)$ is dense in $C_\alpha(X)$ with the topology of uniform convergence (cf. [14]). In general, the algebra $M(F)$ need not be closed under uniform convergence (cf. [14]).

For compactifications αX and γX of X such that $\alpha X \leq \gamma X$, define

$$m_{\gamma\alpha}(X) = \min\{|F| : M(C_\alpha(X) \cup F) = C_\gamma(X)\}.$$

Clearly, $\varepsilon_{\gamma\alpha}(X) \leq m_{\gamma\alpha}(X)$. The cardinal number $m_{\gamma\alpha}(X)$ has the following interesting property:

8. Theorem *Let αX and γX be compactifications of X such that $\alpha X \leq \gamma X$. If $m_{\gamma\alpha}(X)$ is countable, then $\varepsilon_{\gamma\alpha}(X)$ is finite.*

Proof: Take a countable set $F = \{f_1, f_2, \ldots\} \subseteq C^*(X)$ such that $M(C_\alpha(X) \cup F) = C_\gamma(X)$. Suppose, if possible, that for each positive integer n, the set $C_\alpha(X) \cup \{f_1, \ldots, f_n\}$ does not generate γX. Then, for each $n \in \mathbb{N}$, there exists $y_n \in \alpha X \setminus X$ such that the set $\{f_1^\gamma, \ldots, f_n^\gamma\}$ does not separate points of the fibre $\pi_{\gamma\alpha}^{-1}(y_n)$. There exist an infinite set $\mathbb{N}_0 \subseteq \mathbb{N}$ and a collection $\{V_n : n \in \mathbb{N}_0\}$ of pairwise disjoint open subsets of αX, such that $y_n \in V_n$ for $n \in \mathbb{N}_0$. Put $U_n = \pi_{\gamma\alpha}^{-1}(V_n)$ and $Y_n = \pi_{\gamma\alpha}^{-1}(y_n)$ for $n \in \mathbb{N}_0$. Since, for $n \in \mathbb{N}_0$, the family $\{f_1^\gamma, \ldots, f_n^\gamma\}$ does not separate points of Y_n, there exists a function $g_n^* \in C(Y_n)$ which cannot be represented in the form $\phi \circ \triangle_{i=1}^n f_i^\gamma \restriction Y_n$ with $\phi \in C^*(\mathbb{R}^n)$. We may assume that $0 \leq g_n^* \leq 1$. For $n \in \mathbb{N}_0$, define

$$h_n(x) = \begin{cases} g_n^*(x) & \text{when } x \in Y_n, \\ 0 & \text{when } x \in \gamma X \setminus U_n. \end{cases}$$

As the compact set $Y_n \cup (\gamma X \setminus U_n)$ is C^*-embeddable in γX, we can extend the function h_n to a continuous function $\tilde{h}_n : \gamma X \to I$. Let

$$h = \sum_{n \in \mathbb{N}_0} \frac{1}{2^n} \tilde{h}_n.$$

Then $h \in C(\gamma X)$. Since $M(C_\alpha(X) \cup F) = C_\gamma(X)$, for some $k, m \in \mathbb{N}$, there exist functions $g_1, \ldots, g_k \in C_\alpha(X)$ and $\psi \in C^*(\mathbb{R}^{k+m})$, such that

$$h \restriction X = \psi \circ [(\triangle_{i=1}^k g_i) \triangle (\triangle_{i=1}^m f_i)].$$

Choose $n \in \mathbb{N}_0$ such that $n \geq m$. For $\langle z_1, \ldots, z_{k+n} \rangle \in \mathbb{R}^{k+n}$, define

$$\psi^*(\langle z_1, \ldots, z_{k+n} \rangle) = \psi(\langle z_1, \ldots z_{k+m} \rangle).$$

The function $\psi^* \in C^*(\mathbb{R}^{k+n})$ is such that

$$h \restriction X = \psi^* \circ [(\triangle_{i=1}^k g_i) \triangle (\triangle_{i=1}^n f_i)].$$

Then

$$h \restriction Y_n = \psi^* \circ [(\triangle_{i=1}^k g_i^\gamma) \triangle (\triangle_{i=1}^n f_i^\gamma)] \restriction Y_n.$$

The functions g_i are constant on Y_n. Take any $z \in Y_n$ and, for $\langle z_1, \ldots z_n \rangle \in \mathbb{R}^n$, define

$$\phi^*(\langle z_1, \ldots, z_n \rangle) = \psi^*(\langle g_1(z), \ldots, g_k(z), z_1, \ldots, z_n \rangle).$$

