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ON THE SQUARE OF WALLACE SEMIGROUPS AND TOPOLOGICAL FREE ABELIAN GROUPS

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Abstract

We show under $MA_{countable}$ the existence of a counterexample for the Wallace problem whose square is not countably compact. We show under $MA_{\sigma-centered}$ that there exists a free Abelian group endowed with a countably compact group topology whose square is not countably compact.

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0. Introduction.

Wallace Semigroups: It is known since long ago that every two-sided cancellative semigroup endowed with a compact semigroup topology is a topological group. In 1955, A. D. Wallace [Wa] asked what happens if we replace "compact" by "countably compact", that is whether every countably compact two-sided cancellative semigroup is a topological group (see the surveys [Co1], [Co2] and [CHR] for a discussion on the Wallace Problem). Many researchers have obtained various results related to this question by adding either a topological or a algebraic condition. For instance, Mukherjea and Tserpes [MT] showed that a first countable sequentially compact two-sided cancellative semigroup is a topological group and Pfister [Pf] showed that a counterexample for Wallace's question could not be algebraically a group ([Gr1], [Gr2] and [Re] contains generalizations to these results.).

However, only recently was it shown that there are (consistent) counterexamples to Wallace's question (such semigroups will be called Wallace). Robbie and Svetlichny [RS1] showed that if a topological group satisfies certain properties then it contains a subsemigroup which is Wallace. The conditions required in [RS1] were known to be satisfied only by Tkačenko's group constructed under the Continuum Hypothesis (we discuss more about this group later). Thus, Robbie and Svetlichny [RS1] solved Wallace's question only under this axiom.

In [To1] we showed that the existence of a Wallace semigroup which is initially ω_1-compact is independent of ZFC. In particular we constructed a Wallace semigroup using a strong form of the Baire Category theorem:

(#) *The circle is not the union of fewer than* $\mathfrak{c}$ *many closed meager subsets,* which is equivalent to Martin's Axiom restricted to countable partial orders ($MA_{countable}$).

Countable compactness in free Abelian groups: It is also an old result that a free Abelian group cannot be endowed with a compact group topology. In [Tk], Tkačenko showed that the free Abelian group of size $\mathfrak{c}$ can be endowed with a countably compact group topology without non-trivial convergent sequences (by [RS1] the existence of such a group implies the existence of a Wallace semigroup).

In [To3] we showed that the existence of an initially ω_1-compact group topology on some free Abelian group is independent of $\mathfrak{c} = \aleph_2$. In particular, we showed that a group as Tkačenko's could be constructed under $MA_{\sigma-centered}$.

As a shortening, by a "countably compact free Abelian group" we mean a "free Abelian group endowed with a compatible group topology which is countably compact".

Products and powers: The existence of two countably compact topological groups whose product is not countably compact is a difficult question which has not been solved yet in ZFC (see the surveys [Co2],[CHR]). The first example of two countably compact groups whose product is not countably compact was obtained by van Douwen [vD] under Martin's Axiom. Later, Hart and van Mill [HvM] showed under $MA_{countable}$ the existence of a countably compact group whose square is not countably compact.

This motivates the study of countable compactness in the products of Wallace semigroups and the products of free Abelian groups endowed with a countably compact group topologies. In [Tk], Tkačenko mentioned that using an argument similar to that in [vD], one can construct under CH two countably compact free Abelian groups whose product is not countably compact. In [RS2] Robbie and Svetlichny also used an argument similar to that in [vD] to obtain two Wallace semigroups whose product is not countably compact.

In this work, we give a complete proof of the existence, under $MA_{countable}$, of a Wallace semigroup whose square is not countably compact. We will then sketch the construction, under $MA_{\sigma-centered}$, of a group topology on the free Abelian group of size $\mathfrak{c}$ which makes this group countably compact and its square not countably compact.

In this paper, $\mathbb{T}$ will denote the unitary circle group with the usual metric inherited from the plane and $\mathbb{T}^{\mathfrak{c}}$ will be endowed with the Tychonoff product topology. Given $y \in \mathbb{T}^{\mathfrak{c}}$, we denote by $supp\, y$ the set $\{\alpha < \mathfrak{c} : y(\alpha) \neq 0\}$, the support of y. The set of all non negative integers will be denoted by $\mathbb{N}$.

1 Wallace semigroups

In [To1], we showed under $MA_{countable}$ the existence of an $x \in \mathbb{T}^{\mathfrak{c}}$ such that the subsemigroup S of $\mathbb{T}^{\mathfrak{c}}$ generated by x and $G = \{g \in \mathbb{T}^{\mathfrak{c}} : supp\, g$ is bounded in $\mathfrak{c}\}$ is a Wallace semigroup ($S = \{nx + g : n \in \mathbb{N}$ and $g \in G\}$).

One can show that S^{ω} cannot be countably compact. However we were not able to decide whether the square of S is countably compact. Thus, instead of a single x, we use countably many elements of $\mathbb{T}^{\mathfrak{c}}$ to construct the Wallace semigroup below.

