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NEAR METRIC PROPERTIES OF HYPERSPACES

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Abstract

Near metric properties of the hyperspace of closed, compact and finite subsets of a space X are examined. In particular, the properties of monotone normality and stratifiability are investigated.

1 Introduction

This paper aims to examine when a hyperspace of a topological space X possesses certain general metric properties. Given a space X, which we will henceforth assume to be Tychonoff, we define a hyperspace of X to be one of the following:

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$\mathcal{H}(X) = \{A \subseteq X : A$ is non–empty and closed in $X\}$,
$\mathcal{K}(X) = \{A \in \mathcal{H}(X) : A$ is compact$\}$ and
$\mathcal{F}(X) = \{A \in \mathcal{H}(X) : A$ is finite$\}$.

The set $\mathcal{H}(X)$ is given the Vietoris topology, that is, the topology generated by sets of the form $[U_1 \ldots U_n] = \{A \in \mathcal{H}(X) : A \subseteq \bigcup U_i$ and $A \cap U_i \neq \emptyset\}$, where each U_i is a non–empty open subset of X. The subsets $\mathcal{K}(X)$ and $\mathcal{F}(X)$ of $\mathcal{H}(X)$ are given the subspace topology. The map $x \mapsto \{x\}$ embeds X as a closed subspace of $\mathcal{H}(X)$.

We determine when the hyperspaces $\mathcal{H}(X), \mathcal{K}(X), \mathcal{F}(X)$ are monotonically normal, stratifiable or cosmic (definitions are given below). The case of $\mathcal{H}(X)$ is brief — X must be compact and metrisable. On the other hand, properties of $\mathcal{F}(X)$ will be seen to mirror those of (finite powers of) the space X. The situation for $\mathcal{K}(X)$ is murkier. We essentially show that $\mathcal{K}(X)$ is monotonically normal if and only if either $\mathcal{K}(X) = \mathcal{F}(X)$ or $\mathcal{K}(X)$ is stratifiable. In addition, we give a consistent and independent criterion for $\mathcal{K}(X)$ to be cosmic. From this follows a consistent and independent criterion for a space to be separable metrisable. To achieve this we additionally present some results on function spaces in the compact–open and the pointwise topologies.

A space X is monotonically normal (MN) if, for each pair (A, B) of disjoint closed subsets of X, there is an open subset $H(A, B)$ of X such that

(i) $A \subseteq H(A, B) \subseteq \overline{H(A, B)} \subseteq X \backslash B$
(ii) $A \subseteq A'$ and $B' \subseteq B$ implies $H(A, B) \subseteq H(A', B')$.

It will be convenient, however, to use a "local" characterisation of MN:

for every point x and open neighbourhood U of x, there is an open neighbourhood $V(x, U)$ of x such that $V(x, U) \cap V(x', U') \neq \emptyset \Rightarrow x \in U'$ or $x' \in U$.

Without loss of generality, we can assume that $V(x, U) \subseteq U$ and that $V(x, U) \subseteq V(x, U')$ for $U \subseteq U'$. It is sufficient for V

to be defined for members of a base for X. Monotone normality is an hereditary property and is preserved by closed maps.

Stratifiability can be thought of as "monotone perfect normality". A space X is stratifiable if, for each closed subset A of X, and natural number n, there is an open subset $G(A,n)$ of X containing A such that

(i) $G(A,m) \subseteq G(B,n)$ whenever $A \subseteq B$ and $m \geq n$

(ii) $A = \bigcap_n G(A,n)$ and (iii) $A = \bigcap_n \overline{G(A,n)}$.

A space is stratifiable if and only if it is both MN and a σ–space (possesses a σ–discrete network). Recall that a space is cosmic if it possesses a countable network. Then a space is separable and stratifiable if and only if it is both monotonically normal and cosmic. A result of Heath's, which we shall use in the sequel, states that if S is a space with a countable limit point, then $X \times S$ is MN only if X is stratifiable.

The reader is referred to [6] for a survey of these and other generalised metric properties. References for all the facts quoted above will be found there.

