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ON QUASI-LINDELÖF FIRST COUNTABLE SPACES

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Abstract

We show it is consistent with ZFC that there are quasi-Lindelöf first countable Hausdorff spaces and Lindelöf T_1 spaces with points G_δ of all cardinalities except possibly for singulars of uncountable cofinality.

A space X is *quasi-Lindelöf* if for each open cover $\mathcal{U}$, there is a countable $\mathcal{V} \subseteq \mathcal{U}$ such that $\bigcup\{\overline{U} : U \in \mathcal{V}\}$ covers X. For regular spaces, quasi-Lindelöfness coincides with Lindelöfness, but this need not be the case for Hausdorff spaces. In view

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of Arhangel'skiĭ's theorem [A] that $2^{\aleph_0}$ bounds the cardinality of first countable Lindelöf Hausdorff spaces, it is natural to ask [BC] whether the same bound works for first countable quasi-Lindelöf Hausdorff spaces. By a recent result of Bella and Yaschenko [BY] and, independently, Schröder and Watson (unpublished), it does not; below the first measurable cardinal they construct first countable quasi-Lindelöf Hausdorff spaces of sizes each strong limit cardinal of countable cofinality. It is still open whether in ZFC one can construct such spaces of regular cardinality; we shall prove it consistent with ZFC that such spaces exist at all regular cardinals.

Although the details of our proofs are non-trivial, they are not new; we are rather taking minor variations of other authors' proofs and tying them together in a new way. Thus we will merely indicate the variations and point the reader in the right directions, rather than essentially repeat published proofs.

First of all, let's prove

Theorem 1 *It is consistent with ZFC that there is a first countable quasi-Lindelöf Hausdorff space of size* $\aleph_2 = \left(2^{\aleph_0}\right)^+$.

Proof: We use an idea we used in [T] for Lindelöf spaces with points G_δ. We start with a model of GCH, adjoin $\aleph_1$ Cohen reals and then collapse each $\aleph_n$ to $\aleph_2$ with conditions of size less than $\aleph_2$. This will make the space which exists on $\beth_\omega$ in the ground model have cardinality $\aleph_2$. It clearly remains first countable and Hausdorff. But a straightforward modification of the proof in [D] that adding $\aleph_1$ Cohen reals makes ground model Lindelöf spaces indestructible under countably closed forcing shows that that adjunction also makes ground model quasi-Lindelöf spaces indestructible under countably closed forcing.

This idea can be extended to prove

Theorem 2 *it is consistent with ZFC that there is a first countable quasi-Lindelöf Hausdorff space of cardinality* κ *for* *every cardinal κ which is not singular of uncountable cofinality.*

Proof: We start with a model of GCH with no inaccessible cardinals, so that by [BY] there are first countable quasi-Lindelöf Hausdorff spaces on every cardinal of countable cofinality.

For each regular cardinal κ, define $SS(\kappa)$ the *singular successor* of κ to be the least singular cardinal above κ. Define $\mathrm{M}(\kappa)$ to be the least λ such that $SS(\kappa) = SS(\lambda)$.

Let P be the result of first adding $\aleph_1$ Cohen reals and then taking the product over all regular cardinals κ of the product of the collapses of κ to $M(SS(\kappa))$ with conditions of size $< M(SS(\kappa))$. (We could actually work below the first measurable, in which case we would also take the product of the collapses of $\aleph_{\kappa+n}$'s for $\aleph_\kappa$ inaccessible.) By standard techniques [E], this class forcing produces a model of ZFC + GCH in which for every regular uncountable cardinal κ, there is an ordinal σ of that cardinality which was an uncountable cardinal of countable cofinality in the ground model. P may be regarded as a product of adding the Cohen reals, collapsing σ to κ, and a countably closed partial order. The product of the latter two is countably closed so we can argue as in Theorem 1. Actually, to argue that way would require a finer analysis of Dow's proof – which was for *set* partial orders – than we are prepared to engage in, so let us instead note that the third factor in the product is κ^+–closed. But indeed κ-closed partial orders preserve the Lindelöfness of spaces of size $\leq \kappa$ [BT, Lemma 34] and the same proof works for quasi-Lindelöfness.

At singular cardinals of countable cofinality, by GCH we apply [BY].

Incidentally, exactly the same argument, using Juhasz' Lindelöf T_1 space with points G_δ which exists at every strong limit of countable cofinality below the first measurable, yields

Theorem 3 *It is consistent with GCH that there exist Lindelöf T_1 spaces with points G_δ on every cardinal which is not singular of uncountable cofinality.*

In [BT] we proved

Proposition *If it is consistent that there is a huge cardinal, it is consistent with GCH that every quasi-Lindelöf first countable space of size $\aleph_2$ includes a quasi-Lindelöf subspace of size $\aleph_1$.*

I do not see how to combine the constructions of these two papers so I ask:

Is it consistent with GCH that there is a quasi-Lindelöf first countable Hausdorff space of size $\aleph_2$ and each such space includes a quasi-Lindelöf subspace of size $\aleph_1$?

