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ON DOW'S REFLECTION THEOREM FOR METRIZABLE SPACES

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Abstract

We prove that if X is a space in which every subspace of cardinality at most $\aleph_1$ is metrizable, and X has density $\aleph_1$, then X has weight $\aleph_1$. We also extend a reflection theorem of Alan Dow by proving that if X is a space in which every subspace of cardinality at most $\aleph_1$ is metrizable, and X has a dense set, conditionally compact in X, then X is metrizable. Known examples show that the compactness-like condition on X cannot be weakened to pseudocompactness or feeble compactness.

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1 Introduction

In [3] Alan Dow defined "for the purposes of [3] only" a space to be SSM if it is regular, non-metrizable and every $Y \in [X]^{\leq\omega_1}$ is metrizable (in the subspace topology), where $[X]^{\leq\omega_1}$ denotes $\{Y \subset X : |Y| \leq \omega_1\}$. We modify this definition. We say that a space X is *ss-metrizable* provided every $Y \in [X]^{\leq\omega_1}$ is metrizable (in the subspace topology). Thus the difference between these two definitions is that our definition includes no separtion axioms and allows metrizable spaces to be ss-metrizable. SSM and ss-metrizable are short for "small subspaces metrizable."

In this paper we review some known results about ss-metrizable spaces, and prove the following two new results.

Theorem 1.1 *If X is ss-metrizable with density ω_1 then X has weight ω_1.*

Theorem 1.2 *If X is ss-metrizable and has a dense set, conditionally compact in X, then X is metrizable.*

Theorem 1.1 is based on the proof of, and Theorem 1.2 is a corollary to, the following remarkable result of Dow [4, Prop. 3.1] in which no separation axioms are explicitly assumed (note that T_1 is implicit).

Theorem 1.3 (Dow) *If X is ss-metrizable and countably compact, then X is metrizable.*

It is natural to ask if "countably compact" in Dow's theorem can be replaced with "feebly compact," or the weaker property "pseudocompact" (see [9, 1.11(d)]). That the answer is "no" follows immediately from a theorem of D. B. Shakhmatov [10].

Theorem 1.4 (Shakhmatov) *If Y is a $T_{3.5}$-space in which every set of cardinality $\leq \kappa$ is closed, then Y can be embedded into a pseudocompact $T_{3.5}$-space X in which every subset of cardinality $\leq \kappa$ is closed.*

To apply Shakhmatov's theorem, take Y to be a discrete space of uncountable cardinality, take $\kappa \geq \omega_1$, and let X be the space guaranteed by Shakhmatov's theorem. Clearly X is ss-metrizable, pseudocompact (and feebly compact by $T_{3.5}$ [9, 1.11(d)]). The space X is not metrizable because a pseudocompact metric space is compact, hence has a countable base, and Y is a subspace not having a countable base.

In § 4 we prove Theorem 1.2. We show that in Dow's Theorem 1.3, we can replace "countably compact" with "has a dense set conditionally compact in X," a well-known property weaker than countable compactness and stronger than 'feebly compactness (and pseudocompactness).

Recall that a subset Y of a space X is *conditionally compact in* X [1] (or relatively countably compact [6]) provided that every infinite subset of Y has a limit point in X.

We also need the following special case of a reflection theorem of A. Hajnal and I. Junáz [7]:

Theorem 1.5 (Hajnal and Juhász) *If X is a space in which every subset $Y \in [X]^{\leq\omega_1}$ has a countable base (in the subspace topology), then X has a countable base.*

In § 3 we prove Theorem 1.1. In general we follow the terminology in [5], and we do not assume any separation axioms unless explicitly stated. We employ the approach to elementary submodels used by S. Watson and others (see [11]).

2 ss-metrizable spaces

In this section, we review some known results related to our theorems, and give proofs for completeness. Dow's first results about ss-metrizable spaces were proved using some separation axioms (such as Hausdorff [2] or regular [3]), but in [4], Dow showed that for ss-metrizable spaces of density ω_1 these results need no explicitly stated separation axioms at all (T_1 is implicit as we noted). We begin with the following simple example.

Example 2.1 *There exists an ss-metrizable, non-T_2-space.*

Define $L(\omega_2+1)$ to be the set of ordinals $\leq \omega_2$ with the smallest topology larger than the order topology in which all $\alpha < \omega_2$ are isolated. Let p be a point not in $L(\omega_2+1)$. Put $X = L(\omega_2+1) \cup \{p\}$, and define for each $\alpha < \omega_2$ a basic neighborhood of p to be $N(p,\alpha) = \{p\} \cup (\alpha, \omega_2)$. In X, p and ω_2 do not have disjoint neighborhoods. Every $Y \in [X]^{\leq \omega_1}$ is discrete because if we have $p, \omega_2 \in Y$ then p and ω_2 are isolated in Y; so X is ss-metrizable and not T_2.