Example 1. ($MA_{countable}$) *There exists a countable subset X of $\mathbb{T}^{\mathfrak{c}}$ such that the semigroup generated by X and $G = \{g \in \mathbb{T}^{\mathfrak{c}} : supp\, g$ is bounded in $\mathfrak{c}\}$ is a Wallace semigroup whose square is not countably compact.*

The sketch of the construction. To make the semigroup generated by X and G countably compact, we will use an argument similar to one used in [HvM] and [To1]. Consider the following condition

($*$) *every sequence in the semigroup generated by X has an accumulation point in G.*

We will show that if X and G satisfy ($*$) then the semigroup generated by X and G is countably compact.

To be sure that the semigroup S generated by X and G is not a group, we use a modification of an argument from [To1]. We will construct X to satisfy

(**) for any distinct $x_1, \dots x_k \in X$ and positive integers $n_1, \dots, n_k$, the support of $\sum_{i=1}^{k} n_i x_i$ is unbounded in $\mathfrak{c}$.

To make the square of the semigroup S not countably compact, we will make sure that the sequence of pairs $\{(x_{2n}, x_{2n+1}) : n \in \omega\}$ is closed and discrete in the semigroup S, where $X = \{x_n : n \in \omega\}$. For this, we will construct X to satisfy

$(***)$ for all e_0 and e_1 on the semigroup generated by X and for each $\beta < \mathfrak{c}$, there exist $\alpha \in [\beta, \mathfrak{c})$ and $M \in \mathbb{N}$ such that $\{n \in \omega : \forall j \in 2 \ |x_{2n+j}(\alpha) - e_j(\alpha)| < \dfrac{1}{M+1}\}$ is finite.

Let us show now that $(*) - (***)$ imply all other properties we are interested in.

First, we recall the concept of p-limit. Given a free ultrafilter p over ω and a topological space X, we say that $x \in X$ is a p-limit of a sequence $\{x_n : n \in \omega\}$ if for each open neighbourhood U of x, the set $\{n \in \omega : x_n \in U\}$ belongs to p. A space X is p-compact (for this ultrafilter p) if every sequence in X has a p-limit. We use in the proof of Lemma 2 below the following facts:

- for each sequence $\{x_n : n \in \omega\} \subseteq X$ and for each accumulation point x of this sequence, there exists an ultrafilter p such that x is the p-limit of $\{x_n : n \in \omega\}$;

- ω-boundedness of G (the closure of every countable subset is compact) is equivalent to "G is p-compact, for every free ultrafilter p over ω";

- in a Hausdorff topological semigroup, the p-limits are unique and the sum of p-limits of two sequences equals the p-limit of the sum of those sequences;

-the p-limits of a sequence are, in particular, accumulation points.

Lemma 2. *If X satisfies $(*)$ then the semigroup generated* *by X and G is countably compact.*

Proof: Let E be the semigroup generated by X and let S be the semigroup generated by X and G. Let $\{s_n :, n \in \omega\} \subseteq S$ be arbitrary. Then there exists a sequence $\{e_n : n \in \omega\}$ in E and a sequence $\{g_n : n \in \omega\}$ in G such that $s_n = e_n + g_n$ for each $n \in \omega$. If $\{e_n : n \in \omega\}$ contains a constant subsequence then $\{e_n : n \in \omega\}$ has an accumulation point. Otherwise, there exists a subsequence $\{e_{n_k} : k \in \omega\}$ such that for distinct $k, l \in \omega$; we have $e_{n_k} \neq e_{n_l}$. Then, by $(*)$, $\{e_{n_k} : k \in \omega\}$ has an accumulation point in $G \subseteq S$.

In any case, there exists a p-limit of $\{e_n : n \in \omega\}$ in S, for some ultrafilter p over ω. Since G is ω-bounded, the sequence $\{g_n : n \in \omega\}$ has a p-limit in G. Thus, the sequence $\{s_n : n \in \omega\}$ $=\{e_n + g_n : n \in \omega\}$ has a p-limit in $S + G = S$. This proves that every sequence in S has an accumulation point. □

Lemma 3. *If X satisfies $(**)$ then the semigroup generated by X and G is not a group.*

Proof: Every non-zero element of the semigroup E generated by X is of the form $\sum_{i=1}^{k} n_i x_i$, for some finite subset $\{x_1, \ldots, x_k\}$ of X and positive integers $n_1, \ldots, n_k$.

