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TIGHTNESS OF FREE ABELIAN TOPOLOGICAL GROUPS AND OF FINITE PRODUCTS OF SEQUENTIAL FANS

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Abstract

Let $A(X)$ be the free abelian topological group on a Tychonoff space X, and $S(\kappa)$ the sequential fan with κ-many spines, which is the quotient space obtained from $C(\kappa)$ which is the disjoint union of κ-many convergent sequences by identifying all the limit points to a single point. Then we proved that the tightness of $A_{2n}(C(\kappa))$ is equal to that of $S(\kappa)^n$ for each $n \in \mathbb{N}$. As a corollary, we get if κ is an infinite cardinal with $\kappa = \omega$ or $cf(\kappa) > \omega$, $n \in \mathbb{N}$ and X is a metrizable space

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such that the weight of the set X' of all non-isolated points in X is κ, then $t(S(\kappa)^n) = t(A_{2n}(C(\kappa))) \leq t(A_{2n}(X)) \leq t(A(X)) \leq w(X') = \kappa$. This result partially answers the question raised by Arhangel'skiĭ, Okunev and Pestov in [2]. Furthermore, we get that for a metrizable space X such that the weight of X' is ω_1 or ω, $t(A_4(X)) = t(A(X)) = w(X')$, and always $t(A_3(X)) \leq \omega$ for any metrizable space X.

0 Introduction

This paper discusses the relations between the tightness of free abelian topological groups $A(X)$ on metrizable spaces X, the tightness of the subspaces $A_n(X)$ of $A(X)$ formed by reduced words whose lengths are less than or equal to n, $n = 1, 2, \ldots$, and the tightness of finite products of sequential fans $S(\kappa)$.

The tightness of a space X is denoted by $t(X)$ and the weight of X is denoted by $w(X)$. The set of all natural numbers is denoted by $\mathbb{N}$. For an infinite cardinal κ, let $C(\kappa)$ be the topological sum of κ-many convergent sequences with their limits points. Then we prove that $t(A_{2n}(C(\kappa))) = t(S(\kappa)^n)$ for each $n \in \mathbb{N}$. In [2], Arhangel'skiĭ, Okunev and Pestov proved that for a metrizable space X, $t(A(X)) \leq w(X')$, where X' is the set of all non-isolated points in X, and asked whether the converse inequality is true. In this paper, applying the above result, we give the following partial answer to the question.

Let κ be an infinite cardinal such that its cofinality $cf(\kappa)$ is larger than ω and X a metrizable space such that $w(X') = \kappa$. If $t(S(\kappa)^n) = \kappa$ for some $n \in \mathbb{N}$, then $t(A_{2n}(X)) = t(A(X)) = w(X') = \kappa$.

On the other hand, in the same paper Arhangel'skiĭ, Okunev and Pestov proved that if $t(A(X)) = \omega$, the converse inequality is true, i.e. $t(A(X)) = w(X')$. Since Gruenhage and

Tanaka [6] proved that $t(S(\omega_1)^2) = \omega_1$, applying our result, we can improve their result as follows; if $t(A_4(X)) = \omega_1$ or ω, $t(A_4(X)) = t(A(X)) = w(X')$. Furthermore we show that $t(A_3(X)) \leq \omega$ for any metrizable space X.

1 Notations and fundamental results

All topological spaces in this paper are assumed to be non-discrete and Tychonoff. Our terminologies and notations follow [4].

Let $A(X)$ be the *free abelian topological groups* on a space X in the sense of Markov [8]. For each $n \in \mathbb{N}$, let $A_n(X)$ formed by reduced words whose lengths are less than or equal to n (by definition $A_0(X) = \{0\}$, where 0 is the unit element of $A(X)$). Define the mapping i_n from $(X \oplus -X \oplus \{0\})^n$ onto $A_n(X)$ by $i_n((x_1, x_2, \ldots, x_n)) = x_1 + x_2 + \cdots + x_n$ for each $x_i \in X \oplus -X \oplus \{0\}$. The following fundamental properties related to $A(X)$ are used often in this paper (see [1],[7] and [10]).