2 The Space of Closed Subsets

An instance of a theorem of Fedorchuk (Theorem 4.20) in [7] states that a compact Hausdorff space X is metrisable if and only if $\mathcal{H}(X)$ is hereditarily normal. Since normality of $\mathcal{H}(X)$ implies the compactness of X (in fact, the two are equivalent, as proved by Veličko in [14]), we have the result proved directly in [2] :

Theorem 2.1 (Brandsma and van Mill) *The hyperspace $\mathcal{H}(X)$ is MN if and only if X is compact metrisable.*

For the same reason, $\mathcal{H}(X)$ is cosmic or stratifiable if and only if X is compact metrisable. Thus, in this respect, $\mathcal{H}(X)$ is "too large" to possess interesting near metric properties without it being compact and metrisable.

3 The Space of Finite Subsets

In contrast with the case for $\mathcal{H}(X)$, properties of $\mathcal{F}(X)$ are much closer to those of X. We will use two main facts:

(1) that $\mathcal{F}(X)$ is the (countable) union of its closed subspaces $\mathcal{F}_n(X)$, for $n \in \omega$, where $\mathcal{F}_n(X) = \{A \in \mathcal{F}(X) : |A| \leqslant n\}$.

(2) that the mapping $\pi_n : X^n \to \mathcal{F}_n(X)$ defined by $\pi_n(x_1 \ldots x_n) = \{x_1, \ldots, x_n\}$ is a closed, continuous surjection (and therefore transfers a "point and open set" MN operator on X^n to one on $\mathcal{F}_n(X)$ in the standard way).

Theorem 3.1 *Let X be a space. Then the following are equivalent:*

(1) *X^2 is MN*
(2) *X^n is MN for all $n \in \omega$*
(3) *$\mathcal{F}(X)$ is MN.*

Proof: The equivalence of (1) and (2) is shown by Gartside in [3], who proves the more general result that the product of a finite collection of spaces is MN, if the product of any pair is MN. In the above case, let us suppose that X^2 has MN operator V^2, and write

$V^2((x,y), U_x \times U_y) = V_x^2((x,y), U_x \times U_y) \times V_y^2((x,y), U_x \times U_y)$.

We may suppose that V^2 is symmetric in the sense that $V_x^2((x,y), U_x \times U_y) = V_x^2((y,x), U_y \times U_x)$. (This can be achieved by re-defining $V^2((x,y), U_x \times U_y)$ as $V^2((x,y), U_x \times U_y) \cap V^2((y,x), U_y \times U_x)^{-1}$).

Then $V^n((x_1 \ldots x_n), U_1 \times \ldots \times U_n) =$

$\bigcap_{i=1}^n V_{x_1}^2((x_1, x_i), U_1 \times U_i) \times \ldots \times \bigcap_{i=1}^n V_{x_n}^2((x_n, x_i), U_n \times U_i)$
is an MN operator on X^n, as required.

(1)$\Rightarrow$(3) By the above, we have an MN operator V^n, defined on each X^n, and by fact (2), each $\mathcal{F}_n(X)$ is therefore MN with an MN operator V_n. We define what we shall show is an MN operator on $\mathcal{F}(X)$ by

$V_\omega(\{x_1 \dots x_n\}, [U_1 \dots U_n]) =$
$\left[\bigcap_{i=1}^n V_{x_1}^2((x_1, x_i), U_1 \times U_i), \dots, \bigcap_{i=1}^n V_{x_n}^2((x_n, x_i), U_n \times U_i)\right]$
where $[U_1 \dots U_n]$ is a basic open set containing $\{x_1 \dots x_n\}$, and the U_i are pairwise disjoint with $x_i \in U_i$.

We prove that V_ω is an MN operator by comparing V_n and the restriction of V_ω to $\mathcal{F}_n(X)$; specifically, we check that, for $k \leqslant n$,

$$V_\omega(\{x_1 \dots x_k\}, [U_1 \dots U_k]) \cap \mathcal{F}_n(X) \subseteq V_n(\{x_1 \dots x_k\}, [U_1 \dots U_k]). \qquad (*)$$

For suppose that$(*)$ holds, and that
$V_\omega(\{x_1 \dots x_k\}, [U_1 \dots U_k]) \cap V_\omega(\{y_1 \dots y_m\}, [W_1 \dots W_m]) \neq \varnothing$. Then this non–empty intersection is witnessed by a point in $\mathcal{F}_n(X)$, where $n \geqslant \max(k, m)$ (the inequality can be strict). By $(*)$,
$V_n(\{x_1 \dots x_k\}, [U_1 \dots U_k]) \cap V_n(\{y_1 \dots y_m\}, [W_1 \dots W_m]) \neq \varnothing$. Since we know that V_n is an MN operator on $\mathcal{F}_n(X)$, either

$$\{x_1 \dots x_k\} \in [W_1 \dots W_m] \quad \text{or} \quad \{y_1 \dots y_m\} \in [U_1 \dots U_k].$$

Hence V_ω is indeed an MN operator on $\mathcal{F}(X)$.