Let x be a non-zero element of E. We will show that x does not have the inverse in the semigroup S generated by X and G. Indeed, suppose that there exist $e \in E$ and $g \in G$ such that $x + e + g = 0$. Then, $x + e$ cannot have unbounded support in $\mathfrak{c}$, thus by $(**)$, we have $x + e = 0$. This contradicts $(**)$, since x is a non-zero element of E. □

Lemma 4. *If X satisfies $(***)$ then the square of the semigroup generated by X and G is not countably compact.*

Proof: Let E be the semigroup generated by X and S be the semigroup generated by X and G. Let (s_0, s_1) be an arbitrary point of $S \times S$. We will show that (s_0, s_1) is not an accumulation point of the sequence $\{(x_{2n}, x_{2n+1}) : n \in \omega\}$. There exist $e_0, e_1 \in E$ and $g_0, g_1 \in G$ such that $s_j = e_j + g_j$ for $j \in$

2. Let $\beta < \mathfrak{c}$ be large enough such that $supp\, g_0 \cup supp\, g_1 \subseteq \beta$. By $(***)$, there exist $\alpha \in [\beta, \mathfrak{c})$ and $M \in \mathbb{N}$ such that $\{n \in \omega : \forall j \in 2 |x_{2n+j}(\alpha) - e_j(\alpha)| < \frac{1}{M+1}\}$ is finite. Since $g_0(\alpha) = g_1(\alpha) = 0$, we have $s_j(\alpha) = e_j(\alpha)$ for $j \in 2$. Thus the set $\{n \in \omega : \forall j \in 2\ |x_{2n+j}(\alpha) - s_j(\alpha)| < \frac{1}{M+1}\}$ is finite. This means that (s_0, s_1) has an open neighbourhood which misses all but finitely many elements of the sequence $\{(x_{2n}, x_{2n+1}) : n \in \omega\}$. Since $(s_0, s_1) \in S \times S$ was arbitrary, the sequence $\{(x_{2n}, x_{2n+1}) : n \in \omega\}$ does not have an accumulation point in $S \times S$. □

We will now start the construction of a set X satisfying properties $(*) - (***)$. We recall that such a set X cannot be constructed in ZFC (see [To1]).

Further details. The construction of $X = \{x_n : n \in \omega\}$ will be by induction. At stage $\alpha + 1 < \mathfrak{c}$, we define $x_n(\alpha)$ for each $n \in \omega$. Thus at stage $\beta < \mathfrak{c}$, we know what the restricton of x_n to β is, and by abuse of notation, we will denote this element of T^β by $x_n|_\beta$.

If $g : \emptyset \longrightarrow \mathbb{N} \setminus \{0\}$ is the empty function and $\beta \leq \mathfrak{c}$ then $\sum_{n \in F} g(n)(x_n|_\beta)$ will be the the zero of $\mathbb{T}^\beta$. Note that if f is a function from a finite subset F of ω into $\mathbb{N} \setminus \{0\}$, then $\sum_{n \in F} f(n)(x_n|_\beta) = (\sum_{n \in F} f(n) x_n)|_\beta$. During the inductive construction we will use the functions f as above to code the semigroup generated by X.

To make $(**)$ true, we fix an enumeration $\{f_\alpha : \alpha < \mathfrak{c}\}$ of the family of all non-empty functions from a finite subset of ω into the positive integers such that each element of this countable family appears $\mathfrak{c}$ times. Then, at stage $\alpha + 1$, we will make sure that $\sum_{n \in dom\, f_\alpha} f_\alpha(n) x_n(\alpha) \neq 0$.

To guarantee that $(***)$ is satisfied, we fix an enumeration $\{(g^0_\alpha, g^1_\alpha) : \alpha < \mathfrak{c}\}$ of the family of all pairs of functions from a finite subset (which can be empty) of ω into the positive integers such that each element of the family appears $\mathfrak{c}$ many times. Then, at stage $\alpha+1$, we will ensure that $\{n \in \omega : \forall j \in 2\ |x_{2n+j}(\alpha) - \sum_{k \in dom\, g^j_\alpha(k)} g^j_\alpha(k) x_k(\alpha)| < \frac{1}{M+1}\}$is finite for some $M \in \mathbb{N}$.

To make $(*)$ true, let $\{\mathcal{F}_\alpha : \alpha < \mathfrak{c}\}$ be an enumeration of all countably infinite subsets of the family of functions from a finite subset of ω into the positive integers. Then for every countably infinite subset A of the semigroup generated by X there exists $\alpha < \mathfrak{c}$ such that $A = \{\sum_{n \in dom\, f} f(n) x_n : f \in \mathcal{F}_\alpha\}$.

At stage $\alpha+1$, we will assign a function $h_\alpha \in G$ to be an accumulation point of the sequence $\{\sum_{n \in dom\, f} f(n) x_n : f \in \mathcal{F}_\alpha\}$. If we keep this promise then $h_\alpha|_\beta$ will be an accumulation point for $\{\sum_{n \in dom\, f} f(n) x_n|_\beta : f \in \mathcal{F}_\alpha\}$ for each $\beta < \mathfrak{c}$.

In particular, this must hold for $\beta \geq \alpha$. By Lemma 5 below, this suffices to guarantee that h_α be an accumulation point of A.