Lemma 1.1 *Let X be a space. Then the following properties hold:*

(1) *Let $\mathcal{T}_1$ be a group topology for $A(X)$ which induces the original topology for X; then $\mathcal{T}_1 \leq \mathcal{T}$.*

(2) *X and $A_n(X)$, $n \in \mathbb{N}$, are closed subspaces of $A(X)$.*

(3) *$A_0 = \{g = x_1 - y_1 + x_2 - y_2 + \cdots + x_n - y_n : x_i, y_i \in X, n \in \mathbb{N}\}$ is a clopen subgroup of $A(X)$, and hence it is a clopen neighborhood of 0. Furthermore, $A(X)$ can be represented as a disjoint union of the clopen sets $\bigcup\{A_{2n}(X) \setminus A_{2n-1}(X) : n \in \mathbb{N}\} \cup \{0\}$ and $\bigcup\{A_{2n-1}(X) \setminus A_{2n-2}(X) : n \in \mathbb{N}\}$.*

(4) *The mapping i_n is continuous for each $n \in \mathbb{N}$.*

(5) *For each $n \in \mathbb{N}$, the restriction mapping of i_n to $i_n^{-1}(A_n(X)\backslash A_{n-1}(X))$ is an $n!$-to-1, open and closed mapping.*

(6) *Let Y be a closed P-embedded subspace of a space X; then $A(Y)$ is embedded into $A(X)$ as a closed topological subgroup. In fact, the homomorphic extension of the inclusion mapping from Y to X over $A(Y)$ is a closed embedding. Thus, $A_n(Y)$ is also embedded into $A_n(X)$ as a closed subspace for each $n \in \mathbb{N}$.*

For a space X, Let $\mathcal{D}(X)$ be the set of all continuous pseudometric on X. For each $d \in \mathcal{D}(X)$, we denote the Graev's extension of d over $A(X)$ by $\overline{d}$ (cf.[5]). Then, Tkačenko [10] showed that $\{V_d = \{g \in A(X) : \overline{d}(0,g) < 1\} : d \in \mathcal{D}(X)\}$ is a neighborhood base of 0 in $A(X)$. On the other hand, using the universal uniformity $\mathcal{U}_X$ of X, another representation of a neighborhood base of 0 in $A(X)$ was given in [9] and [11]. Furthermore, in [11], the author constructed a neighborhood base of 0 in $A_{2n}(X), n \in \mathbb{N}$. Here, we introduce an improved style of the neighborhood base of 0 in $A_{2n}(X)$.

Lemma 1.2 *Fix an $n \in \mathbb{N}$. For each $U \in \mathcal{U}_X$, put $V_n(U) = \{x_1 - y_1 + x_2 - y_2 + \cdots + x_k - y_k : (x_i, y_i) \in U, k \leq n\}$. Then, $\{V_n(U) : U \in \mathcal{U}_X\}$ is a neighborhood base of 0 in $A_{2n}(X)$.*

Furthermore, as a corollary of the above Lemma, we have the following, which is often used in our main results.

For a space X and each $n \in \mathbb{N}$, we define a mapping j_n from X^{2n} to $A_{2n}(X)$ as follows;

$$j_n(((x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n))) =$$

$$x_1 - y_1 + x_2 - y_2 + \cdots + x_n - y_n$$

for each $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ in X^n.

Corollary 1.3 ([11]) *Let X be a space, $n \in \mathbb{N}$ and E a subset of $A_{2n}(X)$. Then, $0 \in \overline{E}$ if and only if $j_n^{-1}(E) \cap U^n \neq \emptyset$ for each $U \in \mathcal{U}_X$,*
where $U^n = \{((x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n)) \in X^{2n} : (x_i, y_i) \in U, i = 1, 2, \ldots, n\}$.