So, it remains to check that $(*)$ holds. Now $V_n(\{x_1 \dots x_k\}, [U_1 \dots U_k])$ is defined to be $\pi_n^\#\Big(\bigcup_{(a_1 \dots a_n) \in \pi_n^{-1}(\{x_1 \dots x_n\}} V^n((a_1 \dots a_n), \pi_n^{-1}([U_1 \dots U_n]))\Big)$ (where $\pi_n^\#(S) = \{F \in \mathcal{F}_n(X) : \pi_n^{-1}F \subseteq S\}$). Let $\{z_1 \dots z_m\} \in V_\omega(\{x_1 \dots x_k\}, [U_1 \dots U_k]) \cap \mathcal{F}_n(X)$ (so $k \leqslant m \leqslant n$). Let $(b_1 \dots b_n) \in \pi_n^{-1}(\{z_1 \dots z_k\})$ (so every b is some z, and every z is some b).

We want to show that $(b_1 \dots b_n) \in V^n((a_1 \dots a_n), \pi_n^{-1}([U_1 \dots U_n]))$ for some $(a_1 \dots a_n) \in \pi_n^{-1}(\{x_1 \dots x_n\})$.

Each z is in one set of the form $\bigcap_{i=1}^k V_{x_j}^2((x_j, x_i), U_j \times U_i)$ (these sets are disjoint, since the U_i were assumed to be), and for every j, $1 \leqslant j \leqslant k$, some z (possibly more than one) is in $\bigcap_{i=1}^k V_{x_j}^2((x_j, x_i), U_j \times U_i)$.

Choose a_l to be the x_j such that $b_l \in \bigcap_{i=1}^k V_{x_j}^2((x_j, x_i), U_j \times U_i)$ for $1 \leqslant l \leqslant n$. Then $\{a_1 \dots a_n\} = \{x_1 \dots x_k\}$, so $(a_1 \dots a_n) \in$

$\pi_n^{-1}(\{x_1 \dots x_k\})$ as required.

Denoting by p_{a_j} the index such that $a_j = x_{p_{a_j}}$, we have that

$$b_j \in \bigcap_{i=1}^k V_{a_j}^2((a_j, a_i), U_{p_{a_j}} \times U_{p_{a_i}}) = \bigcap_{i=1}^n V_{a_j}^2((a_j, a_i), U_{p_{a_j}} \times U_{p_{a_i}}).$$

Finally, looking at the definition of V^n in terms of V^2, we can see that $(b_1 \dots b_n) \in V^n((a_1 \dots a_n), U_{p_{a_1}} \times \dots \times U_{p_{a_n}}) \subseteq V^n((a_1 \dots a_n), \pi_n^{-1}([U_1 \dots U_k]))$. Thus we have proved $(*)$, and the proof is complete.

(3)$\Rightarrow$(1) We will use only the fact that $\mathcal{F}_2(X)$ has an MN operator V. Write, for $x \neq y$,

$V(\{x,y\},[U_x,U_y]) = \Big[V_x(\{x,y\},[U_x,U_y]), V_y(\{x,y\},[U_x,U_y])\Big]$,

where $x \in V_x(\{x,y\},[U_x,U_y]) \subseteq U_x$ and similarly for y; and

$$V(\{x\},[U_x]) = [V(x,U_x)].$$

Then define V^2 by

$$V^2((x,y), U_x \times U_y) = \Big(V_x(\{x,y\},[U_x,U_y]) \cap V(x,U_x)\Big) \times \Big(V_y(\{x,y\},[U_x,U_y]) \cap V(y,U_y)\Big),$$

where U_x and U_y are disjoint if $x \neq y$, and equal if $x = y$. It can easily be checked that V^2 is an MN operator on X^2. $\square$

Following immediately from this is a result of [11]:

Corollary 3.2 (Mizokami and Koiwa) *X is stratifiable if and only if $\mathcal{F}(X)$ is stratifiable.*

Proof: The reverse implication is obvious. For the direct implication, let X be stratifiable. Then every finite power of X is stratifiable and therefore MN, thus $\mathcal{F}(X)$ is MN. Moreover, each finite power of X is, since stratifiable, a σ–space. Then $\mathcal{F}(X)$ is the countable union of closed σ–spaces, and therefore a σ–space. Hence $\mathcal{F}(X)$ is stratifiable. $\square$

Corollary 3.3 *If $\mathcal{F}(X)$ is MN then $\mathcal{F}(X)^n$ is MN for every $n \in \omega$.*

Proof: Observe that $\mathcal{F}(X)^n$ embeds in $\mathcal{F}(X \oplus \ldots \oplus X)$. If $\mathcal{F}(X)$ is MN, then X^2 is MN by 3.1, and therefore so is $(X \oplus \ldots \oplus X)^2$. By 3.1 again, $\mathcal{F}(X \oplus \ldots \oplus X)$ is MN, and hence $\mathcal{F}(X)^n$ is MN. □

For comparison we have the following example [3]:

Example 3.4 *There is a space X such that all finite powers of X are MN, but X is not linearly stratifiable. Thus $\mathcal{F}(X)$ is MN but not (linearly) stratifiable.*

(A space is κ–stratifiable if it is stratifiable as defined above, but where the open sets are indexed by a cardinal κ instead of ω; and linearly stratifiable if it is κ–stratifiable for some infinite cardinal κ. Every linearly stratifiable space is MN.)

4 The Space of Compact Subsets

If a space X contains no infinite compact subsets, then $\mathcal{K}(X) = \mathcal{F}(X)$, and we are back in the situation of the preceding section. The first result of this section says that if $\mathcal{K}(X) \neq \mathcal{F}(X)$ and $\mathcal{K}(X)$ is MN, then X is stratifiable. This suggests that if $\mathcal{K}(X) \neq \mathcal{F}(X)$ and $\mathcal{K}(X)$ is MN, then $\mathcal{K}(X)$ should be stratifiable. The second and third results of this section show that this conjecture is true under additional assumptions.

Proposition 4.1 *If $\mathcal{K}(X) \neq \mathcal{F}(X)$ and $\mathcal{K}(X)$ is MN, then X is stratifiable.*

Proof: Since $\mathcal{K}(X)$ is MN, so too is $\mathcal{F}(X)$, and all finite powers of X are MN. As $\mathcal{K}(X) \neq \mathcal{F}(X)$, there is a countably infinite non–discrete subset S of X. Now $X \times S$ is MN (as a subspace of X^2), and X is stratifiable by Heath's result. □

Call a space X non–trivial if every non–empty open subset of X contains an infinite compact subset. When $\mathcal{K}(X)$

is MN, as stratifiable compact spaces are metrisable, this is equivalent to saying: every non–empty open subset of X contains a convergent sequence, $x_n \to x_\omega$, such that $x_\alpha \neq x_\beta$ for $\alpha < \beta < \omega + 1$.

Theorem 4.2 *Let X be non–trivial, and $\mathcal{K}(X)$ MN. Then $\mathcal{K}(X)$ is stratifiable.*

Proof: Let us observe first that if X is compact, then $\mathcal{K}(X) = \mathcal{H}(X)$, and thus is compact and metrisable. Otherwise it will suffice to show that $\mathcal{K}(X)$ is a σ–space. If X is not compact, then, since X is stratifiable, there exists an infinite discrete family $\{U_n : n \in \omega\}$ of non–empty open subsets of X. For $n \in \omega$, define $\mathcal{K}_n = \{K \in \mathcal{K}(X) : K \cap U_n = \emptyset\}$.

Then, as the complement of the basic open set $[X, U_n]$ in $\mathcal{K}(X)$, each $\mathcal{K}_n$ is closed. Also, $\mathcal{K}(X) = \bigcup_n \mathcal{K}_n$ by the discreteness of $\{U_n : n \in \omega\}$.