Lemma 5. *Let $\{h_n : n \in \omega\} \cup \{h\}$ be a subset of $\mathbb{T}^\mathfrak{c}$. Then for a limit $\alpha \leq \mathfrak{c}$, $h|_\alpha$ is an accumulation point of $\{h_n|_\alpha : n \in \omega\}$ if and only if $h|_\beta$ is an accumulation point of $\{h_n|_\beta : n \in \omega\}$ for each $\beta < \alpha$.*

Let us fix the enumerations $\{(g^0_\alpha, g^1_\alpha) : \alpha < \mathfrak{c}\}$ and $\{\mathcal{F}_\alpha : \alpha < \mathfrak{c}\}$ as above. Instead of fixing an enumeration $\{f_\alpha : \alpha < \mathfrak{c}\}$ we can use the enumeration $\{g^0_\alpha : \alpha < \mathfrak{c} \wedge g^0_\alpha \neq \emptyset\}$. Here we resume what we have concluded so far:

Lemma 6. *Suppose that there exists a family $X = \{x_n : n \in \omega\} \subseteq \mathbb{T}^\mathfrak{c}$ and a family $\{h_\alpha : \alpha < \mathfrak{c}\} \subseteq G$ such that for each $\alpha < \mathfrak{c}$:*

(i) if $dom\, g^0_\alpha \neq \emptyset$ then $\sum_{k \in dom\, g^0_\alpha} g^0_\alpha(k) x_k(\alpha) \neq 0$;

(ii) $h_\alpha|_\alpha$ is an accumulation point of the sequence $\{\sum_{k \in dom\, f} f(k) x_k|_\alpha : f \in \mathcal{F}_\alpha\}$;

(iii) if $\beta \leq \alpha$ *and* $h_\beta|_\alpha$ *is an accumulation point of the sequence* $\{\sum_{k \in dom\, f} f(k)x_k|_\alpha : f \in \mathcal{F}_\beta\}$ *then* $h_\beta|_{\alpha+1}$ *is an accumulation point of the sequence* $\{\sum_{k \in dom\, f} f(k)x_k|_{\alpha+1} : f \in \mathcal{F}_\beta\}$*;*

(iv) the set $\{n \in \omega : |x_{2n+j}(\alpha) - \sum_{k \in dom\, g^j_\alpha} g^j_\alpha(k)x_k(\alpha)| < \frac{1}{M+1}, \forall j \in 2\}$ *is finite for some* $M \in \mathbb{N}$.

Then the semigroup generated by X *and* G *is a Wallace semigroup whose square is not countably compact.*

Lemma 7. $(MA_{countable})$ *There exist the sequences* $\{x_n : n \in \omega\}$ *and* $\{h_\alpha : \alpha < \mathfrak{c}\}$ *satisfying the conditions of Lemma 6.*

Proof. At stage $\gamma = 0$, we have nothing to do, since $x_n|_0 = \emptyset$. Suppose we have constructed $x_n|_\beta$ and h_β for all $n \in \omega$ and $\beta < \gamma$ satisfying the conditions $(i) - (iv)$.

At a limit stage γ, define $x_n|_\gamma = \bigcup_{\beta<\gamma} x_n|_\beta$ for each $n \in \omega$. Clearly all four conditions are trivially satisfied.

At stage $\gamma = \alpha + 1$, we first define the function $h_\alpha \in G$. The sequence $\{\sum_{k \in dom\, f} f(k)x_k|_\alpha : f \in \mathcal{F}_\alpha\} \subseteq \mathbb{T}^\alpha$ is already defined, thus one can fix an accumulation point $y \in \mathbb{T}^\alpha$ for this sequence. Define $h_\alpha = y \cup 0|_{[\alpha,\mathfrak{c})}$. Then, (ii) is satisfied by α.

We will construct a function $\phi : \omega \longrightarrow \mathbb{T}$ so that if we define $x_n(\alpha) = \phi(n)$ for each $n \in \omega$ then conditions $(i), (iii)$ and (iv) are satisfied by α. To deal with condition (iii), we fix the following notation.

Definition 8. *For each* $\beta < \gamma$*, a finite subset* F *of* α *and* $m \in \mathbb{N}$*, let* $E(\beta, F, m) = \{f \in \mathcal{F}_\beta : \forall \xi \in F\ |\sum_{k \in dom\, f} f(k)x_k(\xi) - h_\beta(\xi)| < \frac{1}{m+1}\}$.

Clearly, every set $E(\beta, F, m)$ is infinite if and only if $h_\beta|_\alpha$ is an accumulation point of the sequence $\{\sum_{k \in dom\, f} f(k)x_k|_\alpha : f \in \mathcal{F}_\beta\}$ for each $\beta \leq \alpha$.

It is left to the reader to check that conditions $(i), (iii)$ and (iv) will be satisfied if ϕ satisfies the properties listed in Lemma

9 below.

Lemma 9. ($MA_{countable}$) *There exists a function* $\phi : \omega \longrightarrow \mathbb{T}$ *satisfying the following properties:*

(*A*) *if* $dom\, g^0_\alpha \neq \emptyset$ *then* $\sum_{k \in dom\, g^0_\alpha} g^0_\alpha(k)\phi(k) \neq 0$.