Corollary 1.4 ([11]) *Let X be a space such that every open neighborhood of the diagonal Δ_X of X^2 is contained in $\mathcal{U}_X$ (in particular, X is paracompact), and E a subset of $A_2(X)$. Then, $0 \in \overline{E}$ if and only if $\overline{j_1^{-1}(E)} \cap \Delta_X \neq \emptyset$.*

Let X be a space. The *tightness* $t(X)$ of X is the smallest cardinal number κ such that for every point $x \in X$ and $A \subset X$, if $x \in \overline{A}$, then $x \in \overline{B}$ for some $B \subset A$ with $|B| \leq \kappa$. For a infinite cardinal κ, let $C(\kappa)$ be the topological sum of κ-many convergent sequences with their limit points. The *sequential fan* $S(\kappa)$ with κ-many spines is the quotient space obtained from $C(\kappa)$ by identifying all the limit points to a single point. Though $S(\kappa)$ is a concrete and simple space, it is difficult to investigate the cardinality of $t(S(\kappa)^n)$, $n \in \mathbb{N}$. In fact Dow and Todorčević asked whether ZFC implies that $t(S(\omega_2)^2) = \omega_2$. On the other hand, there were some results for the tightness. For example, Gruenhage and Tanaka [6] proved that $t(S(\omega_1)^2) = \omega_1$. And recently, Eda, Gruenhage, Koszmider, Tamano and Todorčević [3] obtained some (set theoretic) equivalent statements to $t(S(\kappa)^2) = \kappa$.

2 Tightness of $A_2(X)$

For a space X, since $A_1(X) = X \oplus -X \oplus \{0\}$, it easy to see that $t(A_1(X)) = t(X)$. For the tightness of $A_2(X)$, we can show the following.

Theorem 2.1 *Let X be a space such that every open neighborhood of the diagonal Δ_X of X^2 is contained in the universal uniformity $\mathcal{U}_X$ of X. Then $t(A_2(X)) = t(X^2)$.*

Proof: First, we suppose that $t(X^2) \le \kappa$. Let $E \subset A_2(X)$ and $g \in \overline{E}$. If $g \in (X \oplus -X) \cup (A_2(X) \setminus A_1(X))$, by Lemma 1.1, we have that $g \in \overline{E \cap (X \oplus -X)} \cup \overline{E \cap (A_2(X) \setminus A_1(X))}$. Since $t(X \oplus -X) \le \kappa$ and by the property (5) in Lemma 1.1, we can take a subset C of E such that $g \in \overline{C}$ and $|C| \le \kappa$. Thus, let $g = 0$. Since A_0 is a clopen neighborhood of 0 in $A(X)$, $g \in \overline{F}$, where $F = E \cap A_0$ and $F \subset A_2(X) \setminus A_1(X)$. Therefore, by Corollary 1.4, $\overline{f_1^{-1}(F)} \cap \Delta_X \neq \emptyset$. Take a point $(x, x) \in \overline{f_1^{-1}(F)} \cap \Delta_X$. Since $t(X^2) \le \kappa$, there is a subset C of $j_1^{-1}(F)$ such that $(x, x) \in \overline{C}$ and $|C| \le \kappa$. By Corollary 1.4, we have that $0 \in \overline{j_1(C)}$, and also $j_1(C)$ is a subset of $j_1 j_1^{-1}(F) = F$ and $|f_1(C)| \le \kappa$. Consequently, we get $t(A_2(X)) \le \kappa$.

Next, suppose that $t(A_2(X)) \le \kappa$. Let $E \subset X^2$ and $\mathbf{x} \in \overline{E}$. The proof is in two cases.

Case 1: $\boldsymbol{x} \notin \Delta_x$.