Next we show that, for each n, $\mathcal{K}_n \times (\omega+1)$ embeds in $\mathcal{K}(X)$. By non-triviality of X, pick a sequence $(x^n_\alpha)_{\alpha<\omega+1}$ contained in U_n. Then it is straightforward to check that $A \times \{\alpha\} \mapsto A \cup \{x_\alpha\}$ defines an embedding of $\mathcal{K}_n \times (\omega + 1)$ into $\mathcal{K}(X)$.

Thus $\mathcal{K}_n \times (\omega + 1)$ is MN, hence $\mathcal{K}_n$ is stratifiable, and therefore is a σ–space. Thus $\mathcal{K}(X)$, as a countable union of closed σ–spaces, is a σ–space, as required. □

Note that the proof shows that, if $\mathcal{K}(X)$ is MN and X contains an infinite discrete family of open sets, each containing a convergent sequence, then $\mathcal{K}(X)$ is stratifiable.

For the next result, in the separable case, recall that space is $\aleph_0$ if it possesses a countable family $\mathcal{N}$ of subsets of X such that for every compact subset K of X contained in an open set U, there exists an $N \in \mathcal{N}$ such that $K \subseteq N \subseteq U$. It is a result of Ntantu (page 363 of [12]) that

Proposition 4.3 *$\mathcal{K}(X)$ is cosmic if and only if X is $\aleph_0$.*

Theorem 4.4 *Let $\mathcal{K}(X)$ be separable and MN, and suppose that X contains two disjoint convergent sequences S_1 and S_2. Then $\mathcal{K}(X)$ is stratifiable.*

Proof: It is sufficient to show that $\mathcal{K}(X)$ is cosmic. Since the S_i are disjoint compact sets in X, they are separated by open sets U_i. Define $\mathcal{K}_i = \mathcal{K}(X\backslash U_i)$. Then $\mathcal{K}_i \times (\omega+1)$ embeds in $\mathcal{K}(X)$ as above, thus $\mathcal{K}_i$ is separable and stratifiable. Hence it is cosmic, and so $X\backslash U_i$ is $\aleph_0$. Then $X = (X\backslash U_1) \cup (X\backslash U_2)$ is the union of two closed $\aleph_0$ subspaces and hence $\aleph_0$. Thus $\mathcal{K}(X)$ is cosmic. $\Box$

The following non-monotone version of Theorem 4.2 follows the lines of the classic proof by Katětov [8]:

Theorem 4.5 *Let X be non–trivial, and $\mathcal{K}(X)$ hereditarily normal. Then points of $\mathcal{K}(X)$ are G_δ.*

Proof: Pick $A \in \mathcal{K}(X)$ and suppose that $A \neq X$ (if $A = X$ then $\mathcal{K}(X)$ is compact and metrisable). Then, since X is normal, there exists a non–empty open subset of X with closure disjoint from A. Inside this closure, pick a non–trivial convergent sequence, $x_n \to x_\omega$, with $x_\alpha \neq x_\beta$ for $\alpha < \beta < \omega + 1$. Let $S = \{x_\alpha : \alpha < \omega + 1\}$.

Define $\mathcal{P} = \{A \cup F : F \subseteq [S\backslash\{x_\omega\}]^{<\omega}\}$ and $\mathcal{Q} = \{K \in \mathcal{K}(X) : K \cap S = \{x_\omega\}$ and $K\backslash\{x_\omega\} \neq A\}$.

Then $C \in \overline{\mathcal{Q}}$ implies $\{x_\omega\} \in C$ so $\mathcal{P} \cap \overline{\mathcal{Q}} = \emptyset$. Also, $C \in \overline{\mathcal{P}}$ implies $C = A \cup I$ for some $I \subseteq S$, so $\overline{\mathcal{P}} \cap \mathcal{Q} = \emptyset$.

So by hereditary normality, there exists an open set $\mathcal{G}$ in $\mathcal{K}(X)$ s.t. $\mathcal{P} \subseteq \mathcal{G}$ and $\mathcal{G} \cap \overline{\mathcal{Q}} = \emptyset$.

For $\mathcal{U} = [U_1 \dots U_m]$, with $A \cup \{x_n\} \in \mathcal{U} \subseteq \mathcal{G}$, satisfying $x_n \in U_m$, $U_m \cap A = \emptyset$, and $x_n \notin U_i$ for $i \neq m$, define $\mathcal{U}^{-n} = [U_1 \dots U_{m-1}]$. Then $A \in \mathcal{U}^{-n}$, and also $B \in \mathcal{U}^{-n} \Rightarrow B \cup \{x_n\} \in \mathcal{U}$, and $x_n \notin B$. For $n \in \omega$, define $\mathcal{G}_n = \bigcup\{\mathcal{U}^{-n} : \mathcal{U}$ as above$\}$. Then $\mathcal{G}_n$ is open and contains A.