We have $\phi^* \in C^*(\mathbb{R}^n)$ and

$$h \restriction Y_n = \phi^* \circ \triangle_{i=1}^n f_i^\gamma \restriction Y_n.$$

This implies that g_n^* is of the form $\phi \circ \triangle_{i=1}^n f_i^\gamma \restriction Y_n$ for some $\phi \in C^*(\mathbb{R}^n)$ because $h \restriction Y_n = \frac{1}{2^n} g_n^*$. But this is impossible. The contradiction obtained proves that $C_\alpha(X) \cup \{f_1, \ldots, f_n\}$ generates $C_\gamma(X)$ for some $n \in \mathbb{N}$. In consequence, $\varepsilon_{\gamma\alpha}(X)$ is finite. □

We do not know if $m_{\gamma\alpha}(X)$ must be countable when $\varepsilon_{\gamma\alpha}(X)$ is finite.

Our next theorem is related to the theory of dimension. We refer the reader to [8] for more information about dimensions.

9. Theorem *Let αX and γX be compactifications of X such that $\alpha X \leq \gamma X$. If αX is metrizable and finite-dimensional, then the following conditions are equivalent:*

(i) $m_{\gamma\alpha}(X)$ *is finite;*

(ii) $m_{\gamma\alpha}(X)$ *is countable;*

(iii) $\varepsilon_{\gamma\alpha}(X)$ *is finite;*

(iv) γX *is metrizable and finite-dimensional.*

Proof: Implication $(i) \Rightarrow (ii)$ is obvious. That (ii) implies (iii) is a consequence of Theorem 8. Implication $(iii) \Rightarrow (iv)$ follows from Proposition 1.

Assume (iv). Then $\varepsilon(\gamma X)$ is finite; hence, we can choose a finite set $F = \{f_1, \ldots, f_n\} \subseteq C_\gamma(X)$ which generates γX. Then every function $h \in C_\gamma(X)$ is of the form $h = \phi \circ \triangle_{i=1}^n f_i$ for some $\phi \in C^*(\mathbb{R}^n)$ (cf. [2] and [14]), which implies that $M^n(F) = C_\gamma(X)$. Accordingly, $M(C_\alpha(X) \cup F) = C_\gamma(X)$ and thus $(iv) \Rightarrow (i)$.

Remark. Let X be an arbitrary locally compact non-pseudocompact space. Then the Hilbert cube I^ω is the remainder of a compactification γX of X (cf. [12]). Obviously, for the one-point compactification ωX of X, we have $\varepsilon_{\gamma\omega}(X) = \omega$. Hence, if X is second countable and finite-dimensional, then ωX is metrizable and finite-dimensional, but γX is a metrizable space which is not finite-dimensional. This shows that, in condition (ii) of Theorem 9, the cardinal number $m_{\gamma\alpha}(X)$ cannot be replaced by $\varepsilon_{\gamma\alpha}(X)$.

Example 4.8 of [1] shows that it may happen that $\varepsilon(\gamma X)$ is finite, while $\varepsilon(\alpha X)$ is infinite for some $\alpha X \leq \gamma X$. To state a more general fact, we will need the following lemma:

10. Lemma *For every metrizable compactification αX of a metrizable separable finite-dimensional space X, there exists a metrizable compactification γX of X such that $\alpha X \leq \gamma X$ and γX preserves the dimension of X.*

Proof: In view of Theorem 3.3 of [16], there exists a compatible totally bounded metric d on X such that αX is generated by the collection $U_d^*(X)$ of all bounded uniformly continuous with respect to d functions $f : X \to \mathbb{R}$. By Theorem 1.7.2 of [8], there exists a compatible totally bounded metric $\tilde{d}$ on X such that $d(x,y) \leq \tilde{d}(x,y)$ for any $x, y \in X$, and the metric completion γX of the metric space $(X, \tilde{d})$ is a compactification of X which preserves the dimension of X. Making use of Theorem 3.3 of [16] once again, we deduce that γX is generated by the collection $U_{\tilde{d}}^*(X)$ of all bounded uniformly continuous with respect to $\tilde{d}$ functions $f : X \to \mathbb{R}$. Since $d(x,y) \leq \tilde{d}(x,y)$ for any $x, y \in X$, we have $U_d^*(X) \subseteq U_{\tilde{d}}^*(X)$. This implies that $\alpha X \leq \gamma X$. □