In this case, it is easy to see that $i_2(\mathbf{x}) \in A_2(X) \setminus A_1(X)$. Then we can take an open set U in X^2 and an open set V in $A_2(X)$ such that

$$\boldsymbol{x} \in U \text{ and } i_2(\boldsymbol{x}) \in i_2(U) \subset V \subset \overline{V} \subset A_2(X) \setminus A_1(X).$$

Let $F = U \cap E$; then $\boldsymbol{x} \in \overline{F}$ and $i_2(\overline{F}) \subset i_2(\overline{U}) \subset A_2(X) \setminus A_1(X)$. By the property (5) in Lemma 1.1, we have that $t(\overline{U}) \le \kappa$, and hence we can take a subset C of F such that $\boldsymbol{x} \in \overline{C}$ and $|C| \le \kappa$.

Case 2: $\boldsymbol{x} \in \Delta_X$.

If $\boldsymbol{x} \in \overline{E \cap \Delta_X}$, the proof is finished. For, the diagonal Δ_X is homeomorphic to X and X is a subset of $A_2(X)$. It follows that $t(\Delta_X) \le \kappa$. Then, we may assume that $\boldsymbol{x} \in \overline{F}$, where $F = E \setminus \Delta_X$. We define a subset A of Δ_X, as follows;

$$A = \{\boldsymbol{y} \in \Delta_X : \text{ there is } C_{\boldsymbol{y}} \subset F \text{ such that } \boldsymbol{y} \in \overline{C_{\boldsymbol{y}}} \text{ and } |C_{\boldsymbol{y}}| \le \kappa\}.$$

Since $\boldsymbol{x} \in \Delta_X$, $\overline{F} \cap \Delta_X \neq \emptyset$, and hence $0 \in \overline{j_1(F)}$. By the assumption, there is a subset D of $j_1(F)$ such that $0 \in \overline{D}$ and

$|D| \leq \kappa$. Note that the mapping $j_1|_{X^2 \setminus \Delta_X}$ is one-to-one. Then we have $j_1^{-1}(D) \subset F$ and $\Delta_X \cap \overline{j_1^{-1}(D)} \neq \emptyset$. This implies that the set A is not empty. Next we show that $\boldsymbol{x} \in \overline{A}$. On the contrary, there is an open set U of X^2 such that $\boldsymbol{x} \in U$ and $\overline{U} \cap A = \emptyset$. Since $\boldsymbol{x} \in \overline{F \cap U}$, $0 \in \overline{j_1(F \cap U)}$. In the same way, we can show that there is a subset C of $F \cap U$ such that $\overline{C} \cap \Delta_X \neq \emptyset$ and $|C| \leq \kappa$. Take a point $\boldsymbol{y} \in \overline{C} \cap \Delta_X$; then $\boldsymbol{y} \in A$. But this contradicts to $\boldsymbol{y} \in \overline{U}$. Therefore, we have that $\boldsymbol{x} \in \overline{A}$. Since $t(\Delta_X) \leq \kappa$, we can take a subset B of A such that $\boldsymbol{x} \in \overline{B}$ and $|B| \leq \kappa$. Thus, $\boldsymbol{x} \in \overline{\bigcup\{C_{\boldsymbol{y}} : \boldsymbol{y} \in B\}}$, and also $\bigcup\{C_{\boldsymbol{y}} : \boldsymbol{y} \in B\}$ is a subset of F such that $|\bigcup\{C_{\boldsymbol{y}} : \boldsymbol{y} \in B\}| \leq \kappa$.

Consequently, we have that $t(X^2) \leq \kappa$. $\square$

Corollary 2.2 *Let X be a paracompact space. Then $t(A_2(X)) = t(X^2)$.*

The following example shows that the hypothesis of a space X in Theorem 2.1 cannot be omitted. It was proved by Ohta and is presented here with his kind permission.