Claim $\{A\} = \bigcap_n \mathcal{G}_n \cap (\mathcal{K}(X)\backslash\{A \cup \{x_\omega\}\})$

For Suppose $B \in \mathcal{G}_n \cap (\mathcal{K}(X)\setminus\{A \cup \{x_\omega\}\})$, $B \neq A$. Then $\forall n \in \omega$, $B \in \mathcal{G}_n$ so $B \cup \{x_n\} \in \mathcal{G}$, and $x_n \notin B$. Then $B \cup \{x_\omega\} \in \overline{\mathcal{G}}$. But $B \cup \{x_\omega\} \cap S = \{x_\omega\}$ and $(B \cup \{x_\omega\})\setminus\{x_\omega\} \neq A$ so $B \cup \{x_\omega\} \in \mathcal{Q}$ - contradicting $\overline{\mathcal{G}} \cap \mathcal{Q} = \emptyset$.

Hence points of $\mathcal{K}(X)$ are G_δ. $\square$

5 Interlude—Function Spaces

Let X and Y be spaces, and write $C(X,Y)$ for the set of all continuous functions of X into Y. We abbreviate $C(X,\mathbb{R})$ by $C(X)$. For $A \subseteq X$ and $U \subseteq Y$, define $B(A,U) = \{f \in C(X,Y) : f[A] \subseteq U\}$. Letting A range over finite (respectively, compact) subsets of X and U range over open subsets of Y, the $B(A,U)$ form a subbase for the topology of pointwise convergence (respectively, compact–open topology). Write $C_p(X,Y)$ for $C(X,Y)$ with the topology of pointwise convergence, and $C_k(X,Y)$ for $C(X,Y)$ with the compact–open topology.

We explain how the space $C_k(X,Y)$ may be related to the spaces $C_p(X',Y')$, $\mathcal{K}(X)$ and $\mathcal{K}(Y)$. Since the topology of pointwise convergence has been intensively studied, this provides a useful tool for investigating the compact–open topology.

Let f be a continuous function of X to Y. Lift f to a continuous function $\mathcal{K}f : \mathcal{K}(X) \to \mathcal{K}(Y)$ by defining $\mathcal{K}f(K) = f[K]$. This gives a map of $C_k(X,Y)$ into $C_p(\mathcal{K}(X),\mathcal{K}(Y))$. It is straightforward to check that this map is a (topological) embedding.

Proposition 5.1 *The map $f \mapsto \mathcal{K}f$ embeds $C_k(X,Y)$ into $C_p(\mathcal{K}(X),\mathcal{K}(Y))$.*

Let us apply Proposition 5.1 to cardinal invariants of $C_k(X)$. From Proposition 5.1, $C_k(X)(= C_k(X,\mathbb{R}))$ embeds in $C_p(\mathcal{K}(X), \mathcal{K}(\mathbb{R}))$. Now $\mathcal{K}(\mathbb{R})$ is a separable metric space, and so embeds in $\mathbb{R}^\omega$; also, $C_p(\mathcal{K}(X),\mathbb{R}^\omega) = C_p(\mathcal{K}(X))^\omega$. Hence, $C_k(X)^\omega$

embeds in $C_p(\mathcal{K}(X))^\omega$. The following result is a consequence of this last fact, and the results II.1.3, II.5.6 and II.5.10 in Arkhangel'skii's book [1] on C_p–theory.

Corollary 5.2 *If, for all $n \in \omega$, $\mathcal{K}(X)^n$ is:*
(1) Lindelöf, (2) hereditarily Lindelöf, (3) hereditarily separable, (4) hereditarily ccc,
then, for all $n \in \omega$, $C_k(X)^n$ is:
(1) countably tight, (2) hereditarily separable, (3) hereditarily Lindelöf, (4) hereditarily ccc (respectively).

Part (4) of the above result plays a vital role in the next section.

