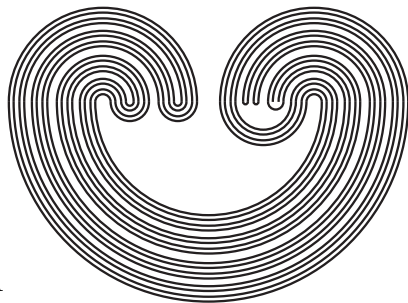


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APOSYNDETTIC PROPERTIES OF SYMMETRIC PRODUCTS OF CONTINUA

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Abstract

We show the following: (1) The n^{th} symmetric product of a continuum is countable closed aposyndetic. (2) If X is a chainable continuum such that its second symmetric product is mutually aposyndetic then X is homeomorphic to $[0, 1]$. (3) A chainable continuum X is indecomposable if and only if its second symmetric product is strictly nonmutually aposyndetic.

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1 Introduction

Let X be a topological space and let n be a positive integer. Then $\mathcal{F}_n(X) = \{A \subset X \mid A \text{ has at most } n \text{ points}\}$ is called the n^{th} *symmetric product of X* . Symmetric products were defined by Borsuk and Ulam (see [B–U]) in 1931. Since then several papers have been written about symmetric products see for example [G–I], [I], [Ma], and [Mo]. In [Bor], Borsuk claimed that the third symmetric product of the unit circle, $\mathcal{S}^1$, was homeomorphic to $\mathcal{S}^1 \times \mathcal{S}^2$, where $\mathcal{S}^2$ is the two sphere, but Bott showed that actually the third symmetric product of $\mathcal{S}^1$ is homeomorphic to the three sphere, $\mathcal{S}^3$ (see [Bot]).

R. W. FitzGerald (see [F], Corollary 2.1) showed that the product of two continua is countable closed aposyndetic. We will show that

Theorem 8. *If X is a continuum, $n > 1$ and $\mathcal{K}$ is any closed, countable subset of $\mathcal{F}_n(X)$, then $T_{\mathcal{F}_n(X)}(\mathcal{K}) = \mathcal{K}$, that is, $\mathcal{F}_n(X)$ is countable closed aposyndetic. In particular, $\mathcal{F}_n(X)$ is m -point aposyndetic for each $m \in \mathbb{N}$.*

L. E. Rogers (see [R], Theorem 1) proved that if X and Y are two chainable continua such that $X \times Y$ is mutually aposyndetic then X and Y are homeomorphic to $[0, 1]$. We prove that

Theorem 15. *If X is a chainable continuum such that $\mathcal{F}_2(X)$ is mutually aposyndetic then X is homeomorphic to $[0, 1]$.*

C. L. Hagopian (see [H], Theorem 10) showed that if X is a chainable continuum, then X is indecomposable if and only if the topological product $X \times X$ is strictly nonmutually aposyndetic. We show that

Theorem 16. *If X is a chainable continuum then X is indecomposable if and only if $\mathcal{F}_2(X)$ is strictly nonmutually aposyndetic.*

Definitions. If (Z, d) is a metric space, then given $A \subset Z$ and $\varepsilon > 0$, the open ball around A of radius ε is denoted by $\mathcal{V}_\varepsilon^Z(A)$, the interior of A is denoted by A° , its closure will be denoted by $\overline{A}$ and its boundary by $\partial(A)$. The product of Z with itself n times will be denoted by Z^n , and we will consider the metric D_n on Z^n given by $D_n((z_1, \dots, z_n), (z'_1, \dots, z'_n)) = \max\{d(z_j, z'_j) \mid j \in \{1, \dots, n\}\}$. Δ_{Z^n} will denote the diagonal of Z^n , i. e., $\Delta_{Z^n} = \{(z, z, \dots, z) \in Z^n \mid z \in Z\}$. The symbol $\mathbb{N}$ will denote the set of positive integers.

A *continuum* is a nonempty, compact, connected, metric space. A *subcontinuum* of a space Z is a continuum contained in Z . A continuum is said to be *colocally connected*, provided that each point of it has a local base of open sets whose complements are connected. A continuum X is *connected im kleinen at the point x of X* , if for each open subset U of X there exists a connected neighborhood V such that $x \in V^\circ \subset V \subset U$. If in the previous definition, we ask V to be open then we say that X is *locally connected at x* .

A continuum X is said to be *aposyndetic at x with respect to y* , provided that there exists a subcontinuum W of X such that $x \in W^\circ \subset W \subset X \setminus \{y\}$, it is said to be *aposyndetic at x* , if it is aposyndetic at x with respect to any point of $X \setminus \{x\}$, and it is said to be *aposyndetic*, if it is aposyndetic at each of its points. The continuum X is *mutually aposyndetic*, provided that if x and y are two distinct points of X then there exist two disjoint subcontinua W_x and W_y of X such that $x \in W_x^\circ$ and $y \in W_y^\circ$. A continuum X is said to be *strictly nonmutually aposyndetic* if each pair of subcontinua of X which have nonempty interior intersect.

