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A PARTIAL CHARACTERIZATION OF UNIVERSAL IMAGES OF GRAPHS USING DECOMPOSITIONS OF TREES

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Abstract

Let G be a graph. Then a graph G' is said to be a *first order reconstitution graph of* G provided that there exists a partition P of G' such that

1. if $D \in P$, then either $D \subset E_{G'}$ or $|D| = 1$, and
2. P is its decomposition topology is homeomorphic to G.

If G' is a tree, then G' is called a *first order reconstitution tree.*

This paper will show that if $f: G \to T$ is a light universal function such that G minus the inverse images of the endpoints of T contains no simple closed curve,

then T can be imbedded into G. This result will be proven by finding a first order reconstitution tree S of G, showing that $f \circ \pi: S \to T$ is a light universal function, locating an imbedding of T into S, and proving that $\pi \circ i: T \to G$ is the desired imbedding.

1 Definitions

This paper is a continuation of the discussion in [H] concerning the universal images of graphs. In [H] a characterization was determined for the light universal images of trees. In this paper it will be shown that if a certain condition of the light universal function holds, then the same result as the characterization in [H] can be inferred even though the domain is a graph but not a tree. This opening section will dispense of all the definitions to be used in this paper.

Definition 1 *A* continuum *is a compact, connected metric space. A subspace D of a continuum C is called a* subcontinuum *if D is a continuum. An* arc *is a continuum that is homeomorphic to the subset* $[0,1]$ *of the real line. A* graph *is a continuum which can be written as a finite union of arcs, any two of which meet at most at one point. A* tree *is a graph that contains no simple closed curve.*

The next two definitions are of terms discussed in [N], although the first definition is a specialization of the definition in [N].

Definition 2 *Let G be a graph. Then $x \in G$ is said to have* order n in G, *denoted* $\operatorname{ord}(x,G) = n$, *provided that x has arbitrarily small open neighborhoods with boundary of cardinality n and n is the smallest such positive integer. x is said to be an* endpoint *of G if* $\operatorname{ord}(x,G) = 1$. *The set of endpoints of G is denoted E_G.*

Definition 3 *Let G be a graph. Then $x \in G$ is said to have* component number n in G, *denoted* $\mathrm{c}(x, G) = n$, *provided that the number of components of $G - \{x\}$ equals n.*

Note that if T is a tree, then if $x \in T$, then $\mathrm{c}(x, T) = \mathrm{ord}(x, T)$.

Definition 4 *Let X be a space and P be a partition of X. Let $\pi: X \to P$ be the function that sends the point $x \in X$ to the element of P in which x is contained. Then the* decomposition topology of P *is defined as follows: $V \subset P$ is open in P if and only if $\pi^{-1}[V]$ is open in X.*

Definition 5 *Let G be a graph. Then a graph G' is said to be a* first order reconstitution graph of G *provided that there exists a partition P of G' such that*

1. *if $D \in P$, then either $D \subset E_{G'}$ or $|D| = 1$, and*
2. *P is its decomposition topology is homeomorphic to G.*

If G' is a tree, then G' is called a first order reconstitution tree.

Definition 6 *Let $f: X \to Y$ be a continuous function. Then f is said to be* light *provided that for each $y \in Y$, $f^{-1}(y)$ is totally disconnected.*

Definition 7 *Let $f: X \to Y$ be a continuous function. Then f is said to be* universal *provided that for each continuous function $g: X \to Y$, there exists $x \in X$ such that $f(x) = g(x)$.*

Definition 8 *Let G be a graph and T be a tree. Let $f: G \to T$ be a continuous function. Then f is said to be* minimal universal *provided that if G' is a subcontinuum of G such that $f\,|G' : G' \to T$ is universal, then $G' = G$.*

2 Preliminaries and Examples

The need to change the domains from trees to graphs is the only preliminary work that must be done. The results of this section were proven for functions from trees to trees in [EF] and [H, pp. 18, 22–25]. The first three need to be generalized to allow the domain to be graphs. However, the generalized proofs are very similar to the original proofs.

