Topology Proceedings

Web: http://topology.auburn.edu/tp/
Mail: Topology Proceedings
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E-mail: topolog@auburn.edu
ISSN: 0146-4124

TOPOLOGY
PROCEEDINGS
Volume 22, 1997

COUNTABLY FAN-TIGHT SUBSPACES OF A COUNTABLE PRODUCT OF LASNEV SPACES ARE METRIZABLE

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Abstract

We show that a subspace of a countable product of Lašnev spaces is metrizable if and only if it has countable fan-tightness, which gives an affirmative answer to Arhangel'skii and Bella's question.

1 Introduction

A space is called a *Lašnev space* if it is the image of a metric space under a closed continuous map.

*This work was done while the first author was visiting Yokohama National University

A space X is *Fréchet* if for any subset A of X and a point $p \in X$ with $p \in \mathrm{cl}A$, there exists a sequence in A which converges to p. A space X has *countable tightness* if for any $A \subset X$ and $p \in X$ with $p \in \mathrm{cl}A$, there is a countable subset B of A with $p \in \mathrm{cl}B$. A space X has *countable fan-tightness* if for any sequence $\{A_i : i \in \omega\}$ of subsets of X satisfying $x \in \bigcap_{i\in\omega}\mathrm{cl}A_i$, there is a sequence $\{K_i : i \in \omega\}$ of finite sets satisfying that $K_i \subset A_i$ for any $i \in \omega$, and $x \in \mathrm{cl}(\bigcup_{i\in\omega}K_i)$. The notion of countable fan-tightness was introduced in [A] during the investigation of the topological properties of function spaces with the topology of pointwise convergence. It was shown that $C_p(X)$ has countable fan-tightness if and only if X^n is a Hurewicz space for any natural number n.

The following theorems are known:

Theorem 1 [T] *A subspace of the product of countably many Lašnev space is a Lašnev space if and only if it is Fréchet.*

Theorem 2 [AB] *A Lašnev space is metrizable if and only if it has countable fan-tightness.*

In this paper, we show the following theorem, which gives a positive answer to Question 4 in [AB]:

Theorem 3 *A subspace of the product of countably many Lašnev spaces is metrizable if and only if it has countable fan-tightness.*

Recall that a family $\{A_\lambda : \lambda \in \Lambda\}$ of subsets of a space X is hereditarily closure preserving if whenever $B_\lambda \subset A_\lambda$ for any $\lambda \in \Lambda$, we have $\mathrm{cl}(\bigcup\{B_\lambda : \lambda \in \Lambda\}) = \bigcup\{\mathrm{cl}B_\lambda : \lambda \in \Lambda\}$.

All spaces are assumed to be regular T_1-spaces.

2 Proof

To show Theorem 3, we need the following lemmas:

Lemma 4 *Let X be a countably fan-tight subspace of a product $Y \times Z$ of spaces and let $\pi : X \to Y$ be the projection. Let $p = (p_0, p_1) \in X$. Suppose that $\mathcal{S}$ is a hereditarily closure preserving family of subsets of Y such that $p_0 \notin S$ for any $S \in \mathcal{S}$. The following properties hold:*

(i) *There are only finitely many $S \in \mathcal{S}$ such that $p \in \mathrm{cl}(\pi^{-1}S)$.*

(ii) *If $\mathcal{S}$ is countable and $p \notin \mathrm{cl}(\pi^{-1}S)$ for any $S \in \mathcal{S}$, then $p \notin \mathrm{cl}(\pi^{-1}(\cup\mathcal{S}))$.*

Proof: (i): Suppose not. Take a countable subfamily $\{S_i\}_{i\in\omega}$ of $\mathcal{S}$, such that $p \in \cap_{i\in\omega}\mathrm{cl}(\pi^{-1}S_i)$. Since X has countable fan-tightness, there is a finite set $K_i \subset \pi^{-1}S_i$ for each $i \in \omega$ satisfying $p \in \mathrm{cl}(\cup_{i\in\omega}K_i)$. Note that each $\pi(K_i)$ is a finite set (hence a closed set) of Y included in S_i. Since $\{S_i\}_{i\in\omega}$ is hereditarily closure preserving, we have that $\pi(\cup_{i\in\omega}K_i)$ is a closed set of Y which does not contain p_0, which contradicts the fact that $p \in \mathrm{cl}(\cup_{i\in\omega}K_i)$.

(ii): Assume the contrary and suppose that $\mathcal{S} = \{S_i\}_{i\in\omega}$ and $S = \cup_{i\in\omega}S_i$ satisfy the following:

(1) $p \notin \mathrm{cl}(\pi^{-1}S_i)$ for any $i \in \omega$, and

(2) $p \in \mathrm{cl}(\pi^{-1}S)$.