Given a continuum X , we define the set function $\mathcal{T}_X$ as follows: if $A \subset X$ then

$$X \setminus \mathcal{T}_X(A) = \{x \in X \mid \text{there exists a subcontinuum } W \text{ of } X \\ \text{such that } x \in W^\circ \subset W \subset X \setminus A\}$$

Bellamy wrote a good survey about the function $\mathcal{T}$ (see [B2]). Let us observe that for any subset A of X , $\mathcal{T}_X(A)$ is closed subset of X and $A \subset \mathcal{T}_X(A)$.

We say that the continuum X is *apосyndetic at the closed subset* A , if $\mathcal{T}_X(A) = A$. In particular, X is said to be *m-point apосyndetic (or countable closed apосyndetic)*, if X is apосyndetic at each subset of it with exactly m points (which is closed and countable, respectively).

A *chain* is a finite collection $\{U_1, \dots, U_m\}$ of open sets such that $U_j \cap U_k \neq \emptyset$ if and only if $|j - k| \leq 1$. The elements of a chain are called *links*. For $\varepsilon > 0$ an ε -*chain* is a chain in which each link has diameter less than ε . A continuum is *chainable* if for each $\varepsilon > 0$, it can be covered by an ε -chain. It is known that every subcontinuum of a chainable continuum is chainable (see [C-V], Theorem (9.C.4)).

Given a continuum X , we define its *hyperspaces* as the following sets:

$$\begin{aligned} 2^X &= \{A \subset X \mid A \text{ is closed and nonempty}\}, \\ \mathcal{C}(X) &= \{A \in 2^X \mid A \text{ is a connected}\}, \\ \mathcal{F}_n(X) &= \{A \in 2^X \mid A \text{ has at most } n \text{ points}\} \ (n \in \mathbb{N}), \\ \mathcal{F}(X) &= \{A \in 2^X \mid A \text{ is finite}\}. \end{aligned}$$

Let us observe that for each $n \in \mathbb{N}$, $\mathcal{F}_n(X) \subset \mathcal{F}_{n+1}(X)$ and that $\mathcal{F}(X) = \bigcup_{n=1}^{\infty} \mathcal{F}_n(X)$. It is known that 2^X is a metric space with the Hausdorff metric, $\mathcal{H}$, defined as follows:

$$\mathcal{H}(A, B) = \inf\{\varepsilon > 0 \mid A \subset \mathcal{V}_\varepsilon^X(B) \text{ and } B \subset \mathcal{V}_\varepsilon^X(A)\},$$

(see [N1], (0.1)), and in fact, 2^X and $\mathcal{C}(X)$ are arcwise connected continua (see [N1], (1.13)). It is also known that each $\mathcal{F}_n(X)$ is a continuum (see [B-U]), and then $\mathcal{F}(X)$ is a connected space. On the other hand, 2^X can be topologized with the *Vietoris Topology*, defined as follows: given a finite collection, $U_1, U_2, \dots, U_m$, of open sets of X , we define

$$\langle U_1, \dots, U_m \rangle = \{A \in 2^X \mid A \subset \bigcup_{k=1}^m U_k \text{ and } A \cap U_k \neq \emptyset \text{ for each } k \in \{1, \dots, m\}\}.$$

It is known that the family of all subsets of 2^X of the form $\langle U_1, \dots, U_m \rangle$, as defined above, form a basis for a topology for 2^X (see [N1], (0.11)) called *Vietoris Topology*, and that the Vietoris Topology and the Topology induced by the Hausdorff metric coincide (see [N1], (0.13)). To simplify notation, $\langle U_1, \dots, U_m \rangle_n$ will denote the intersection of the open set $\langle U_1, \dots, U_m \rangle$, of the Vietoris Topology, with $\mathcal{F}_n(X)$.

It is important to note that given a continuum X , then $\mathcal{F}_1(X)$ is an isometric copy of X (using the Hausdorff metric) contained in each of the hyperspaces defined above.

A map $f: X \rightarrow Y$ between continua is said to be *open* if it sends open subsets of X into open subsets of Y . The map f is said to be *confluent*, provided that for each subcontinuum Q of Y , each component of $f^{-1}(Q)$ is mapped onto Q by f .

2 The Theorems

We begin with some elementary results.

Lemma 1. *Let X be a continuum and let n be a positive integer. If D_n denotes the metric on X^n given by*

$$D_n((x_1, \dots, x_n), (x'_1, \dots, x'_n)) = \max\{d(x_1, x'_1), \dots, d(x_n, x'_n)\},$$

where d is the metric on X , then the function $f_n: X^n \rightarrow \mathcal{F}_n(X)$ given by

$$f_n((x_1, \dots, x_n)) = \{x_1, \dots, x_n\}$$

is onto and satisfies the following inequality:

$$\begin{aligned} \mathcal{H}(f_n((x_1, \dots, x_n)), f_n((x'_1, \dots, x'_n))) \\ \leq D_n((x_1, \dots, x_n), (x'_1, \dots, x'_n)), \end{aligned}$$

for each $(x_1, \dots, x_n)$ and $(x'_1, \dots, x'_n)$ in X^n .