However I can get this at the cost of violating CH:

Theorem 4 *If it is consistent that there is a weakly compact cardinal, it is consistent that $2^{\aleph_0} = \aleph_2$, that every quasi-Lindelöf first countable space of size $\aleph_2$ has a quasi-Lindelöf subspace of size $\aleph_1$, and that there are quasi-Lindelöf first countable Hausdorff spaces of every cardinality except possibly singulars of uncountable cofinality.*

Proof: We first add $\aleph_1$ Cohen reals. This is a "mild" forcing, so preserves weakly compact cardinals [K, 10.16]. We then Mitchell-collapse [M] the first weakly compact κ to $\aleph_2$ (which makes $2^{\aleph_0} = \aleph_2$). We then perform the class forcing we did earlier, below the first measurable if any, but at the first stage only collapsing the $\aleph_n$'s, $n > 3$, to $\aleph_3$ with conditions of size less than $\aleph_3$. This will give us quasi-Lindelöf spaces of size $\geq \aleph_3$ except for singular cardinals of uncountable cofinality, since the first weakly compact cardinal is below the first measurable [K, 5.15]. There are of course quasi-Lindelöf first countable Hausdorff spaces of cardinality $\leq 2^{\aleph_0}$ in ZFC.

Since the class forcing adds no new spaces of size $\leq \aleph_2$, it does not affect the reflection argument. By standard arguments [DTW], [DJW], to obtain the reflection it suffices to prove that quasi-Lindelöfness is a Π^1_1 property in first countable spaces, and that it is preserved by Mitchell forcing. The latter is the same proof as for the Lindelöf case in [D]: first prove that adding $\geq \aleph_1$ Cohen reals makes ground model quasi-Lindelöf spaces indestructible by countably closed forcing. Then use that Mitchell forcing produces a model which is a submodel of the result of first adding Cohen reals and then doing countably closed forcing. Let $\mathcal{U}$ be an open cover of X produced by the Mitchell forcing. Without loss of generality, it is comprised of ground model open sets. The larger forcing preserves quasi-Lindelöfness, so there is a countable $\mathcal{V} \subseteq \mathcal{U}$ with closures covering in the bigger model. Then by CCC and countably closed, there is a countable $\mathcal{W} \supseteq \mathcal{V}$ in the ground model such that Cohen real forcing and hence Mitchell forcing forces $\mathcal{W} \subseteq \mathcal{U}$. But then $\bigcup\{\overline{W} : W \in \mathcal{W}\}$ covers in the bigger model and hence in the Mitchell model.

To see that quasi-Lindelöfness is Π^1_1 in first countable spaces, let the space sit on a cardinal λ and let a basis be $\mathcal{B} : \lambda \times \lambda \times \omega \to 2$, $\mathcal{B}(\alpha, \beta, n) = 1$ if and only if $\alpha \in$ the nth basic open set about β. A collection of basic open sets of size $\leq \lambda$ can then be coded by $\mathcal{U} : \lambda \to \lambda \times \omega$. A countable subcollection of $\mathcal{U}$ can be coded as $\mathcal{V} : \omega \to \lambda$. We then have

$$\begin{aligned}(\forall \mathcal{U} : \lambda \to \lambda \times \omega)[(\forall x \in \lambda)(\exists y \in \lambda)(\exists n \in \omega)(\mathcal{U}(\gamma) = \langle y, n\rangle \ \& \\ \mathcal{B}(x, y, n) = 1) \to (\exists \mathcal{V} : \omega \to \lambda)(\forall x \in \lambda)(\exists k \in \omega)(\exists y \in \lambda) \\ (\mathcal{U}(\mathcal{V}(k)) = \langle y, n\rangle \ \& (\forall m \in \omega)(\exists z \in \lambda) \\ (\mathcal{B}(x, z, m)) = 1 \ \& \ \mathcal{B}(y, z, n) = 1)].\end{aligned}$$

This is Π^1_1.

I do not see how to get the reflection to hold above $\aleph_2$.

References

[A] A.V. Arhangel'skiĭ, *The power of bicompacta with first axiom of countability*, Sov. Math. Dokl., **10** (1969), 951-955.

[BC] A Bella and F. Cammaroto, *On the cardinality of Urysohn spaces*, Canad. Math. Bull., **31** (1988), 153-158.

[BT] J.E. Baumgartner and F.D. Tall, *Reflecting Lindelöfness*, preprint.

[BY] A. Bella and I.V. Yaschenko, *Embeddings into first countable spaces with H-closed like properties*, Top. Appl., to appear.

[D] A. Dow, *Two applications of forcing and reflection to topology*, 155-172 in General Topology and its Relations to Modern Analysis and Algebra VI, Proc. Sixth Prague Top. Symp. 1986, ed. Z. Frolik, Heldermann, Berlin, 1986.

[DJW] A. Dow, I. Juhász and W. Weiss, *Integer - valued functions and increasing unions of first countable spaces*, Israel J. Math., **67** (1989), 181-192.

[DTW] A. Dow, F.D. Tall, W.A.R. Weiss, *New proofs of the consistency of the normal Moore space conjecture*, II, Top. Appl., **37** (1990), 115-129.

[E] W.B. Easton, *Powers of regular cardinals*, Ann. Math. Logic, **1** (1970), 139-178.

[J] I. Juhász, *Cardinal functions in topology - ten years later*, Math. Centre, Amsterdam, 1980.

[M] W. Mitchell, *Aronszajn trees and the independence of the transfer property*, Ann. Math. Logic, **5** (1972), 21-46.

[T] F.D. Tall, *On the cardinality of Lindelöf spaces with points G_δ*, Top Appl., **63** (1995), 21-38.

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