Proposition 1 *Let G be a graph and T be a tree. Let $f\colon G \to T$ be a continuous function. Let F be a finite subset of G. If f is not universal, then there exists a continuous function $g\colon G \to T$ such that for each $x \in G$, $f(x) \neq g(x)$ and $g[F] \subset E_T$.*

Proof: Since F is a finite set, it suffices to prove the case $|F| = 1$, since then the general case can be proven by repeating the argument a finite number of times.

Let $F = \{b\}$. Since f is not universal, there exists a continuous function $h\colon G \to T$ such that for each $x \in G$, $f(x) \neq h(x)$. If $h(b) \in E_T$, then there is nothing to prove. Thus, without loss of generality, $h(b) \notin E_T$. Since T is a metric space, there exist connected open sets V_1 and V_2 of T such that $h(b) \in V_1$, $f(b) \in V_2$, $\overline{V_1} \cap \overline{V_2} = \emptyset$, and $V_2 \cap E_T \subset \{f(b)\}$. Since f and h are continuous and G is a metric space, there exists a connected open set W of G such that $b \in W$, $f\left[\overline{W}\right] \cap h\left[\overline{W}\right] = \emptyset$, $f\left[\overline{W}\right] \subset V_2$, $h\left[\overline{W}\right] \subset T - V_2$, and $\overline{W}$ is a tree. Since V_2 contains at most one endpoint of T, the component C of $T - V_2$ containing $h\left[\overline{W}\right]$ contains at least one endpoint of T, say z. Now, $\overline{W} = \bigcup_{i=1}^{n} A_i$ where each A_i is an arc with endpoints b and a boundary point of $\overline{W}$ and there exists an arc B of T with endpoints z and $h(b)$ lying in C. Then there exists a continuous function $h'\colon \overline{W} \to h\left[\overline{W}\right] \cup B$ such that h' is linear on each A_i, $h'(b) = z$, and $h' \left| \mathrm{Bd}\left(\overline{W}\right) = h \right| \mathrm{Bd}\left(\overline{W}\right)$. Define $g\colon G \to T$ by

$$g(x) = \begin{cases} h(x) & \text{if } s \in G - \overline{W} \\ h'(x) & \text{if } s \in \overline{W}. \end{cases}$$

Since $\overline{G - \overline{W}} = (G - \overline{W}) \cup \mathrm{Bd}(\overline{W})$, g is continuous by the Pasting Lemma. Since $(h[\overline{W}] \cup B) \cap f[\overline{W}] = \emptyset$ and for each $x \in G$, $f(x) \neq h(x)$, for each $x \in G$, $f(x) \neq g(x)$. Finally, $g(b) = z \in E_T$.□

Lemma 1 *Let G be a graph and T be a tree. Then, if $f: G \to T$ is a universal function, then there exists a subgraph G' of G such that $f|G' : G' \to T$ is minimal universal.*

Proof: Let G' be the intersection of a maximal tower of subgraphs G_α of G such that for each α, $f|G_\alpha : G_\alpha \to T$ is universal. Suppose that $f|G' : G' \to T$ is not universal. Then, there exists a continuous function $g: G' \to T$ such that for each $x \in G'$, $f(x) \neq g(x)$. Since T is an absolute retract [B], there exists a continuous function $g': G \to T$ that is an extension of g. Since G' is a subgraph of G, there exists $\epsilon > 0$ such that for each $x \in G'$, $d(f(x), g(x)) > \epsilon$. Since f and g' are continuous, there exists $\delta > 0$ such that if $x, y \in G'$ such that $d(x,y) < \delta$, then $d(f(x), f(y)) < \epsilon/3$ and $d(g'(x), g'(y)) < \epsilon/3$. Now, there exists G_β such that G_β is a subset of the δ ball of G'. Now, let $x \in G_\beta$. Then, there exists $y \in G'$ such that $d(x,y) < \delta$. Now, $\epsilon < d(f(y), g(y)) \leq d(f(y), f(x)) + d(f(x), g'(x)) + d(g'(x), g'(y)) < 2\epsilon/3 + d(f(x), g'(x))$. Thus, for each $x \in G_\beta$, $d(f(x), g'(x)) > 0$. Thus, for each $x \in G_\beta$, $f(x) \neq g'(x)$.$\rightarrow\leftarrow$ Thus, $f|G' : G' \to T$ is universal. Now, since the tower is maximal, G' is an element of the tower and $f|G' : G' \to T$ is minimal universal.□