Proof: Clearly the map f_n is onto. Let $(x_1, \dots, x_n)$ and $(x'_1, \dots, x'_n)$ be two points of X^n . Suppose that $D_n((x_1, \dots, x_n), (x'_1, \dots, x'_n)) = r$ and let $\varepsilon > 0$ be given. Hence $D_n((x_1, \dots, x_n), (x'_1, \dots, x'_n)) < r + \varepsilon$. This implies that for each $j \in \{1, \dots, n\}$, $d(x_j, x'_j) < r + \varepsilon$. Thus, for each $x_j \in f_n((x_1, \dots, x_n))$, we have that $x'_j \in f_n((x'_1, \dots, x'_n))$ and $d(x_j, x'_j) < r + \varepsilon$. This shows that $f_n((x_1, \dots, x_n)) \subset \mathcal{V}_{r+\varepsilon}^X(f_n((x'_1, \dots, x'_n)))$. Similarly, $f_n((x'_1, \dots, x'_n)) \subset \mathcal{V}_{r+\varepsilon}^X(f_n((x_1, \dots, x_n)))$. Thus $\mathcal{H}(f_n((x_1, \dots, x_n)), f_n((x'_1, \dots, x'_n))) \leq r + \varepsilon$. Since the ε was arbitrary, we obtain that $\mathcal{H}(f_n((x_1, \dots, x_n)), f_n((x'_1, \dots, x'_n))) \leq r$. $\square$

Lemma 2. *If X is a continuum and $n \in \mathbb{N}$ then X is locally connected if and only if $\mathcal{F}_n(X)$ is locally connected.*

Proof: Clearly if X is locally connected then $\mathcal{F}_n(X)$ is locally connected. Thus, suppose that $\mathcal{F}_n(X)$ is locally connected and $n > 1$. Let $x_0 \in X$ and let U be an open subset of X containing x_0 . Since $\mathcal{F}_n(X)$ is locally connected, there exists a connected open subset $\mathcal{W}$ of $\mathcal{F}_n(X)$ such that $\{x_0\} \in \mathcal{W} \subset \overline{\mathcal{W}} \subset \langle U \rangle_n$. Let $\varepsilon > 0$ be given such that $\mathcal{V}_\varepsilon^{\mathcal{F}_n(X)}(\{x_0\}) \cap \mathcal{F}_1(X) \subset \mathcal{W} \cap \mathcal{F}_1(X)$.

Since $\overline{\mathcal{W}}$ is a subcontinuum of 2^X and $\overline{\mathcal{W}} \cap \mathcal{C}(X) \neq \emptyset$, then $\cup \overline{\mathcal{W}}$ is a subcontinuum of X (see [N1], (1.49)). On the other hand, since $\overline{\mathcal{W}} \subset \langle U \rangle_n$, we have that $\cup \overline{\mathcal{W}} \subset U$. Let $x \in \mathcal{V}_\varepsilon^X(x_0)$, then $\{x\} \in \mathcal{V}_\varepsilon^{\mathcal{F}_n(X)}(\{x_0\}) \cap \mathcal{F}_1(X) \subset \mathcal{W} \cap \mathcal{F}_1(X)$. Hence, $x \in \cup \overline{\mathcal{W}}$. Therefore, $\mathcal{V}_\varepsilon^X(x_0) \subset \cup \overline{\mathcal{W}}$, and $\cup \overline{\mathcal{W}}$ is a connected neighborhood of x_0 contained in U . Thus, X is connected im kleinen at each of its points and then X is locally connected (see [H-Y], Theorem 3-11). $\square$

In her B. S. Thesis (see [G]), R. García, essentially, showed that if X is a continuum then $\mathcal{F}(X)$ is colocally connected, a modification of her proof, shows that for each $n > 1$, $\mathcal{F}_n(X)$ is colocally connected, we include that modification for completeness.

Lemma 3. *Let X be a continuum with metric d , let $n > 1$ be given. Let $\{x_1, \dots, x_k\} \in \mathcal{F}_n(X)$, and let $\varepsilon > 0$ be given such that $d(x_j, x_\ell) > \varepsilon$ for each $x_j, x_\ell \in \{x_1, \dots, x_k\}$ and $x_j \neq x_\ell$. Then $\mathcal{F}_n(X) \setminus \langle \mathcal{V}_\varepsilon^X(x_1), \dots, \mathcal{V}_\varepsilon^X(x_k) \rangle_n$ is connected.*

Proof. Let $\mathcal{A} = \langle \mathcal{V}_\varepsilon^X(x_1), \dots, \mathcal{V}_\varepsilon^X(x_k) \rangle_n$ and let p be a point of X such that $d(p, x_j) > \varepsilon$ for each $j \in \{1, \dots, k\}$. Let $\{b_1, \dots, b_m\} \in \mathcal{F}_n(X) \setminus \mathcal{A}$. We are going to show that there is a connected set in $\mathcal{F}_n(X) \setminus \mathcal{A}$ containing $\{b_1, \dots, b_m\}$ and $\{p\}$, this will prove the Lemma.