(*B*) *for every* $\beta < \gamma$, *a finite subset* F *of* α *and* $m \in \mathbb{N}$, *the set* $\{f \in E(\beta, F, m) : |\sum_{k \in dom\, f} f(k)\phi(k) - h_\beta(\alpha)| < \frac{1}{m+1}\}$ *is infinite.*

(*C*) *There exists* $M \in \mathbb{N}$ *such that the set* $\{n \in \omega : \forall j \in 2\ |\phi(2n+j) - \sum_{k \in dom\, g^j_\alpha} g^j_\alpha(k)\phi(k)| < \frac{1}{M+1}\}$ *is finite.* □

The construction of ϕ requires the use of a partial order and dense sets, so the proof of Lemma 9 will be done in the next two subsections.

The partial order. Let $\mathcal{B}$ be a countable basis for $\mathbb{T}$ consisting of non-empty connected open sets and such that $\mathbb{T} \in \mathcal{B}$. Fix $M \in \omega$ such that $2M \supseteq dom\, g^0_\alpha \cup dom\, g^1_\alpha$. For each $i < 2M$, fix $V_i \in \mathcal{B}$ and $W_0, W_1 \in \mathcal{B}$ whose length is at most π such that:

(*a*) if $dom\, g^0_\alpha \neq \emptyset$ then $0 \notin \sum_{k \in dom\, g^0_\alpha} g^0_\alpha(k)V_k$ and $\sum_{k \in dom\, g^0_\alpha} g^0_\alpha(k)V_k \subseteq W_0$;

(*b*) if $dom\, g^1_\alpha \neq \emptyset$ then $\sum_{k \in dom\, g^1_\alpha} g^1_\alpha(k)V_k \subseteq W_1$;

(*c*) if $j \in 2$ and $dom\, g^j_\alpha = \emptyset$ then $0 \in W_j$.

Definition 10. Let $(\mathbb{P}, \leq)$ be a partial order whose underlying set $\mathbb{P}$ consists of functions p for which $\exists n \geq M$ such that $dom\, p = 2n, rng\, p \subset \mathcal{B}$ and $(\forall k \in 2M)$ $(p(k) \subseteq V_k)$.

Denote by M_p the unique integer such that $dom\, p = 2M_p$. Given $p, q \in \mathbb{P}$, we define $p \leq q$ if and only if $dom\, p \supseteq dom\, q$, for each $k \in dom\, p \setminus dom\, q$ either $p(k) = q(k)$ or $\overline{p(k)} \subseteq q(k)$ and for each $m \in [M_q, M_p)$ we have $p(2m) \times p(2m+1) \cap W_0 \times W_1 = \emptyset$.

Clearly $(\mathbb{P}, \leq)$ is a countable partial order. By abuse of notation, this partial order will be denoted by $\mathbb{P}$. We will find a generic set $\mathcal{G}$ for a family of suitable dense subsets of $\mathbb{P}$ which we will define later; and for each $k \in \omega$, we will choose $\phi(k)$ as an element of $\bigcap_{p\in\mathcal{G},k\in dom\, p} \overline{p(k)} = \bigcap_{p\in\mathcal{G},k\in dom\, p} p(k)$. The last equality holds because of the ordering of $\mathbb{P}$ and the fact that $\mathcal{G}$ is a filter.

Suppose that $\phi : \omega \longrightarrow \mathbb{T}$, that is, for each $k \in \omega$, there exist, $p \in \mathcal{G}$ such that $k \in dom\, p$. We will check now that conditions (A) and (C) are satisfied.

We have to worry about condition (A) only if $dom\, g_\alpha^0 \neq \emptyset$. In this case we have chosen V_i's so that $0 \notin \sum_{k\in dom\, g_\alpha^0} g_\alpha^0(k) V_k$. Furthermore, $\phi(k) \in V_k$ for each $k \in dom\, g_\alpha^0$ thus $\sum_{k\in dom\, g_\alpha^0} g_\alpha^0(k)\phi(k) \in \sum_{k\in dom\, g_\alpha^0} g_\alpha^0(k) V_k \subseteq \mathbb{T} \setminus \{0\}$.

For condition (C), first note that $\sum_{k\in dom\, g_\alpha^j} g_\alpha^j(k)\phi(k) \in W_j$ for $j \in 2$. Thus, we can fix $m \in \mathbb{N}$ such that $B(\sum_{k\in dom\, g_\alpha^j} g_\alpha^j(k)\phi(k), \frac{1}{m+1}) \subseteq W_j$ for $j \in 2$, where $B(x, r)$ is the open ball of center $x \in \mathbb{T}$ and radius r in the usual metric of the plane. Let q be an arbitrary element of $\mathcal{G}$. We claim that

$$\{n \in \omega : \forall j \in 2\, |\phi(2n+j) - \sum_{k\in dom\, g_\alpha^j} g_\alpha^j(k)\phi(k)| < \frac{1}{m+1}\} \subseteq M_q.$$

Indeed, let $n \geq M_q$, Then there exists $p \leq q$ in $\mathcal{G}$ such that $2n \in dom\, p$. Since $n \in [M_q, M_p)$, we have $p(2n) \times p(2n+1) \cap W_0 \times W_1 = \emptyset$. However, $(\phi(2n), \phi(2n+1)) \in p(2n) \times p(2n+1)$. Therefore $\phi(2n+i) \notin W_i$ for some $i \in 2$ and hence (C) is satisfied.