Example 2.3 *There exists a space X such that $\omega = t(X^2) < t(A_2(X)) = \omega_1$.*

Proof: Let $X = \{(\alpha, \beta) \in \omega_1 \times \omega_1 : \alpha < \beta\}$, where ω_1 is the space of countable ordinal numbers with the order topology. Then $t(X^2) = \omega$, because X satisfies the first axiom of countability. To show that $t(A_2(X)) = \omega_1$, let $f : \omega_1 \to \omega_1$ be a map such that $f(\alpha) > \alpha$ for each $\alpha \in \omega_1$ and $f(\alpha) < f(\alpha')$ if $\alpha < \alpha'$, and put $C = \{\alpha \in \omega_1 : f(\beta) \leq \alpha$ for each $\beta < \alpha\}$. Since C is closed unbounded in ω_1, C can be decomposed into two disjoint stationary sets $C_1 = \{\alpha_\gamma : \gamma \in \omega_1\}$ and $C_2 = \{\beta_\gamma : \gamma \in \omega_1\}$, where $\alpha_\gamma < \alpha_{\gamma'}$ and $\beta_\gamma < \beta_{\gamma'}$ if $\gamma < \gamma'$. Put $x_\gamma = (\alpha_\gamma, f(\alpha_\gamma))$ and $y_\gamma = (\beta_\gamma, f(\beta_\gamma))$ for each $\gamma \in \omega_1$, and define $H = \{x_\gamma - y_\gamma : \gamma \in \omega_1\} \subset A_2(X)$. Suppose that $0 \notin \overline{H}$ in $A_2(X)$. Then there is a continuous pseudometric d on X

such that $d(x_\gamma, y_\gamma) > 1$ for each $\gamma \in \omega_1$. For each $\gamma \in \omega_1$, there are $\lambda_\gamma < \alpha_\gamma$ and $\mu_\gamma < \beta_\gamma$ such that $d((\alpha, f(\alpha_\gamma)), x_\gamma) < 1/3$ for each $\lambda_\gamma < \alpha \leq f(\alpha_\gamma)$ and $d((\beta, f(\beta_\gamma)), y_\gamma) < 1/3$ for each $\mu_\gamma < \beta \leq f(\beta_\gamma)$. Since C_1 and C_2 are stationary, we can find an increasing sequence $\{\gamma(n) : n \in \omega\} \subset \omega_1$ and $\nu < \alpha_{\gamma(0)}$ such that $\alpha_{\gamma(n)} < \beta_{\gamma(n)} < \alpha_{\gamma(n+1)}$, $\lambda_{\gamma(n)} < \nu$ and $\mu_{\gamma(n)} < \nu$ for each $n \in \omega$. Since $\{f(\alpha_{\gamma(n)})\}_{n\in\omega}$ and $\{f(\beta_{\gamma(n)})\}_{n\in\omega}$ have the same limit, there is $m \in \omega$ such that $d(p, q) < 1/3$, where $p = (\nu, f(\alpha_{\gamma(m)}))$ and $q = (\nu, f(\beta_{\gamma(m)}))$. Then, $d(x_{\gamma(m)}, y_{\gamma(m)}) \leq d(x_{\gamma(m)}, p) + d(p, q) + d(q, y_{\gamma(m)}) < 1/3 + 1/3 + 1/3 = 1$, which is a contradiction. Hence, $0 \in \overline{H}$ in $A_2(X)$. On the other hand, the set $D = \{(\alpha, f(\alpha)) : \alpha \in C\} (= \{x_\gamma,\ y_\gamma : \gamma \in \omega_1\})$ is discrete and closed in X. For, if $(\alpha, \beta) \in X$ and $\alpha \in C$, then $[0, \alpha] \times (\alpha, \beta]$ contains at most one element of D. For each $\gamma \in \omega_1$, since $\{(\alpha, \beta) \in X : \alpha < \beta \leq \gamma\}$ is metrizable, we can find a continuous pseudometric d_γ on X such that $d_\gamma(x_\delta, y_\delta) > 1$ for each $\delta \leq \gamma$. This means that $0 \notin \overline{H'}$ in $A_2(X)$ for each countable set $H' \subset H$. Hence, $t(A_2(X)) = \omega_1$. $\square$

3 Tightness of $A_{2n}(C(\kappa))$ and of $S(\kappa)^n$

Let X be a space and fix an $n \in \mathbb{N}$. When we take a subset H of $A_n(X)$ and a word $g \in A_n(X)$ such that $g \in \overline{H}$ to investigate the tightness of $A_n(X)$, by Lemma 1.1, it suffices to consider the following three cases:

(1) $g \in A_k(X) \setminus A_{k-1}(X)$ and $g \in \overline{H \cap (A_k(X) \setminus A_{k-1}(X))}$ for some $k \leq n$.