Recall that, by definition, $K \in \mathcal{A}$ if and only if $K \subset \bigcup_{j=1}^k \mathcal{V}_\varepsilon^X(x_j)$ and $K \cap \mathcal{V}_\varepsilon^X(x_j) \neq \emptyset$ for each $j \in \{1, \dots, k\}$. Since

$\{b_1, \dots, b_m\} \notin \mathcal{A}$, we have that either $\{b_1, \dots, b_m\} \not\subset \bigcup_{j=1}^k \mathcal{V}_\varepsilon^X(x_j)$

or $\{b_1, \dots, b_m\} \subset \bigcup_{j=1}^k \mathcal{V}_\varepsilon^X(x_j)$ and $\{b_1, \dots, b_m\} \cap \mathcal{V}_\varepsilon^X(x_j) = \emptyset$ for some $j \in \{1, \dots, k\}$.

Suppose that $\{b_1, \dots, b_m\} \not\subset \bigcup_{j=1}^k \mathcal{V}_\varepsilon^X(x_j)$. Suppose first that

$m = 1$. Then $b_1 \notin \bigcup_{j=1}^k \mathcal{V}_\varepsilon^X(x_j)$. Let $f_{\{b_1\}}$ and $f_{\{p\}}$ be functions

from X to $\mathcal{F}_n(X)$ given by $f_{\{b_1\}}(x) = \{x\} \cup \{b_1\}$ and $f_{\{p\}}(x) = \{x\} \cup \{p\}$. By construction, for each $x \in X$, $f_{\{b_1\}}(x)$ and $f_{\{p\}}(x)$ both have at most two elements. Hence both functions are well defined. It is easy to see that they are continuous. Let $\mathcal{K}_1 = f_{\{b_1\}}(X)$ and $\mathcal{K}_2 = f_{\{p\}}(X)$, then $\mathcal{K}_1$ and $\mathcal{K}_2$ are subcontinua of $\mathcal{F}_n(X)$. Since $\{p, b_1\} \in \mathcal{K}_1 \cap \mathcal{K}_2$ ($f_{\{b_1\}}(p) = \{p, b_1\} = f_{\{p\}}(b_1)$), we have that $\mathcal{K} = \mathcal{K}_1 \cup \mathcal{K}_2$ is a subcontinuum of $\mathcal{F}_n(X)$. Let us observe that for each $K \in \mathcal{K}$, either $b_1 \in K$ or $p \in K$, hence $\mathcal{K} \cap \mathcal{A} = \emptyset$. Therefore $\mathcal{K} \subset \mathcal{F}_n(X) \setminus \mathcal{A}$.

Now suppose that $m > 1$ and $\{b_1, \dots, b_m\} \not\subset \bigcup_{j=1}^k \mathcal{V}_\varepsilon^X(x_j)$.

Then there exists $j \in \{1, \dots, m\}$ such that $b_j \in \{b_1, \dots, b_m\}$ and $b_j \notin \mathcal{V}_\varepsilon^X(x_\ell)$ for any $\ell \in \{1, \dots, k\}$. Without loss of generality, we may assume that $j = m$, i. e., $b_m \notin \bigcup_{\ell=1}^k \mathcal{V}_\varepsilon^X(x_\ell)$. Let

$f_{\{b_1\}}: X \rightarrow \mathcal{F}_n(X)$ be given by $f_{\{b_1\}}(x) = \{x\} \cup (\{b_1, \dots, b_m\} \setminus \{b_1\})$. Then $f_{\{b_1\}}$ is well defined and continuous. Let $\mathcal{K}_1 = f_{\{b_1\}}(X)$, then $\mathcal{K}_1$ is a subcontinuum of $\mathcal{F}_n(X)$. Observe that $\{b_1, \dots, b_m\} = f_{\{b_1\}}(b_1)$ and that $f_{\{b_1\}}(p) = \{p, b_2, \dots, b_m\}$. Since $b_m \in \{b_1, \dots, b_m\} \setminus \{b_1\}$, then for each $K \in f_{\{b_1\}}(X)$, we obtain that $b_m \in K$, thus $\mathcal{K}_1 \cap \mathcal{A} = \emptyset$. Hence we have $\mathcal{K}_1$ is a subcontinuum of $\mathcal{F}_n(X) \setminus \mathcal{A}$ containing $\{b_1, \dots, b_m\}$ and $\{p, b_2, \dots, b_m\}$.

Let $f_{\{b_2\}}: X \rightarrow \mathcal{F}_n(X)$ be given by $f_{\{b_2\}}(x) = \{x\} \cup (\{p, b_2, \dots, b_m\} \setminus \{b_2\})$. Then $f_{\{b_2\}}$ is well defined and continuous. Let $\mathcal{K}_2 = f_{\{b_2\}}(X)$, then $\mathcal{K}_2$ is a subcontinuum of $\mathcal{F}_n(X)$ such that $\{p, b_2, \dots, b_m\} \in \mathcal{K}_2$ ($f_{\{b_2\}}(b_2) = \{p, b_2, \dots, b_m\}$). Observe that $f_{\{b_2\}}(p) = \{p, b_3, \dots, b_m\}$, $\mathcal{K}_2 \cap \mathcal{A} = \emptyset$ and $\{p, b_2, \dots, b_m\} \in \mathcal{K}_1 \cap \mathcal{K}_2$. Thus, $\mathcal{K}_1 \cup \mathcal{K}_2$ is a subcontinuum of $\mathcal{F}_n(X) \setminus \mathcal{A}$ containing $\{b_1, b_2, \dots, b_m\}$ and $\{p, b_3, \dots, b_m\}$.