Thus, all we have to do now is to show that ϕ has domain ω and that condition (B) holds.

Dense sets. To show that, ϕ has domain ω it suffices to prove the following.

Lemma 11. *The set* $\mathcal{D}_n = \{p \in \mathbb{P} : M_p \geq n\}$ *is dense in* $\mathbb{P}$ *for each for* $n \in \omega$.

Proof: Fix $n \in \omega$ and let $q \in \mathbb{P}$ be arbitrary. We can assume that $n > M_q$. We will extend q to some p such that $M_q = n$. For each $m \in [M_q, n)$ and $j \in 2$, fix $U_{2m+j} \in \mathcal{B}$ such that $U_{2m+j} \cap W_j = \emptyset$. Then $p = q \cup \{\langle k, U_k \rangle : 2M_q \leq k < 2n\}$ is an element of $\mathcal{D}_n$. □

Definition 12. For each $\beta < \gamma$, for each finite subset F of α and for each $m \in \mathbb{N}$ fix a partition $\{E(\beta, F, m, l) : l \in \omega\}$ of the set $E(\beta, F, m)$ from Definition 8 such that each element of the partition is infinite.

We will now define dense sets which will take care of condition (B) from Lemma 9. Note that (B) is satisfied if the sets $\{f \in E(\beta, F, m, l) : \sum_{k \in dom\, f} f(k)\phi(k) \in B(h_\beta(\alpha), \frac{1}{m+1})\}$ are not empty.

Lemma 13. *The set* $\mathcal{S}(\beta, F, m, l) = \{p \in \mathbb{P} : \exists f \in E(\beta, F, m, l)$ *s.t.* $dom\, f \subseteq dom\, p$ *and* $\sum_{k \in dom\, f} f(k)p(k) \subseteq B(h_\beta(\alpha), \frac{1}{m+1})\}$ *is dense in* $\mathbb{P}$, *for each* $\beta < \gamma$, *for each finite subset* F *of* α, *for each* $m \in \mathbb{N}$ *and for each* $l \in \omega$.

Proof: Fix β, F, m, and l as above and let q be an arbitrary element of $\mathbb{P}$. Let us consider two cases:

Case 1. $dom\, f \setminus dom\, q$ is not empty for some $f \in E(\beta, F, m, l)$. In this case, let n be the largest element of $dom\, f$ and let $n = 2t + i$, where $t \in \omega$ and $i \in 2$. Let $j = 2 - i$ and fix $U_j \in \mathcal{B}$ such that $U_j \cap W_j = \emptyset$. We can extend q to some q_1 whose domain is $2t$. Then define $\tilde{q} = q_1 \cup \{\langle 2t + j, U_j \rangle, \langle 2t + i, \mathbb{T} \rangle\}$.

Case 2. $dom\, f \subseteq dom\, q$ for all $f \in E(\beta, F, m, l)$. Then there exists a finite subset F of ω and an infinite subset E of $E(\beta, F, m, l)$ such that $dom\, f = F$ for all $f \in E$. By finite number of refinements one can find an infinite subset E_1 of E such that either $\{f(s) : f \in E_1\}$ is constant or pairwise distinct for each $s \in F$. In particular, there exists $n \in F$ such that $\{f(n) : f \in E_1\}$ are pairwise distinct. Let us extend q to any $\tilde{q}$ such that $dom\,\tilde{q} \supseteq F$. Since $\tilde{q}(n)$ is a non-empty open subset of $\mathbb{T}$, there exists a positive integer K such that $K\tilde{q}(n) = \mathbb{T}$. Fix $f \in E_1$ such that $f(n) \geq K$.

Thus, in either case, there exists $\tilde{q}$ extending q such that $dom\,f \subseteq dom\,\tilde{q}$ and $f(n)\tilde{q}(n) = \mathbb{T}$ for some $n \in dom\,f$. Fix $a_s \in \tilde{q}(s)$ for each $s \in dom\,f \setminus \{n\}$. Since $f(n)\tilde{q}(n) = \mathbb{T}$ there exists $a_n \in \tilde{q}(n)$ such that $\sum_{s\in dom\,f} f(s)a_s \in B(h_\beta(\alpha), \dfrac{1}{m+1})$. Clearly, there exists $O_l \in \mathcal{B}$ for each $l \in dom\,f$ such that $a_l \in O_l \subseteq \overline{O}_l \subseteq \tilde{q}(l)$ and $\sum_{l\in dom\,f} f(l)O_l \subseteq B(h_\beta(\alpha), \dfrac{1}{m+1})$. Let $p = \{\langle l, \tilde{q}(l)\rangle : l \in dom\,\tilde{q} \setminus dom\,f\} \cup \{\langle l, O_l\rangle : l \in dom\,f\}$. Clearly $p \leq \tilde{q} \leq q$ and $p \in \mathcal{S}(\beta, F, m, l)$. □

Note that we have defined less than $\mathfrak{c}$ many dense sets, thus applying $MA_{countable}$, there exists a generic filter for these dense sets. Therefore the proof of Lemma 9 is complete and Example 1 is complete.