Lemma 2 *Let G be a graph and T be a tree. Then, if $f: G \to T$ is a minimal universal function, then for each $x \in G$, $\mathrm{c}(x, G) \leq \mathrm{c}(f(x), T) = \mathrm{ord}(f(x), T)$.*

Proof: Let $n = \mathrm{c}(x, G)$ and $m = \mathrm{c}(f(x), T)$. Let $A_1, \ldots, A_n$ be the components of $G - \{x\}$ and $B_1, \ldots, B_n$ be the components of $T - \{f(x)\}$. Since, for each $1 \leq i \leq n$, $\mathrm{Bd}(A_i) = \{x\}$, for each $1 \leq i \leq n$, $G - A_i$ is connected and closed, and since for each $1 \leq j \leq m$, $\mathrm{Bd}(B_j) = \{f(x)\}$, for each $1 \leq j \leq m$, $f(x) \in E_{\overline{B_j}}$. Since f is minimal universal, by proposition 1, for each $1 \leq i \leq n$, there exists a continuous function $g_i: G - A_i \to T$ such that for each $y \in G - A_i$, $f(y) \neq g_i(y)$ and $g_i(x) \in E_T$. Now, suppose $n > m$. Then there exist $1 \leq i < j \leq n$ and $1 \leq k \leq m$ such that $\{g_i(x), g_j(x)\} \subset B_k$. Now, since $f(x) \in E_{\overline{B_k}}$, there exists a connected open set V of T such that $f(x) \in V$ and $B_k - V$ is a connected subset of B_k. Since g_i, g_j, and f are continuous, there exists a connected open set W of G such that $x \in W$, $f\left[\overline{W}\right] \cap g_i\left[\overline{W}\right] = \emptyset$, $f\left[\overline{W}\right] \cap g_j\left[\overline{W}\right] = \emptyset$, $f\left[\overline{W}\right] \subset V$, and $\overline{W}$ is a tree. Now, consider $\overline{W \cap A_i}$. Since $\overline{W}$ is a tree, $\overline{W \cap A_i}$ is a tree with endpoints $t_1, \ldots, t_q$. Then, $\overline{W \cap A_i}$ is the union of arcs $W_1, \ldots, W_q$ such that for each $1 \leq p \leq q$, the endpoints of W_p are x and t_p. Thus, there exists a continuous function $h: \overline{W \cap A_i} \to T$ such that $h(x) = g_i(x)$ and for each $1 \leq p \leq q$, $h(t_p) = g_j(t_p)$. Thus, for each $y \in \overline{W \cap A_i}$, $h(y) \neq f(y)$. Define $g: G \to T$ by

$$g(y) = \begin{cases} h(y) & \text{if } y \in \overline{W \cap A_i} \\ g_i(y) & \text{if } y \in G - A_i \\ g_j(y) & \text{if } y \in A_i - (W \cap A_i). \end{cases}$$

By the Pasting Lemma, g is continuous. However, for each $y \in G$, $g(y) \neq f(y)$.$\rightarrow\leftarrow$ Thus, $m \geq n$.□

The final two results are directly from [EF] and [H, pp. 25–26, 45–50]. Since the domains need not be modified from trees to graphs, the proofs are omitted.

Lemma 3 *Let $f: S \to T$ be a light minimal universal function of trees. Then $f^{-1}[E_T] = E_S$.*

This last lemma gives the characterization of light universal images of trees, and it is this characterization that will be extended to domains that are graphs if the functions satisfy certain conditions.

Lemma 4 *Let $f: S \to T$ be a light universal function of trees. Then there exists an imbedding $i: T \to S$. Furthermore, if f is minimal universal, i can be chosen such that $f \circ i\,|E_T$ is identity.*

Now, an example of a light minimal universal function from a graph to a tree. Notice that this function shows why lemma 2 uses component number and not order. In figure 1, $\text{ord}\,(d, G) = 2$ and $\text{c}\,(d, G) = \text{c}\,(f(d), T) = \text{ord}\,(f(d), T) = 1$. Thus, it need not be so that $\text{ord}\,(x, G) \leq \text{ord}\,(f(x), T)$.