Repeating this process, we can find subcontinua $\mathcal{K}_3, \dots, \mathcal{K}_{m-1}$ of $\mathcal{F}_n(X) \setminus \mathcal{A}$ such that $\{p, b_\ell, \dots, b_m\} \in \mathcal{K}_{\ell-1} \cap \mathcal{K}_\ell$ for each $\ell \in \{2, \dots, m-1\}$ and $\{p\} \in \mathcal{K}_{m-1}$. Hence $\mathcal{K} = \bigcup_{j=1}^{m-1} \mathcal{K}_j$ is a subcontinuum of $\mathcal{F}_n(X) \setminus \mathcal{A}$ containing $\{b_1, b_2, \dots, b_m\}$ and $\{p\}$. This finishes this case.

Suppose that $\{b_1, b_2, \dots, b_m\} \subset \bigcup_{j=1}^k \mathcal{V}_\varepsilon^X(x_j)$ and

$\{b_1, b_2, \dots, b_m\} \cap \mathcal{V}_\varepsilon^X(x_j) = \emptyset$ for some $j \in \{1, \dots, k\}$. We may assume that $\{b_1, b_2, \dots, b_m\} \cap \mathcal{V}_\varepsilon^X(x_k) = \emptyset$. Suppose also that $b_1 \in \mathcal{V}_\varepsilon^X(x_1)$.

Let A be the component of $\mathcal{V}_\varepsilon^X(x_1)$ containing b_1 . By the Boundary Bumping Theorem (see [N2], Theorem 5.7), we have that $\overline{A} \cap \partial(\mathcal{V}_\varepsilon^X(x_1)) \neq \emptyset$. Let $c \in \overline{A} \cap \partial(\mathcal{V}_\varepsilon^X(x_1))$. Let $f: \overline{A} \rightarrow \mathcal{F}_n(X)$ be given by $f(x) = \{x\} \cup (\{b_1, b_2, \dots, b_m\} \setminus \{b_1\})$. Then f is well defined and continuous. Thus $f(\overline{A})$ is a subcontinuum of $\mathcal{F}_n(X)$, $\{b_1, b_2, \dots, b_m\} \in f(\overline{A})$ ($f(b_1) = \{b_1, b_2, \dots, b_m\}$). Since $c \in \overline{A} \cap \partial(\mathcal{V}_\varepsilon^X(x_1))$, we have that $\{c, b_2, \dots, b_m\}$ is not contained in $\bigcup_{j=1}^k \mathcal{V}_\varepsilon^X(x_j)$. Hence, by the first case, there exists a subcontinuum, $\mathcal{L}_1$, of $\mathcal{F}_n(X) \setminus \mathcal{A}$ containing $\{c, b_2, \dots, b_m\}$ and $\{p\}$. By the election of ε , we have that for each $j \in \{2, \dots, k\}$, $\overline{A} \cap \mathcal{V}_\varepsilon^X(x_j) = \emptyset$, thus $f(\overline{A}) \cap \mathcal{A} = \emptyset$. Let $\mathcal{L} = \mathcal{L}_1 \cup f(\overline{A})$. Then $\mathcal{L}$ is a subcontinuum of $\mathcal{F}_n(X) \setminus \mathcal{A}$ containing $\{b_1, b_2, \dots, b_m\}$ and $\{p\}$. This finishes the proof of the second case and the proof of the Lemma. $\square$

As a Corollary of this Lemma we have the following results.

Theorem 4. *If X is a continuum and $n > 1$ then $\mathcal{F}_n(X)$ is colocally connected. In particular, $\mathcal{F}_n(X)$ is aposyndetic.*

Corollary 5. *If X is a continuum and $n > 1$ then for each $\{x_1, x_2, \dots, x_k\} \in \mathcal{F}_n(X)$, $\mathcal{F}_n(X) \setminus \{x_1, x_2, \dots, x_k\}$ is connected.*

Proof: Let $N \in \mathbb{N}$ be such that

$$\frac{1}{N} < \min\{d(x_j, x_\ell) \mid x_j, x_\ell \in \{x_1, x_2, \dots, x_k\} \text{ and } x_j \neq x_\ell\}.$$

Observe that $\{x_1, x_2, \dots, x_k\} = \bigcap_{\ell=N}^{\infty} < \mathcal{V}_{\frac{1}{\ell}}^X(x_1), \dots, \mathcal{V}_{\frac{1}{\ell}}^X(x_k) >_n$.