For any $i \in \omega$, let $B_i = \pi^{-1}(S \setminus \cup_{j\leq i}S_j)$. It follows from (1) and (2) that $p \in \cap_{i\in\omega}\mathrm{cl}B_i$. Since X has countable fan-tightness, there is a finite set $K_i \subset B_i$ for each $i \in \omega$ satisfying $p \in \mathrm{cl}(\cup_{i\in\omega}K_i)$. Let $K = \cup_{i\in\omega}K_i$. Note that $\pi(K) \cap S_i$ is a finite set (hence is a closed set), because $\pi(K_j) \cap S_i = \emptyset$ if $i \leq j$. Since $\mathcal{S}$ is hereditarily closure preserving, we have that $\pi(K)$ is a closed set of Y which does not contain p_0, contradicting the condition $p \in \mathrm{cl}(K)$. □

Lemma 5 *Let X be a separable subspace of a product $Y \times Z$ of spaces and let $\pi : X \to Y$ be the projection. Assume that*

Y is a Lašnev space and X has countable fan-tightness. If $p = (p_0, p_1) \in X$, then there is a countable family $\mathcal{E}$ of subsets of Y which forms a local network at p_0 in Y satisfying that $\pi^{-1}(E)$ is a neighborhood of p in X for each $E \in \mathcal{E}$.

Proof: By replacing Y by $f(X)$, without loss of generality, we may assume that Y is a separable Lašnev space. Hence there is a closed map $f : M \to Y$ from a separable metrizable space M onto Y. Let $F = f^{-1}(p_0)$.
Now define

$H = \{x \in M : p \in \text{cl}(\pi^{-1} f(U \setminus F))$ for any neighborhood U of x in $M\}$.

Observe that H is closed in M and $H \subset F$. We show that H is compact. Suppose not. Then there is a countable, discrete family $\mathcal{U} = \{U_i : i \in \omega\}$ of open sets such that $U_i \cap H \neq \emptyset$ for any $i \in \omega$. For each $i \in \omega$, let $S_i = f(U_i \setminus F)$. Then $p \in \text{cl}(\pi^{-1} S_i)$ for any $i \in \omega$. Since $\{S_i\}_{i \in \omega}$ is the image of a locally finite family under a closed map, it is hereditarily closure preserving. This contradicts Lemma 4 (i). Thus H is compact.

To complete the proof of Lemma 5, we consider two cases.

Case 1. $H = \emptyset$.

In this case, we have $p \notin \text{cl}(\pi^{-1}(Y \setminus \{p_0\}))$. Indeed, since $H = \emptyset$ and M is separable metrizable, there is a countable, locally finite open cover $\mathcal{V} = \{V_i : i \in \omega\}$ of M such that $p \notin \text{cl}\pi^{-1} f(V_i \setminus F)$ for each $i \in \omega$. Apply Lemma 4 (ii) by putting $S_i = f(V_i \setminus F)$. Then we have $p \notin \text{cl}(\pi^{-1}(\cup_{i \in \omega} f(V_i \setminus F))) = \text{cl}(\pi^{-1}(Y \setminus \{p_0\}))$. Let $\mathcal{E} = \{\{p_0\}\}$, which consists of one set consisting of one element. Trivially this works.

Case 2. $H \neq \emptyset$.

Let $\mathcal{W} = \{W_i : i \in \omega\}$ be a countable outer neighborhood base of the compact set H of a metrizable space M. Then it is easy to see that $\mathcal{E} = \{f(W) : W \in \mathcal{W}\}$ is a local network of Y at p_0. We show that $\pi^{-1}(E)$ is a neighborhood of p in X. It suffices to show that if W is a neighborhood of H in M, then $p \notin \mathrm{cl}\pi^{-1}f(M \setminus (F \cup W))$, which implies that $\pi^{-1}f(W)$ is a neighborhood of p in X because $Y = f(W) \cup f(M \setminus (F \cup W))$. It follows from $H \subset W$ that $M \setminus W$ is a closed set of M which misses H. Hence there is a countable, locally finite family $\mathcal{V}' = \{V_i'\}_{i \in \omega}$ of open sets of M such that $p \notin \mathrm{cl}\pi^{-1}f(V_i' \setminus F)$ for any $i \in \omega$ and $\cup\mathcal{V}' \supset M \setminus W$. Apply Lemma 4 (ii) by putting $S_i = f(V_i' \setminus F)$. Then we have $p \notin \mathrm{cl}\pi^{-1}f(\cup_{i\in\omega}(V_i' \setminus F))$. Hence $p \notin \mathrm{cl}\pi^{-1}f(M \setminus (F \cup W))$, which completes the proof of Lemma 5. □

Now we are in a position to prove our theorem:

Proof of Theorem 3: Let X be a countably fan-tight subspace of $\prod_{n\in\omega} Y_n$, where each Y_n, $n \in \omega$, is a Lašnev space. We show that X is metrizable. By Theorem 1 and Theorem 2, it suffices to show that X is Fréchet. Since every space with a countable fan-tightness has countable tightness, we need only show that every countable subspace C of X is first countable. Replaceing C by X, we may assume that X is countable. Let $p = (p_i)_{i\in\omega} \in X$, and let $\pi_n : X \to Y_n$ be the projection for each $n \in \omega$. By Lemma 5, for each $n \in \omega$, there is a countable family $\mathcal{E}_n$ of subsets of Y_n which forms a local network at p_n in Y_n satisfying that $\pi_n^{-1}(E)$ is a neighborhood of p in X for each $E \in \mathcal{E}_n$. Define $\mathcal{B}_n = \{\pi_n^{-1}(E) : E \in \mathcal{E}_n\}$, and $\mathcal{B} = \cup_{n\in\omega}\mathcal{B}_n$. Let $\mathcal{B}'$ be the set of all finite intersections of $\mathcal{B}$. Now it is easy to check that $\mathcal{B}'$ is a neighborhood base of X at p, which completes the proof of Theorem 3. □

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