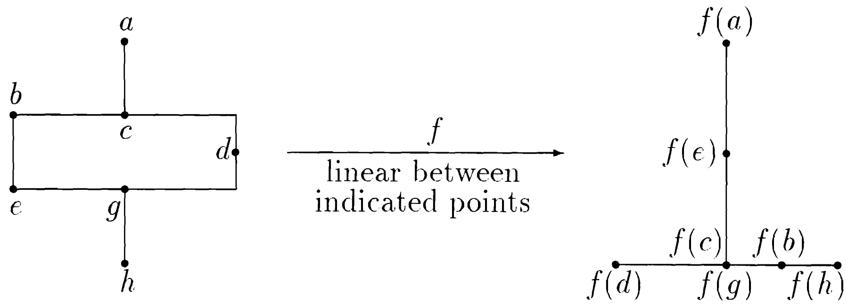

Figure 1: f is light minimal universal

3 The Partial Characterization

With the preliminaries safely out of the way, the extension of lemma 4 to domains that are graphs under certain light universal functions can be shown. The method of proof will be to modify the domain by turning it into a reconstitution graph of the domain, continue the modification until the domain is a reconstitution tree, and then prove that the imbedding into the reconstitution tree can be extended injectively into the original domain. The proof begins by showing the existence of reconstitution graphs of the domain.

Theorem 1 *Let G be a graph and T be a tree. Suppose $f\colon G \to T$ is a light universal function. Then, if there exists $x \in f^{-1}[E_T] - E_G$, then there exists a first order reconstitution graph G' of G such that $\pi^{-1}(x) \subset E_{G'}$ and $|\pi^{-1}(x)| = \operatorname{ord}(x, G)$.*

Proof: Let $\operatorname{ord}(x, G) = n$. Let V be an open set of G such that $x \in V$ and $|\operatorname{Bd}(V)| = n$. Then $V - \{x\}$ consists of n components and for each component, there exists a sequence $\{y_i(k)\}_{k=1}^{\infty}$ that converges to x. Consider the space $G - \{x\}$. Since $f(x) \in E_T$, by lemma 2, $\operatorname{c}(x, G) \leq \operatorname{ord}(f(x), T) = 1$. Thus, $G - \{x\}$ is connected. Let $G' = \{y_1, \ldots, y_n\} \cup G - \{x\}$ such that $|\{y_1, \ldots, y_n\}| = n$, $\{y_1, \ldots, y_n\} \cap (G - \{x\}) = \emptyset$, and for each $1 \leq i \leq n$, $y_i(k) \to y_i$. Then, since the n components of $V - \{x\}$ are disjoint, $\{y_1, \ldots, y_n\} \subset E_{G'}$. Let $P = \{y_1, \ldots, y_n\} \cup \{\{z\} |\ z \in G - \{x\}\}$ be a partition of G'. Then, P in its decomposition topology is G. Thus, G' is a first order reconstitution graph of G such that $\pi^{-1}(x) \subset E_{G'}$ and $|\pi^{-1}(x)| = \operatorname{ord}(x, G)$.□

Corollary 1 *Let G be a graph and T be a tree. Then, if $f\colon G \to T$ is a light universal function such that $G - f^{-1}[E_T]$ contains no simple closed curve, then there exist a finite set $F \subset f^{-1}[E_T] - E_G$ and a first order reconstitution tree T' of G such that for each $x \in G$,*

$$|\pi^{-1}(x)| = \begin{cases} 1 & \textit{if } x \notin F \\ \operatorname{ord}(x, G) & \textit{if } x \in F. \end{cases}$$

Proof: Since G is the union of a finite number of arcs, G contains only a finite number of simple closed curves. Thus, since $G_f^{-1}[E_T]$ contains no simple closed curve, there exists a finite set $F \subset f^{-1}[E_T]$ such that $G - F$ contains no simple closed curve. Now, if $x \in E_G$, then x is a member of no simple closed curve of G. Thus, without loss of generality, $F \subset$