By Lemma 3, we have that for each $\ell \geq N$, $\mathcal{F}_n(X) \setminus < \mathcal{V}_{\frac{1}{\ell}}^X(x_1), \dots, \mathcal{V}_{\frac{1}{\ell}}^X(x_k) >_n$ is connected. Then

$$\mathcal{F}_n(X) \setminus \{x_1, x_2, \dots, x_k\}$$

$$\begin{aligned}
&= \mathcal{F}_n(X) \setminus \bigcap_{\ell=N}^{\infty} \langle \mathcal{V}_{\frac{1}{\ell}}^X(x_1), \dots, \mathcal{V}_{\frac{1}{\ell}}^X(x_k) \rangle_n \\
&= \bigcup_{\ell=N}^{\infty} \left(\mathcal{F}_n(X) \setminus \langle \mathcal{V}_{\frac{1}{\ell}}^X(x_1), \dots, \mathcal{V}_{\frac{1}{\ell}}^X(x_k) \rangle_n \right),
\end{aligned}$$

and this set is connected, being a union of connected sets with nonempty intersection. $\square$

R. W. FitzGerald proved the following result (see [F], Corollary 2.1).

Theorem 6. *If X_1 and X_2 are compact Hausdorff continua and K is any closed, countable subset of $X_1 \times X_2$, then we have that $\mathcal{T}_{X_1 \times X_2}(K) = K$. Hence $X_1 \times X_2$ is n -point aposyndetic for all $n \in \mathbb{N}$.*

D. P. Bellamy showed the following (see [B], Lemma 14).

Theorem 7. *If $f: X \rightarrow Y$ is an onto map between continua and $B \subset Y$ then $\mathcal{T}_Y(B) \subset f\mathcal{T}_X f^{-1}(B)$.*

As a consequence of Theorems 6 and 7 we have the following result.

Theorem 8. *If X is a continuum, $n > 1$ and $\mathcal{K}$ is any closed, countable subset of $\mathcal{F}_n(X)$, then $\mathcal{T}_{\mathcal{F}_n(X)}(\mathcal{K}) = \mathcal{K}$, that is, $\mathcal{F}_n(X)$ is countable closed aposyndetic. In particular, $\mathcal{F}_n(X)$ is m -point aposyndetic for each $m \in \mathbb{N}$.*

Proof. Let $f_n: X^n \rightarrow \mathcal{F}_n(X)$ given by $f_n((x_1, \dots, x_n)) = \{x_1, \dots, x_n\}$. By Lemma 1, f_n is continuous and onto. Observe that for each $A \in \mathcal{F}_n(X)$, we have that $f_n^{-1}(A)$ has at most $n!$ elements.

Since $\mathcal{K}$ is a closed, countable subset of $\mathcal{F}_n(X)$, we have that $f_n^{-1}(\mathcal{K})$ is a closed, countable subset of X^n . By Theorem 6, $\mathcal{T}_{X^n}(f_n^{-1}(\mathcal{K})) = f_n^{-1}(\mathcal{K})$. By Theorem 7, $\mathcal{T}_{\mathcal{F}_n(X)}(\mathcal{K}) \subset f_n \mathcal{T}_{X^n}(f_n^{-1}(\mathcal{K})) = f_n f_n^{-1}(\mathcal{K}) = \mathcal{K}$. Since $\mathcal{K} \subset \mathcal{T}_{\mathcal{F}_n(X)}(\mathcal{K})$, we have that $\mathcal{T}_{\mathcal{F}_n(X)}(\mathcal{K}) = \mathcal{K}$. $\square$

Lemma 9. *If X is a continuum, then the map $f_2: X \times X \rightarrow \mathcal{F}_2(X)$ given by $f_2((x_1, x_2)) = \{x_1, x_2\}$ is open.*

Proof. Let (x_1, x_2) be a point of X^2 and let $\varepsilon > 0$ be given, we will show that $f_2(\mathcal{V}_\varepsilon^{X^2}((x_1, x_2)))$ is open in $\mathcal{F}_2(X)$. Let $\{y_1, y_2\} \in f_2(\mathcal{V}_\varepsilon^{X^2}((x_1, x_2)))$. If $y_1 \neq y_2$, then let $\delta < \min\{d(y_1, y_2), \varepsilon - d(x_1, y_1), \varepsilon - d(x_2, y_2), D_2((y_1, y_2), \Delta_{X^2})\}$. We claim that $\mathcal{U} = \langle \mathcal{V}_\delta^X(y_1), \mathcal{V}_\delta^X(y_2) \rangle_2$ is contained in $f_2(\mathcal{V}_\varepsilon^{X^2}((x_1, x_2)))$. Let $\{z_1, z_2\} \in \mathcal{U}$, without loss of generality we may assume that $z_1 \in \mathcal{V}_\delta^X(y_1)$ and $z_2 \in \mathcal{V}_\delta^X(y_2)$. Then

$$\begin{aligned} d(z_1, x_1) &\leq d(z_1, y_1) + d(y_1, x_1) < \delta + d(y_1, x_1) \\ &\leq \varepsilon - d(x_1, y_1) + d(y_1, x_1) = \varepsilon. \end{aligned}$$

Similarly, we have that $d(z_2, x_2) < \varepsilon$. Hence, we have that $(z_1, z_2) \in \mathcal{V}_\varepsilon^{X^2}((x_1, x_2))$ and $f_2((z_1, z_2)) = \{z_1, z_2\}$. Therefore $\mathcal{U} \subset f_2(\mathcal{V}_\varepsilon^{X^2}((x_1, x_2)))$.