A connected countably compact group whose square is not countably compact. The countably compact group whose square is not countably compact obtained by Hart and van Mill [HvM] is a subgroup of $2^{\mathfrak{c}}$. In particular, the group is zero-dimensional. There are connected groups like this as well.

Example 14. ($MA_{countable}$) There exists a connected countably compact group whose square is not countably compact.

By means of a small modification of Example 1, one can construct under $MA_{countable}$, a countable subset X of $\mathbb{T}^{\mathfrak{c}}$ such that the group generated by X and G is countably compact but its square is not. Note that group $G = \{x \in \mathbb{T}^{\mathfrak{c}} : supp\,x$ is bounded in $\mathfrak{c}\}$ is connected and dense in $\mathbb{T}^{\mathfrak{c}}$. Thus any subspace of $\mathbb{T}^{\mathfrak{c}}$ containing G is also connected.

2. Free Abelian groups

We will give now a sketch of a construction of the following.

Example 15. ($MA_{\sigma-centered}$) There exists a countably compact group topology on the free Abelian group H of size $\mathfrak{c}$ which makes H countably compact but H^2 not countably compact.

We will construct a family $Y = \{y_\alpha : \alpha < \mathfrak{c}\} \subseteq \mathbb{T}^\mathfrak{c}$ such that the group H generated by Y is countably compact, Y is a free basis for H and $\{(y_{2n}, y_{2n+1}) : n \in \omega\}$ witnesses that $H \times H$ is not countably compact. At stage $\alpha + 1 < \mathfrak{c}$ we will define $y_\beta(\alpha)$ for every $\beta \leq \alpha$.

In Example 1, we needed a condition to make S not a group. A similar argument will be used to guarantee that Y is free. To obtain the countably compactness of H, we will enumerate all possible countably infinite subsets of H in length $\mathfrak{c}$ and the α-th sequence will have x_α as an accumulation point. The sequence $\{(y_{2n}, y_{2n+1}) : n \in \omega\}$ will be made closed and discrete in $H \times H$ as in the construction of Example 1.

The enumerations needed in this construction are similar to the ones used to construct Example 1, but the elements of H which we can work with at stage α must be generated by $\{y_\beta : \beta < \alpha\}$, since at this stage we will have only defined $y_\beta|_\alpha$ for each $\beta < \alpha$.

Let $\{g^0_\alpha, g^1_\alpha) : \alpha < \mathfrak{c}\}$ be an enumeration of all pair of functions from a finite subset of $\mathfrak{c}$ into non-zero integers so that $dom\, g^0_\alpha \cup dom\, g^0_\alpha \subseteq \alpha$ for each $\alpha < \mathfrak{c}$.

Let $\{\mathcal{F}_\alpha : \alpha < \mathfrak{c}\}$ be an enumeration of all countable subsets of the family of functions from a finite subset of $\mathfrak{c}$ into non-zero integers so that $\bigcup_{f \in \mathcal{F}_\alpha} dom\, f \subseteq \alpha$ for each $\alpha < \mathfrak{c}$.

It is left to the reader to check that a group H generated by Y in Lemma 16 is a group as in Example 15.

Lemma 16. ($MA(\sigma$-*centered*)) *There exists* $Y = \{y_\alpha : \alpha < \mathfrak{c}\}$ *satisfying*

(*i*) *if* $dom\, g^0_\alpha \neq \emptyset$ *then* $\sum_{\xi \in dom\, g^0_\alpha} g^0_\alpha(\xi) y_\xi(\alpha) \neq 0$;

(ii) $y_\alpha|_\alpha$ *is an accumulation point of* $\{\sum_{\xi\in dom\, f} f(\xi)y_\xi|_\alpha : f \in \mathcal{F}_\alpha\}$*;*

(iii) *if* $\beta \leq \alpha$ *and* $y_\beta|_\alpha$ *is an accumulation point of* $\{\sum_{\xi\in dom\, f} f(\xi)y_\xi|_\alpha : f \in \mathcal{F}_\beta\}$ *then* $y_\beta|_{\alpha+1}$ *is an accumulation point of* $\{\sum_{\xi\in dom\, f} f(\xi)y_\xi|_{\alpha+1} : f \in \mathcal{F}_\beta\}$*;*

(iv) *The set* $\{n \in \omega : |y_{2n+j}(\alpha) - \sum_{\xi\in dom\, g_\alpha^j} g_\alpha^j(\xi)y_\xi(\alpha)| < \frac{1}{M+1}, \forall j \in 2\}$ *is finite, for some* $M \in \mathbb{N}$.