6 Cosmicity of $\mathcal{K}(X)$

This section examines when $\mathcal{K}(X)$ is cosmic, away from the monotone properties considered above. The following theorem requires the following extra–ZFC axiom:

The Open Colouring Axiom (OCA) *If $[X]^2 = K_0 \cup K_1$ is a given partition where $X \subseteq \mathbb{R}$ and where K_0 is open in $[X]^2$, then either there is an uncountable 0–homogeneous set, or else X is the union of countably many 1–homogeneous sets.*

For more details, and applications, of OCA, the reader is referred to [13]. We shall be content to note here that OCA follows from PFA, but that ZFC and (ZFC + OCA) are equiconsistent.

Condition (CK) was defined by Gartside and Reznichenko in [4]: A space X satisfies (CK) if there is a σ–compact subset Y of X such that for every compact subset K of X, there is a compact subset L satisfying $K \subseteq \overline{L \cap Y}$. It is used here through the following

Theorem 6.1 (Gartside and Reznichenko) *X has (CK) if and only if $C_k(X)$ is cometrisable.*

Above, a space Y is cometrisable if there is a coarser metric topology on Y, and for each point of Y a neighbourhood base of metric closed sets. Also for the following theorem and example, observe, as is well known, that a space Y has Y^ω hereditarily ccc if and only Y^n is hereditarily ccc for all $n \in \omega$. (A space is hereditarily ccc if it does not contain any uncountable discrete subspaces.)

Theorem 6.2 (OCA)

(i) *$\mathcal{K}(X)$ is cosmic if and only if $\mathcal{K}(X)^\omega$ is hereditarily ccc and X has (CK).*

(ii) *$\mathcal{K}(X)$ is separable metrisable if and only if $\mathcal{K}(X)^\omega$ is first countable and hereditarily ccc.*

Proof: Let us first suppose that $\mathcal{K}(X)$ is cosmic. Then X is $\aleph_0$ and therefore has (CK) as observed in [4]. Also, cosmicity of $\mathcal{K}(X)$ implies cosmicity of $\mathcal{K}(X)^\omega$, and therefore $\mathcal{K}(X)^\omega$ is hereditarily ccc. Conversely, suppose that $\mathcal{K}(X)^\omega$ is hereditarily ccc and X has (CK). Then $C_k(X)$ is cometrisable, and, by Corollary 5.2, is hereditarily ccc in all of its finite powers. By a theorem of Gruenhage [5], originally proved under PFA, but subsequently shown by Todorčević [13] to hold under OCA, $C_k(X)$ is cosmic, which is equivalent to $C_k(X)$ being $\aleph_0$, by a result of Michael in [10]. Then X is $\aleph_0$, and hence $\mathcal{K}(X)$ is cosmic.

Now suppose $\mathcal{K}(X)$ is separable metrisable, then clearly $\mathcal{K}(X)^\omega$ is first countable and hereditarily ccc. Conversely, if $\mathcal{K}(X)$ is first countable then, by Proposition 18 of [4], X has (CK), and, as above, $\mathcal{K}(X)$ is cosmic. Hence (Proposition 4.3) X is first countable and $\aleph_0$. But first countable $\aleph_0$ spaces are separable metrisable (see [6]). Finally, X is separable metrisable if and only if $\mathcal{K}(X)$ is separable metrisable. □

Example 6.3 *($\mathfrak{b} = \omega_1$) There is an uncountable subset X of the Sorgenfrey Line such that $\mathcal{K}(X)^\omega$ is hereditarily ccc and* *first countable (and hence X has (CK)), but $\mathcal{K}(X)$ and X are not cosmic.*

Using $\mathfrak{b} = \omega_1$, and observing that any discrete subspace is left separated, the space $X = A[\leqslant_{\text{lex}}]$ given by Todorčević in Theorem 3.0 of [13] is a subspace of the Sorgenfrey Line (with left–facing topology) of size ω_1 such that X^n is hereditarily ccc for all n. It is clear that X, and therefore $\mathcal{K}(X)$, does not possess a countable network.

Observe that the compact subsets of X are homeomorphic to countable compact ordinals, and so a basic open neighbourhood of an element A of $\mathcal{K}(X)$ is composed of pairwise disjoint basic open intervals in the Sorgenfrey Line, whose right–hand end–points are points of A. From this one can easily check that $\mathcal{K}(X)$ is first countable.