The following lemma was extracted from the middle of Dow's proof [4, Theorem 3.1]. Richard E. Hodel, and later the referee, noticed that it has a standard proof without elementary submodels. We give a proof implicit in the proof of Dow's theorem.

Lemma 2.2 (Dow) *If X is ss-metrizable with density at most ω_1, then X is T_3, first countable, and $|X| \leq 2^\omega$.*

Proof: Let D be a dense subset of X with $|D| \leq \omega_1$.

Claim 1 *X is regular.* If X is not regular, there exist $x \in X$ and an open set U such that $x \in U$ and for all open V, if $x \in V$ then $cl_X V \not\subset U$. Let M be an elementary submodel with $X, D, x, U \in M$, $D \subset M$ and $|M| = |D|$. (Since there exist non-regular spaces having countable dense metrizable subspaces, we will have to use the hypothesis of the theorem on some subsets besides D; in fact we use it on both $D \cup \{x\}$, and $X \cap M$). Put $Y = D \cup \{x\}$. By elementarity, $Y \in M$. Also, $|Y| \leq \omega_1$; so by ss-metrizable Y is metrizable, hence first countable. By elementarity, there exists a countable family $\mathcal{B} \subset \mathcal{T}$ such that $\{B \cap Y : B \in \mathcal{B}\}$ is a local base for x in Y, $\mathcal{B} \in M$, and $\mathcal{B} \subset M$.

Since $|M| \leq \omega_1$, $X \cap M$ is metrizable, hence regular. Since $x \in U \cap X \cap M$, there exist an open set T in X such that $x \in T$ and

$$cl_{X \cap M}(T \cap X \cap M) \subset U \cap X \cap M.$$

Pick $B \in \mathcal{B}$ such that $B \cap Y \subset T \cap Y$. By denseness of Y we have $cl_X B \subset cl_X T$. To complete the proof (by contradiction) it suffices to show that $cl_X B \subset U$. If this is not true then there exists $p \in cl_X B \setminus U$; so by elementarity there exists such a p in M. Thus $p \in cl_X B \cap (X \cap M)$.

Fact $p \in cl_{X \cap M}(T \cap X \cap M)$: If $p \in W \in \mathcal{T}$, then $p \in W \cap cl_X(B) \subset W \cap cl_X(T)$; so $W \cap T \neq \emptyset$. Since $X \cap M$ is dense in X, we have that $W \cap T \cap X \cap M \neq \emptyset$. This proves the Fact.

This Fact, however, yeilds that $p \in U$, which is a contradiction. Thus we have that $x \in B \subset cl_X B \subset U$, but this contradicts our beginning assumption, and completes the proof of Claim 1.

Claim 2 *X is first countable.* Let $x \in X$. $Y = D \cup \{x\}$ is metrizable, hence first countable; so there exists a countable family of open sets $\mathcal{B}$ such that $\{B \cap Y : B \in \mathcal{B}\}$ is a local base for x in Y. We show that $\mathcal{B}$ is a local base for x in X. Let U be open in X and $x \in U$. By T_3 there exists an open set V such that $x \in V \subset cl_X V \subset U$. There exists $B \in \mathcal{B}$ such $(B \cap Y) \subset (V \cap Y)$. By denseness of Y we have $cl_X B \subset cl_X V$; so we have $x \in B \subset cl_X B \subset U$. This completes the proof of the Lemma. A general version of the proof of Claim 2 can be found in [8, 3.9(c)].

Claim 3 That $|X| \leq 2^\omega$ follows from Claims 1 and 2 and the set-theoretic equality $\omega_1^\omega = 2^\omega$.

Corollary 2.3 (Dow [3]) *Every separable ss-metrizable space X is metrizable.*

Proof: Let D be a countable dense subset of X. For any $Y \in [X]^{\leq\omega_1}$, $Y \cup D$ is separable and (by ss-metrizable) metrizable; hence second countable. Therefore Y is second countable. By the Hajnal-Juhász reflection theorem (Theorem 1.5), X is second countable. By Lemma 2.2, X is T_3, hence X is metrizable, and this completes the proof.

Remark 2.4 (Dow [3]) *Every ss-metrizable, non-metrizable space X contains a discrete subspace of cardinality ω_1. Hence every ss-metrizable space, with countable spread is metrizable.*

Proof: Since a non-metrizable ss-metrizable space X is not separable, it contains a left separated $Y \subset X$ of cardinality ω_1, which is therefore a non-separable, metrizable subspace, hence has uncountable cellularity. Thus Y is not a subspace of a space with countable spread.