$f^{-1}[E_T]-E_G$. Let $F = \{x_1, \ldots, x_n\}$. After n uses of theorem 1, G has a first order reconstitution graph G_n such that $\pi^{-1}[F] \subset E_{G_n}$, for each $1 \leq i \leq n$, $|\pi^{-1}(x_i)| = \text{ord}\,(x_i, G)$, and for each $x \notin F$, $|\pi^{-1}(x)| = 1$. Thus, it suffices to show that G_n is a tree. But, by the proof of theorem 1, G_n is homeomorphic to the union of $G - f^{-1}[E_T]$ and some endpoints of G_n. Since $G - f^{-1}[E_T]$ contains no simple closed curve, G_n contains no simple closed curve. Thus, G_n is a tree.□

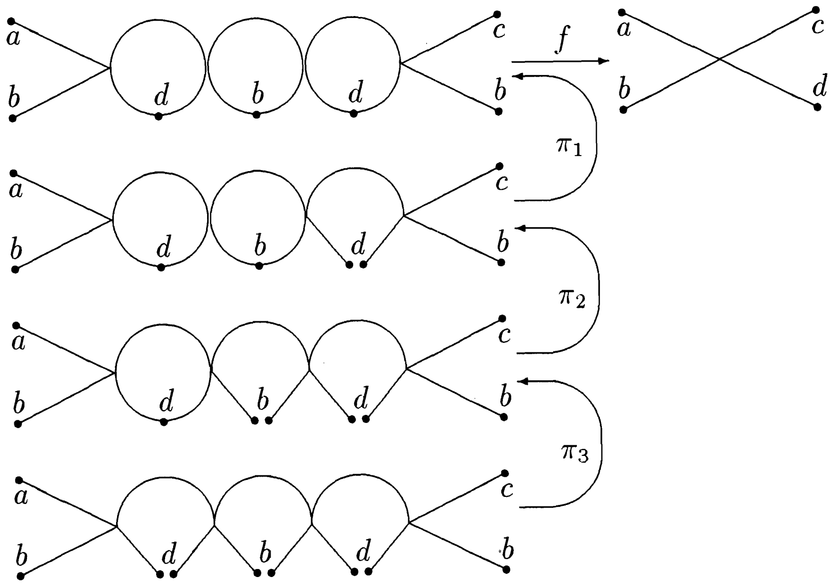

Figure 2: The step-by-step construction of a first order reconstitution tree.

Figure 2 gives an example of the building of a first order reconstitution tree. The function f is a light universal function that is similar to the function in figure 1.

Now with the existence of a reconstitution tree of the domain, a light universal function from the reconstitution tree to the range needs to be found. The next theorem shows that the obvious function from the reconstitution tree to the range satisfies the condition.

Theorem 2 *Let G be a graph and T be a tree. Then, if $f\colon G \to T$ is a light universal function, $F \subset f^{-1}[E_T] - E_G$ is a finite set such that $G - F$ contains no simple closed curve, and G' is a first order reconstitution graph of G such that for each $x \in T$,*

$$\left|\pi^{-1}(x)\right| = \begin{cases} 1 & \text{if } x \notin F \\ \operatorname{ord}(x, G) & \text{if } x \in F, \end{cases}$$

then $f \circ \pi\colon G' \to T$ is light universal.

Proof: The theorem is proven separately for two cases of F.
Case 1 Suppose $F = \emptyset$. Then, for each $x \in G$, $|\pi^{-1}(x)| = 1$. Thus, $\pi\colon G' \to G$ is a homeomorphism. Thus, $f \circ \pi\colon G' \to T$ is light universal.
Case 2 Suppose $F \neq \emptyset$. Let $x \in F$. Then $|\pi^{-1}(x)| > 1$. Since G is the union of a finite number of arcs, $\operatorname{ord}(x, G)$ is finite. Thus, $|\pi^{-1}(x)|$ is finite. Suppose further that $f \circ \pi\colon G' \to T$ is not universal. Then, by proposition 1, there exists a continuous function $g\colon G' \to T$ such that for each $y \in G'$, $(f \circ \pi)(y) \neq g(y)$ and $g[\pi^{-1}(x)] \subset E_T$. Since $f(x) \in E_T$, by the proof of proposition 1, there is no loss in generality in assuming that $|g[\pi^{-1}(x)]| = 1$. Thus, there exists a continuous function $h\colon G \to T$ such that $g = h \circ \pi$. Thus, for each $y \in G'$, $(f \circ \pi)(y) \neq (h \circ \pi)(y)$. Thus, for each $z \in G$, $f(z) \neq h(z)$. Thus, since π is surjective, f is not universal.$\rightarrow\leftarrow$ Thus, $f \circ \pi\colon G' \to T$ is universal. Finally, since π and f are light, then $f \circ \pi$ is light.□