If $y_1 = y_2 = y$ then let $\delta < \min\{\varepsilon - d(x_1, y), \varepsilon - d(x_2, y)\}$, and $\mathcal{W} = \langle \mathcal{V}_\delta^X(y) \rangle_2$. We will show that $\mathcal{W} \subset f_2(\mathcal{V}_\varepsilon^{X^2}((x_1, x_2)))$. Let $\{z_1, z_2\} \in \mathcal{W}$. If $z_1 \neq z_2$, then

$$\begin{aligned} d(z_1, x_1) &\leq d(z_1, y) + d(y, x_1) < \delta + d(y, x_1) \\ &\leq \varepsilon - d(x_1, y) + d(y, x_1) = \varepsilon. \end{aligned}$$

Similarly, $d(z_2, x_2) < \varepsilon$. Hence $(z_1, z_2) \in \mathcal{V}_\varepsilon^{X^2}((x_1, x_2))$ and $f_2((z_1, z_2)) = \{z_1, z_2\}$. Finally, if $z_1 = z_2 = z$ then

$$\begin{aligned} d(z, x_1) &\leq d(z, y) + d(y, x_1) < \delta + d(y, x_1) \\ &\leq \varepsilon - d(x_1, y) + d(y, x_1) = \varepsilon. \end{aligned}$$

Similarly, $d(z, x_2) < \varepsilon$. Thus $(z, z) \in \mathcal{V}_\varepsilon^{X^2}((x_1, x_2))$ and $f_2((z, z)) = \{z\}$. Therefore $\mathcal{W} \subset f_2(\mathcal{V}_\varepsilon^{X^2}((x_1, x_2)))$. $\square$

Lemma 10. *If X is a continuum and $\xi: X^2 \rightarrow X^2$ is the map given by $\xi((x_1, x_2)) = (x_2, x_1)$ then ξ is a homeomorphism and $f_2 \circ \xi = f_2$ and $f_2 \circ \xi^{-1} = f_2$, where f_2 was defined in the previous Lemma.*

From now on, we will use the following notation. If W is a subset of X^2 then $W^* = \xi(W)$. $f_2: X^2 \rightarrow \mathcal{F}_2(X)$ will be the map given by $f_2((x_1, x_2)) = \{x_1, x_2\}$.

Lemma 11. *Let X be a chainable continuum and let x_0 and y_0 be two distinct points of X . If C is a subcontinuum of X^2 containing (x_0, y_0) and (y_0, x_0) then $C \cap \Delta_{X^2} \neq \emptyset$.*

Proof: By Lemma 10, C^* is a continuum. Let $D = C \cup C^*$. Since $(x_0, y_0) \in C \cap C^*$, we have that D is a continuum and $D^* = D$. This equality implies that $\pi_1(D) = \pi_2(D)$, where π_j denotes the projection map from X^2 to the j^{th} factor, $j \in \{1, 2\}$. Note that $\pi_1(D)$ is a chainable continuum (see [C-V], Theorem (9.C.4)). Since any two maps from a continuum onto a chainable continuum have a coincidence point (see [N2], Corollary 12.26), we have that there exists a point $(z_1, z_2) \in D$ such that $z_1 = \pi_1|_D((z_1, z_2)) = \pi_2|_D((z_1, z_2)) = z_2$. Then the point $(z_1, z_1) \in C \cap \Delta_{X^2}$. $\square$

Lemma 12. *Let X be a continuum and let $\mathcal{W}$ be a subcontinuum of $\mathcal{F}_2(X)$. If W is a component of $f_2^{-1}(\mathcal{W})$ then W^* is a component of $f_2^{-1}(\mathcal{W})$ too and $f_2^{-1}(\mathcal{W}) = W \cup W^*$.*

Proof: Let $\mathcal{W}$ be a subcontinuum of $\mathcal{F}_2(X)$ and let W be a component of $f_2^{-1}(\mathcal{W})$. Since $W^* = \xi(W)$, we have that W^* is a subcontinuum of X^2 . By Lemma 9, f_2 is open, hence it is confluent (see [W], Theorem 7.5, p. 148). Hence $f_2(W) = \mathcal{W}$ and $f_2(W^*) = f_2 \circ \xi(W) = f_2(W) = \mathcal{W}$. If C were the component of $f_2^{-1}(\mathcal{W})$ containing W^* then $\xi(C)$ would be a subcontinuum of X^2 containing W and $f_2 \circ \xi(C) = f_2(C) = \mathcal{W}$. Hence $\xi(C) = W$ and $C = W^*$.