Theorem 2.5 *The statement "every ss-metrizable space of density ω_1 is metrizable" is independent and consistent with ZFC*

Proof: Let X be an ss-metrizable space of density ω_1. By the Lemma, $|X| \leq 2^\omega$; so obviously X is metrizable under the assumption $2^\omega = \omega_1$. In the other direction, under MA + $\neg$CH, Dow [3] constructed a family of Lindelöf ss-metrizable non-metrizable spaces (the part of MA that Dow used is "$\mathbf{p} > \omega_1$"). Some of these spaces have density ω_1.

3 Density and weight in ss-metrizable spaces

A simple example, similar to Example 2.1, shows that in ss-metrizable spaces density need not equal weight.

Example 3.1 *An ss-metrizable $T_{3.5}$-space X of density ω_2 and weight greater than ω_2.*

Proof: Take $X = \omega_2 + 1$ with all points isolated except ω_2. Let u be a uniform ultrafilter on ω_2, then u does not have a base of size ω_2. For neighborhoods of ω_2, take all sets of the form $\{\omega_2\} \cup U$ where $U \in u$.

In order to prove Theorem 1.1, we use the technique in the last paragraph of Dow's proof of Theorem 1.3, and which uses the following lemma.

Lemma 3.2 (Dow) [4, Prop. 2.3] *If $(X, \mathcal{T})$ is a space with a point-countable base, and M is any elementary submodel with $X, \mathcal{T} \in M$, then $\mathcal{T} \cap M$ contains a local base in $(X, \mathcal{T})$ for every point of $cl_{(X,\mathcal{T})}(X \cap M)$.*

For a space $(X, \mathcal{T})$ and elementary submodel M with $X, \mathcal{T} \in M$, by elementarity the sets in $\mathcal{T} \cap M$, intersected with $X \cap M$, form a base for a topology on $X \cap M$ which is in general coarser than the subspace topology on $X \cap M$. We call this topology the *submodel topology* on $X \cap M$, and denote it by $(X \cap M, \mathcal{T} \cap M)$. We say that a family of countable elementary submodels $\{M_\alpha : \alpha < \omega_1\}$ is an $\in$-*chain* [4] provided for all $\alpha < \omega_1$, $M_\alpha \in M_{\alpha+1}$, and $M_\lambda = \cup_{\alpha<\lambda} M_\alpha$ for λ a limit ordinal.

Lemma 3.3 *If $\{M_\alpha : \alpha < \omega_1\}$ is an $\in$-chain of countable elementary submodels with $X, \mathcal{T} \in M_0$, and $M = \cup\{M_\alpha : \alpha < \omega_1\}$, and $(X, \mathcal{T})$ is a space such that the submodel topology $(X \cap M, \mathcal{T} \cap M)$ has a point countable base, then for cofinally many $\alpha < \omega_1$, $\mathcal{T} \cap M_\alpha$ contains a local base in $(X, \mathcal{T})$ for every point in $cl_{(X,\mathcal{T})}(X \cap M_\alpha)$.*

Proof: By contradiction, we may assume for every $\alpha < \omega_1$ that $\mathcal{T} \cap M_\alpha$ does not contain a local base in $(X, \mathcal{T})$ for all points in $cl_{(X,\mathcal{T})}(X \cap M_\alpha)$. For each $\alpha < \omega_1$ we construct a point x_α and open set U_α such that

(i) $x_\alpha \in cl_{(X,\mathcal{T})}(X \cap M_\alpha) \cap M_{\alpha+1}$,

(ii) $x_\alpha \in U_\alpha$ and $U_\alpha \in \mathcal{T} \cap M_{\alpha+1}$,

(iii) for every $T \in \mathcal{T} \cap M_\alpha$ with $x_\alpha \in T$, there exists $y \in M_{\alpha+1}$ such that $y \in T \setminus U_\alpha$.

We construct x_α and U_α. By our assumption, there exists $x \in cl_{(X,\mathcal{T})}(X \cap M_\alpha)$ such that $\mathcal{T} \cap M_\alpha$ does not contain a local base in $(X, \mathcal{T})$ for x. Since $M_\alpha \in M_{\alpha+1}$, by elementarity there exists such a point in $M_{\alpha+1}$; pick one such point and call it x_α. Thus there exists an open set U containing x_α such that for all $T \in \mathcal{T} \cap M_\alpha$, $T \setminus U \neq \emptyset$. Again by elementarity there exists such a $U \in M_{\alpha+1}$; so pick one such and call it U_α. Now for any $T \in \mathcal{T} \cap M_\alpha$ with $x_\alpha \in T$, there exists $y = y(T) \in X$ such that $y \in T \setminus U_\alpha$; so by elementarity there is such a y in $M_{\alpha+1}$. This completes the proof of (i), (ii) and (iii).