(2) $g = 0$ and $g \in \overline{H \cap (A_{2k}(X) \setminus A_{2k-1}(X))}$ for some k with $2k \leq n$.

(3) $g \in A_k(X) \setminus A_{k-1}(X)$ and $g \in \overline{H \cap (A_{2m+k}(X) \setminus A_{2m+k-1}(X))}$ for some k and m with $2m + k \leq n$.

In the case (1), We can apply the property (5) in Lemma 1.1. For example, if X is metrizable, by the property, we can take a countable subset C of $H \cap (A_k(X) \setminus A_{k-1}(X))$ such that $g \in \overline{C}$. And in the case (2), we can apply Corollary 1.3, however it is not easy to know the smallest cardinality of subsets of H whose closure contains g. For the case (3), we prepare the following Lemma.

Lemma 3.1 *Let X be a first-countable space such that every open neighborhood of the diagonal Δ_X of X^2 is contained in the universal uniformity $\mathcal{U}_X$ of X and let $m, n \in \mathbb{N}$. Take a set $H \subset A_{2m+n}(X) \setminus A_{2m+n-1}(X)$ and a word $g \in A_n(X) \setminus A_{n-1}(X)$ such that $g \in \overline{H}$. Then, there is subset H_0 of H such that $g \in \overline{H_0}$ and $|H_0| \leq t(A_{2m}(X))$.*

Proof: Put $t(A_{2m}(X)) = \tau$. Since $g \in A_n(X) \setminus A_{n-1}(X)$, let $g = a_1 + \cdots + a_k - a_{k+1} - \cdots - a_n$ be a reduced form such that $0 \leq k \leq n$ and each $a_i \in X$. Put $H_1 = H - g$; then $0 \in \overline{H_1}$. By the property (3) of Lemma 1.1, we may assume that $H_1 \subset A_0$. It follows that if we take $h = \sum_{i=1}^{2m+n} z_i^{\varepsilon_i} \in H$ and $h_1 \in H_1$ with $h_1 = h - g$, then $\sum_{i=1}^{2m+n} \varepsilon_i - k + (n-k) = 0$. Since the operation on $A(X)$ is commutative, we can represent h_1 as follows;

$$
\begin{aligned}
h_1 = &(x_1 + \cdots + x_m) - (y_1 + \cdots + y_m) \\
&+ (x_{m+1} + \cdots + x_{m+k}) - (y_{m+k+1} + \cdots + y_{m+n}) \\
&- (a_1 + \cdots + a_k) + (a_{k+1} + \cdots + a_n) \\
= &\, x_1 - y_1 + \cdots + x_m - y_m + x_{m+1} - a_1 + \cdots + x_{m+k} - a_k \\
&+ a_{k+1} - y_{m+k+1} + \cdots + a_n - y_{m+n},
\end{aligned}
$$

where each $x_i, y_i \in X$. Note that the above form is not necessarily a reduced one, that is, some x_i may be a member of $\{a_1, \ldots, a_k\}$ and some y_i may be a member of $\{a_{k+1}, \ldots, a_n\}$. From this argument, we can put the set H_1 as follows;

$$\begin{aligned}H_1 = \{\boldsymbol{h}_\lambda = x_1^\lambda - y_1^\lambda + \cdots + x_m^\lambda - y_m^\lambda + x_{m+1}^\lambda - a_1 + \cdots \\ + x_{m+k}^\lambda - a_k + a_{k+1} - y_{m+k+1}^\lambda + \cdots + a_n - y_{m+n}^\lambda \\ : \text{ each } x_i^\lambda, y_i^\lambda \in X,\ \lambda \in \Lambda\}.\end{aligned}$$