11. Theorem *For every metrizable compactification δX of a second countable finite-dimensional non-compact Tikhonov space X, there exist metrizable compactifications αX and γX of X, such that $\delta X \leq \alpha X \leq \gamma X$, the cardinal number $\varepsilon(\gamma X)$ is finite, while $\varepsilon(\alpha X)$ is infinite. Furthermore, we may demand that the compactification γX should preserve the dimension of X.*

Proof: Take a point $y_0 \in \delta X \setminus X$ and put $Y = \delta X \setminus \{y_0\}$. Since the space Y is locally compact and non-pseudocompact, every metric continuum is a remainder of Y (cf. [12]). In consequence, there exists a compactification αY of Y such that $\alpha Y \setminus Y = I^\omega$. The space X being dense in Y, we can consider αY as a compactification αX of X. Since $X \subseteq Y$ and δX is the one-point compactification of Y, we have $\delta X \leq \alpha X$. As the space Y is locally compact and second countable, the space αX is also metrizable. Of course, the dimension of αX is infinite, which implies that $\varepsilon(\alpha X) = \omega$. It follows from Lemma 10 that there exists a metrizable compactification γX of X such that $\alpha X \leq \gamma X$ and γX preserves the dimension of X. Then $\varepsilon(\gamma X)$ is finite. □

For compactifications αX and γX of X such that $\alpha X \leq \gamma X$, G. D. Faulkner considered in [9] conditions under which $C_\gamma(X) = \langle C_\alpha(X) \cup \{f\}\rangle$. He proved that if X is locally compact and the set $\bigcup\{\pi_{\gamma\alpha}^{-1}(y) : y \in P_{\gamma\alpha}\}$ is finite, then there exists $f \in C^*(X)$ such that $C_\gamma(X) = \langle C_\alpha(X) \cup \{f\}\rangle$. Moreover, if $C_\gamma(X) = \langle C_\alpha(X) \cup \{f\}\rangle$, then all the fibres of $\pi_{\gamma\alpha}$ are finite. However, he did not know if the set $P_{\gamma\alpha}$ must be finite if $C_\gamma(X) = \langle C_\alpha(X) \cup \{f\}\rangle$. The following theorem gives an answer to Faulkner's question:

12. Theorem *For every cardinal number κ, there exist a locally compact space X and a compactification αX of X, such that $|P_{\beta\alpha}| = \kappa$ and*

$$C^*(X) = \langle C_\alpha(X) \cup \{f\}\rangle$$

for some $f \in C^*(X)$.

Proof: In view of Theorem 3 of [9], it suffices to consider the case when κ is infinite.

For an infinite cardinal number κ, denote by $\omega\mathbb{D}$ the one-point compactification of the discrete space $\mathbb{D}$ of cardinality κ. There exists a locally compact space Y such that $\beta Y \setminus Y = \omega\mathbb{D}$ (cf. [5; Coroll. 4.18]). Let $X = Y \times \{0,1\}$. Then $\beta X = \beta Y \times \{0,1\}$. Denote by αX the compactification of X which arises from βX by identifying each one of the sets $\{(d,0),(d,1)\}$ with a point p_d where $d \in \omega\mathbb{D}$. Then $\pi_{\beta\alpha}^{-1}(p_d) = \{(d,0),(d,1)\}$ for each $p_d \in \alpha X \setminus X$; hence $|P_{\beta\alpha}| = \kappa$. Let us define $f(y,i) = i$ for $y \in Y$ and $i = 0,1$. We shall show that $C^*(X) = \langle C_\alpha(X) \cup \{f\}\rangle$. To this end, for a function $h \in C^*(X)$, let us put $g_0(y,i) = h(y,0)$ and $g_1(y,i) = h(y,1)$ where $y \in Y$ and $i = 0,1$. Then $g_0, g_1 \in C_\alpha(X)$ and, moreover, $h = g_0 + fg_1 - fg_0$, so that $h \in \langle C_\alpha(X) \cup \{f\}\rangle$. This completes the proof. □

Finally, let us notice that if, for compactifications αX and γX of X, there exists a countable set $F \subseteq C_\gamma(X)$ such that $C_\gamma(X) = \langle C_\alpha(X) \cup F \rangle$, then $\alpha X \leq \gamma X$ and the number $m_{\gamma\alpha}(X)$ is countable; hence $\varepsilon_{\gamma\alpha}(X)$ is finite.

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