Now let N be a countable elementary submodel such that $X, T, M, \{M_\alpha : \alpha < \omega_1\} \in N$, and let $\lambda = N \cap \omega_1$. Then by elementarity, and the fact that $\{M_\alpha : \alpha < \omega_1\} \in N$, we have

(iv) $(\mathcal{T} \cap M) \cap N = \mathcal{T} \cap M_\lambda$,

(v) $(X \cap M) \cap N = X \cap M_\lambda$.

By Dow's Lemma 3.2 applied to the submodel topology $(X \cap M, \mathcal{T} \cap M)$, we know that $(\mathcal{T} \cap M) \cap N$ contains a local base in $(X \cap M, \mathcal{T} \cap M)$ for every point in $cl_{(X \cap M, \mathcal{T} \cap M)}(X \cap M \cap N)$. By (i) and (v), $x_\lambda \in cl_{(X \cap M, \mathcal{T} \cap M)}(X \cap M \cap N)$. To get a contradiction, it suffices to show that $(\mathcal{T} \cap M \cap N)$ does not contain a local base for x_λ in $(X \cap M, \mathcal{T} \cap M)$. To see this, consider U_λ. By (ii) $x_\lambda \in U_\lambda \in \mathcal{T} \cap M$. By (iii) and (iv) for any $T \in (\mathcal{T} \cap M \cap N) = \mathcal{T} \cap M_\lambda$ there exists $y \in M_{\lambda+1} \subset M$ such that $y \in T \setminus U_\lambda$; so $y \in (T \cap X \cap M) \setminus (U_\lambda \cap X \cap M)$. This completes the proof.

Proof of Theorem 1.1. Let $D = \{d_\alpha : \alpha < \omega_1\}$ be dense in X, and let $\{M_\alpha : \alpha < \omega_1\}$ be an $\in$-chain of countable elementary submodels, such that $\{d_\beta : \beta < \alpha\} \subset M_\alpha$ for all $\alpha < \omega_1$. Since $|M| = \omega_1$, $X \cap M$ is metrizable in the subspace topology, hence has a point-countable base. By first countability, the submodel topology $\mathcal{T} \cap M$ equals the subspace topology on $X \cap M$; hence the submodel topology $(X \cap M, \mathcal{T} \cap M)$ has a point-countable base. By the previous lemma, there is a cofinal set $A \subset' \omega_1$ such that $\mathcal{T} \cap M_\alpha$ contains a local base in $(X, \mathcal{T})$ for all points in $cl_{(X,\mathcal{T})}(X \cap M_\alpha)$. Thus $\cup\{\mathcal{T} \cap M_\alpha : \alpha \in A\}$ contains a local base in $(X, \mathcal{T})$ for all points in $\cup\{cl_{(X,\mathcal{T})}(X \cap M_\alpha) : \alpha \in A\}$, but this latter set contains D; so by first countability

$$\cup\{cl_{(X,\mathcal{T})}(X \cap M_\alpha) : \alpha \in A\} = X$$

and we are done.

Remark 3.4 *Dow proved that in a model obtained by adding ω_2 Cohen reals to a model of CH, every ss-metrizable space of weight ω_1 is metrizable [4]. By Theorem 1.1 "weight ω_1" can be replaced by "density ω_1" in that result.*

4 Extension of Dow's Theorem

We give a relatively simple proof of Theorem 1.2 that does not involve elementary submodels directly. We do this by calling on both Dow's reflection theorem (1.3) and the Hajnal-Juhász reflection theorem (1.5).

Proof of Theorem 1.2. Let X be ss-metrizable and $D \subset X$ a dense set that is conditionally compact in X.

Claim: For every $Y \in [D]^{\leq \omega_1}$, Y is second countable. To see this, put $Y = \{y_\alpha : \alpha < \omega_1\}$, and $Y_\alpha = \{y_\beta : \beta < \alpha\}$ for every $\alpha < \omega_1$. Now Y_α is conditionally compact in $cl_X(Y_\alpha)$, and therefore $cl_X(Y_\alpha)$ is feebly compact, and separable. Since ss-metrizable is an hereditary property, $cl_X(Y_\alpha)$ is metrizable by Corollary 2.3, and hence compact. Then

$$H = \bigcup_{\alpha < \omega_1} cl_X(Y_\alpha)$$

is countably compact and ss-metrizable, hence metrizable by Dow's Theorem 1.3. Thus H is compact and metrizable, and therefore second countable. Since $Y \subset H$, Y is second countable. This completes the proof of the Claim.

Since every $Y \in [D]^{\leq \omega_1}$ is second countable, by the Hajnal-Juhász Theorem 1.5, D is second countable. Thus D is separable; so X is separable, and thus by Corollary 2.3, X is metrizable (and compact).

Questions 4.1 (1) *Does "$p = \omega_1$" imply that every ss-metrizable space with density ω_1 is metrizable?*

(2) (Dow [3]) *Is every Lindelöf ss-metrizable space first countable?*

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