The equality $f_2^{-1}(\mathcal{W}) = W \cup W^*$ follows from the fact that for each element $A \in \mathcal{F}_2(X)$, $f_2^{-1}(A)$ has at most two elements. $\square$

Corollary 13. *Let X be a chainable continuum and let $\mathcal{W}$ be a subcontinuum of $\mathcal{F}_2(X)$. Then $f_2^{-1}(\mathcal{W})$ is connected if and only if $f_2^{-1}(\mathcal{W}) \cap \Delta_{X^2} \neq \emptyset$.*

Proof: If $f_2^{-1}(\mathcal{W})$ is connected then, by Lemma 11, we have that $f_2^{-1}(\mathcal{W}) \cap \Delta_{X^2} \neq \emptyset$. Suppose $f_2^{-1}(\mathcal{W}) \cap \Delta_{X^2} \neq \emptyset$ and let W be a component of $f_2^{-1}(\mathcal{W})$. By Lemma 9, f_2 is open, hence confluent. Since $f_2^{-1}(\mathcal{W}) \cap \Delta_{X^2} \neq \emptyset$, we have that $\mathcal{W} \cap \mathcal{F}_1(X) \neq \emptyset$. Thus $W \cap \Delta_{X^2} \neq \emptyset$. Then $W = W^*$ and $f_2^{-1}(\mathcal{W}) = W$. $\square$

Lemma 14. *Let X be a continuum and let $\mathcal{W}$ be a subcontinuum of $\mathcal{F}_2(X)$ with nonempty interior. If W is a component of $f_2^{-1}(\mathcal{W})$ then W has nonempty interior.*

Proof: Let $\mathcal{W}$ be a subcontinuum of $\mathcal{F}_2(X)$ with nonempty interior. If $f_2^{-1}(\mathcal{W})$ is connected then the result is clear. Suppose then that $f_2^{-1}(\mathcal{W})$ is not connected and let W be a component of $f_2^{-1}(\mathcal{W})$. Since f_2 is at most 2 to 1, the only other component of $f_2^{-1}(\mathcal{W})$ is W^* . Since W and W^* are interchanged by the homeomorphism of X^2 that switches the factors, W and W^* both have interior if the union does, completing the proof. $\square$

Theorem 15. *If X is a chainable continuum such that $\mathcal{F}_2(X)$ is mutually aposyndetic then X is homeomorphic to $[0, 1]$.*

Proof: Let (x_1, x_2) and (y_1, y_2) be two distinct points of X^2 , and suppose that $f_2((x_1, x_2)) \neq f_2((y_1, y_2))$. Since $\mathcal{F}_2(X)$ is mutually aposyndetic, then there exist two disjoint subcontinua $\mathcal{W}_1$ and $\mathcal{W}_2$ of $\mathcal{F}_2(X)$ such that $f_2((x_1, x_2)) \in \mathcal{W}_1^\circ$ and $f_2((y_1, y_2)) \in \mathcal{W}_2^\circ$. Let W_1 and W_2 be the components of $f_2^{-1}(\mathcal{W}_1)$ and $f_2^{-1}(\mathcal{W}_2)$ containing (x_1, x_2) and (y_1, y_2) , respectively. Then W_1 and W_2 are two disjoint subcontinua of X^2 . By the proof of Lemma 14, we have that $(x_1, x_2) \in W_1^\circ$ and $(y_1, y_2) \in W_2^\circ$.

In order to finish the proof let us observe that in the proof of [R], Theorem 1, Rogers just needed to find disjoint subcontinua of the product of two chainable continua containing points of the form (x, q) and (y, q) (or (q, x) and (q, y)) in their interiors to conclude that those chainable continua were homeomorphic to $[0, 1]$. Since we have shown that we can find disjoint subcon-

tinua having those types of points of X^2 in their interior, then the Theorem follows from the proof of Rogers's Theorem. $\square$

Theorem 16. *If X is a chainable continuum then X is indecomposable if and only if $\mathcal{F}_2(X)$ is strictly nonmutually aposyndetic.*

Proof: Suppose that X is decomposable, then there exists a proper subcontinuum W of X with nonempty interior. Let U be any open subset of X such that $\overline{U} \cap W = \emptyset$. Let $K = W \times W$ and $H = [\overline{U} \times X] \cup [X \times \overline{U}]$. Then K and H are disjoint subcontinua of X^2 each having nonempty interior. Observe that $K = K^*$ and $H = H^*$, hence $f_2(K) \cap f_2(H) = \emptyset$. Since f_2 is open (by Lemma 9), we have that $f_2(K)$ and $f_2(H)$ are disjoint subcontinua of $\mathcal{F}_2(X)$ having nonempty interior. It follows that $\mathcal{F}_2(X)$ is not strictly nonmutually aposyndetic.

If $\mathcal{F}_2(X)$ is not strictly nonmutually aposyndetic then there exist two disjoint subcontinua $\mathcal{K}$ and $\mathcal{H}$ of $\mathcal{F}_2(X)$ having nonempty interior. Let K and H be components of $f_2^{-1}(\mathcal{K})$ and $f_2^{-1}(\mathcal{H})$, respectively. By Lemma 14, K and H have nonempty interior. Therefore, K and H are two disjoint subcontinua of X^2 . Hence, by [H], Theorem 10, X is decomposable. $\square$

Let us observe that in order to show that if $\mathcal{F}_2(X)$ is strictly nonmutually aposyndetic then X is indecomposable, we did not use the fact that X was chainable. Hagopian did not use chainability of X either to show that if X^2 is strictly nonmutually aposyndetic then X is indecomposable. Thus we have the following:

Theorem 17. *Let X be a continuum. If either X^2 or $\mathcal{F}_2(X)$ is strictly nonmutually aposyndetic then X is indecomposable.*

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