Finally, the only thing left to prove is the injective extension of the imbedding. That fact and the previous results are combined to prove the characterization.

Theorem 3 *Let G be a graph and T be a tree. Then, if $f\colon G \to T$ is a light universal function such that $G - f^{-1}[E_T]$ contains* *no simple closed curve. Then, there exists an imbedding of T into G.*

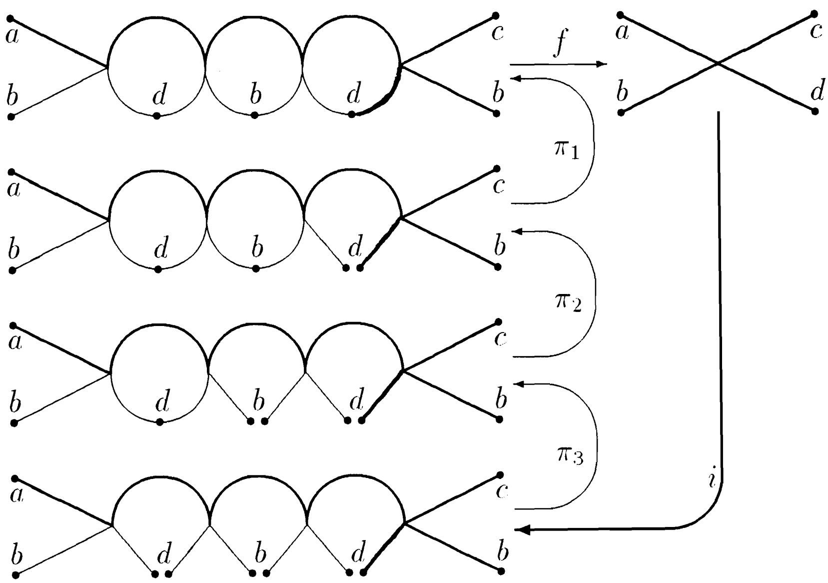

Figure 3: $i \circ \pi_1 \circ \pi_2 \circ \pi_3$ is an imbedding into the original domain.

Proof: By corollary 1 and theorem 2, there exists a first order reconstitution tree T' of G such that $f \circ \pi : T' \to T$ is light universal. Thus, by lemma 1, there exists a subcontinuum S of T' such that $f \circ \pi\, |S : S \to T$ is light minimal universal. Thus, by lemma 4, there exists an imbedding $i : T \to S$ such that $(f \circ \pi \circ i)\, |E_T$ is identity. Let P be the partition of T' such that P in its decomposition topology is homeomorphic to T. Now, suppose $D \in P$ such that $D \subset i[T]$ and $|D| > 1$. Then, there exist distinct points x and y of T such that $\{i(x), i(y)\} \subset D$. Since $|D| > 1$, $D \subset E_{T'}$. Thus, $D \subset E_{i[T]}$. Thus, by lemma 3, $\{x, y\} \subset E_T$. Since $|\pi[D]| = 1$, $|(f \circ \pi \circ i)\, [\{x, y\}]| = 1. \rightarrow\leftarrow$ Thus, for each $D \in P$ such that $D \subset i[T]$, $|D| = 1$. Thus,

$\pi\,|i[T]$ is a continuous injection. Since $i[T]$ is compact and G is Hausdorff, $\pi\,|i[T]$ is an imbedding. Thus, $\pi \circ i\colon T \to G$ is an imbedding.□

Figure 3 shows the imbedding and its continuation from the first order reconstitution tree to the original graph using the function from figure 2.

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