Proof: At stage 0 there is nothing to do, since $x_0|_0$ is the empty function. At limit stage γ define $y_\beta|_\gamma = \bigcup_{\beta<\alpha<\gamma} y_\beta|_\alpha$. In both cases conditions $(i) - (iv)$ are satisfied.

At successor stage $\gamma = \alpha + 1$, fix an accumulation point $y_\alpha|_\alpha$ for the sequence $\{\sum_{\xi\in dom\, f} f(\xi)y_\xi|_\alpha : f \in \mathcal{F}_\alpha\}$. As in Lemma 9, we construct a function $\phi : \gamma \longrightarrow \mathbb{T}$ and define $y_\beta(\alpha) = \phi(\beta)$ for each $\beta < \gamma$. We will only give the partial order and the dense sets and leave the details to the reader.

The partial order. Let $\mathcal{B}$ be a countable basis for $\mathbb{T}$ consisting of non-empty connected open sets and such that $\mathbb{T} \in \mathcal{B}$.

Fix $M \in \omega$ such that $2M \supseteq (dom\, g_\alpha^0 \cup dom\, g_\alpha^1) \cap \omega$. For each $\xi \in (dom\, f \setminus \omega) \cup 2M$, fix an open set $V_\xi \in \mathcal{B}$ and fix $W_0, W_1 \in \mathcal{B}$ which covers at most half of the circle such that:

(a) if $dom\, g_\alpha^0 \neq \emptyset$ then $0 \notin \sum_{\xi\in dom\, g_\alpha^0} g_\alpha^0(\xi)V_\xi$ and $\sum_{\xi\in dom\, g_\alpha^0} g_\alpha^0(\xi)V_\xi \subseteq W_0$;

(b) if $dom\, g_\alpha^1 \neq \emptyset$ then $\sum_{\xi\in dom\, g_\alpha^1} g_\alpha^1(\xi)V_\xi \subseteq W_1$;

(c) if $j \in 2$ and $dom\, g_\alpha^j = \emptyset$ then $0 \in W_j$.

Definition 17. Let $\mathbb{P}$ be a partial order whose underlying set is the family of all finite functions p such that $dom\, p \subseteq \gamma$, $\exists n \in M$ such that $dom\, p \cap \omega = 2n, rng\, p \subseteq \mathcal{B}$ and $p(\xi) \subseteq V_\xi$

for all $\xi \in (dom\, f \setminus \omega) \cup 2M$. Denote by M_p the unique integer such that $dom\, p \cap \omega = 2M_p$. Given $p, q \in \mathbb{P}$, we define:

$p \leq q$ if and only if $dom\, p \supseteq dom\, q$, either $p(\beta) = q(\beta)$ or $p(\beta) \subseteq q(\beta)$ for each $\beta \in dom\, p \setminus dom\, q$ and $p(2m) \times p(2m+1) \cap W_0 \times W_1 = \emptyset$ for each $m \in [M_q, M_p)$.

Let us show that $\mathbb{P}$ is σ-centered. Indeed, consider $\mathcal{B}$ as a discrete space and take a countable dense subset S of $\mathcal{B}^\gamma$. Let F be the family of all finite functions from some $2n \in \omega$ into $\mathcal{B}$. Then $\mathbb{P} = \bigcup_{f \in F, s \in S} \{p \in \mathbb{P} : f = p|_{(dom\, p \cap \omega)} \text{ and } p \subseteq s\}$ is a countable union of centered subsets.

The dense subsets. The following dense sets are used to make $dom\, \phi = \gamma$:

Definition 18. For each $\beta < \gamma$ let $\mathcal{D}_\beta = \{p \in \mathbb{P} : \beta \in dom\, p\}$.

Fix $\beta < \gamma$, a finite subset F of α and $m \in \mathbb{N}$. By hypothesis, the set $\{f \in \mathcal{F}_\beta : \forall \xi \in F \ \sum_{\mu \in dom\, f} f(\mu) y_\mu(\xi) \in B(y_\beta(\alpha), \dfrac{1}{m+1})\}$ is infinite. Let $\{E(\beta, F, m, l) : l \in \omega\}$ be a partition of this set into infinite pieces of infinite size.

Definition 19. For each $\beta < \gamma$, for each finite subset F of α, for each $m \in \mathbb{N}$ and $l \in \omega$ let $\mathcal{S}(\beta, F, m, l) = \{p \in \mathbb{P} : \exists f \in E(\beta, F, m, l)$ such that $dom\, f \subseteq dom\, p$ and $\sum_{\xi \in dom\, f} f(\xi) p(\xi) \subseteq B(y_\beta(\alpha), \frac{1}{m+1})\}$.

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Remark (added in July 1998) A new argument from the author and S. Watson makes it possible to construct a countably compact group without non-trivial convergent sequences under $MA_{countable}$. This argument can be used to modify the construction of Example 15 to obtain a group as in Example 15 under $MA_{countable}$.

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