Suppose that $\mathcal{A}$ were an uncountable discrete subspace of X. Then for every $A \in \mathcal{A}$ there is an open $\mathcal{U}^A = [U_1^A \dots U_{n_A}^A]$ such that $\mathcal{U}^A \cap \mathcal{A} = \{A\}$. We may assume that each $\mathcal{U}^A$ is basic, and, by counting, that $n_A = n$ for all $A \in \mathcal{A}$. Let x_i^A be the first point of A in U_i^A, and y_i^A the corresponding right–hand end–point.

Then, setting $p(A) = (x_1^A, y_1^A, \dots, x_n^A, y_n^A)$, $P = \{p(A) : A \in \mathcal{A}\}$ is a subset of X^{2n}, each point with open neighbourhood $V^A = U_1^A \times U_1^A \times \dots \times U_n^A \times U_n^A$. If P is countable then distinct A, A' give the same points in X^{2n}, and hence $A \in \mathcal{U}^{A'}$. If P is uncountable, then since X^{2n} is hereditarily ccc, P cannot be discrete, so there exist distinct A, A' such that $p(A) \in V^{A'}$, and hence $A \in \mathcal{U}^{A'}$. This contradicts the discreteness of $\mathcal{A}$. Thus $\mathcal{K}(X)$ is hereditarily ccc. By a similar method, $\mathcal{K}(X)^n$ is hereditarily ccc for all $n \in \omega$. □

Corollary 6.4 *The statement:* 'A space X is separable metrisable if and only if $\mathcal{K}(X)^\omega$ is first countable and hereditarily ccc' *is consistent and independent of Set Theory.*

7 Open Questions

The key remaining questions seem to be the following.

Question 7.1 Does $\mathcal{K}(X) \neq \mathcal{F}(X)$, and $\mathcal{K}(X)$ MN, imply that $\mathcal{K}(X)$ is stratifiable?

Question 7.2 Does $\mathcal{K}(X) \neq \mathcal{F}(X)$, and $\mathcal{K}(X)$ hereditarily normal, imply that $\mathcal{K}(X)$ is perfectly normal? What if X is non–trivial?

Problem 7.3 Determine those spaces X such that $\mathcal{K}(X)$ is stratifiable.

Remark: The results of sections 3 and 4 were obtained independently by the first two and last two authors. Section 5 is joint work. The final section is due to the first two authors.

References

[1] A.V. Arkhangel'skii, Topological Function Spaces, Kluwer Academic Publishers Group, 1992.

[2] H. Brandsma and J. van Mill, *Monotonically Normal Hyperspaces are Metrisable*, preprint.

[3] P.M. Gartside, Monotonicity in Analytic Topology, DPhil Thesis, University of Oxford, U.K., 1993.

[4] P.M. Gartside and E. Reznichenko, *Near Metric Properties of Function Spaces*, preprint.

[5] G. Gruenhage, *Cosmicity of cometrisable spaces*, Trans. Amer. Math. Soc., **313** (1989), 301 – 315.

[6] G. Gruenhage, *Generalized Metric Spaces*, Handbook of Set–Theoretic Topology, eds Kunen and Vaughan, 423 – 501, North–Holland, 1984.

[7] H. Herrlich and M. Hušek, *Categorical Topology* Recent Progress in General Topology, eds Hušek and van Mill, 370 – 403, North–Holland, 1992.

[8] M. Katětov, *Complete Normality in Cartesian Products*, Fund. Math., **35** (1948), 271 - 274.

[9] R. McCoy and I. Ntantu, Topological Properties of spaces of continuous functions (Lecture Notes in Mathematics 1315), Springer-Verlag, 1988.

[10] E. Michael, $\aleph_0$*-spaces*, J. Math. Mech., **15** (1966), 983 - 1002.

[11] T. Mizokami and K. Koiwa, *On hyperspaces of compact and finite subsets*, Bull. Joetsu Univ. of Educ., **6** (1987), 1 - 14.

[12] I. Ntantu, *Cardinal Functions on hyperspaces and function spaces*, Top. Proc., **10** (1985), 357 - 375.

[13] S. Todorčević, Partition Problems in Topology, American Mathematical Society, 1989.

[14] N.V. Veličko, *On the space of closed subsets*, Siberian Math. J., **16** (1975), 484 - 486.

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