By Corollary 1.3, we have that

$$j_{m+n}^{-1}(H_1) \cap U^{m+n} \neq \emptyset \ \text{ for each } U \in \mathcal{U}_X. \tag{1}$$

Let P be the permutation group of the set $\{1, 2, \ldots, m+n\}$. For each $\pi \in P$, put $E_\pi = \{\boldsymbol{x} = ((x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(m+n)}), (y_{\pi(1)}, y_{\pi(2)}, \ldots, y_{\pi(n+m)})) \in X^{2(m+n)} : j_{m+n}(\boldsymbol{x}) \in H_1\}$. Then, $j_{m+n}^{-1}(H_1) = \bigcup_{\pi \in P} E_\pi$. Since P is a finite set and $\mathcal{U}_X$ is closed with respect to finite intersections, by (1), we may assume that

$$E \cap U^{m+n} \neq \emptyset \ \text{ for each } U \in \mathcal{U}_X, \tag{2}$$

where $E = \{\boldsymbol{e}_\lambda = ((x_1^\lambda, \ldots, x_m^\lambda, x_{m+1}^\lambda, \ldots, x_{m+k}^\lambda, a_{k+1}, \ldots, a_n), (y_1^\lambda, \ldots, y_m^\lambda, a_1, \ldots, a_k, y_{m+k+1}^\lambda, \ldots, y_{m+n}^\lambda)) \in X^{2(m+n)} : \lambda \in \Lambda\}$.

Put

$$\begin{aligned} B = \{\boldsymbol{b}_\lambda &= ((x_1^\lambda, \ldots, x_m^\lambda), (y_1^\lambda, \ldots, y_m^\lambda)) \in X^{2m} : \lambda \in \Lambda\}, \\ C = \{\boldsymbol{c}_\lambda &= ((x_{m+1}^\lambda, \ldots, x_{m+k}^\lambda, a_{k+1}, \ldots, a_n), \\ &\quad (a_1, \ldots, a_k, y_{m+k+1}^\lambda, \ldots, y_{m+n}^\lambda)) \in X^{2n} : \lambda \in \Lambda\}.\end{aligned}$$

Then, by (2), we have that

$$B \cap U^m \neq \emptyset \ \text{ for each } U \in \mathcal{U}_X \text{ and} \tag{3}$$

$$C \cap U^n \neq \emptyset \ \text{ for each } U \in \mathcal{U}_X. \tag{4}$$

Put $D = \{\boldsymbol{d}_\lambda = (x_{m+1}^\lambda, \ldots, x_{m+k}^\lambda, y_{m+k+1}^\lambda, y_{m+n}^\lambda) \in X^n : \lambda \in \Lambda\}$.

Claim. $(a_1, \ldots, a_n) \in \overline{D}$.

Let V_i be an open neighborhood of a_i in X, $i = 1, \ldots, n$, such that $V_i \cap V_j = \emptyset$ if $a_i \neq a_j$. For every $x \in X \setminus \bigcup_{i=1}^n V_i$ let V_x be an open neighborhood of x in X such that $V_x \cap \{a_1, \dots, a_n\} = \emptyset$. Then the set $U = \bigcup_{i=1}^n V_i \times V_i \cup \bigcup\{V_x \times V_x : x \in X \setminus \bigcup_{i=1}^n V_i\}$ is an open neighborhood of the diagonal Δ_X of X^2. By our hypothesis, $U \in \mathcal{U}_X$, and hence, from the property (4), there is a $\lambda \in \Lambda$ such that $\boldsymbol{c}_\lambda \in U^n$. Then $(x^\lambda_{m+i}, a_i) \in U$ for each $i = 1, \dots, k$ and $(a_i, y^\lambda_{m+i}) \in U$ for each $i = k+1, \dots, n$. From our method of the choices of V_i and V_x, it follows that $x^\lambda_{m+i} \in V_i$ for each $i = 1, \dots, k$ and $y^\lambda_{m+i} \in V_i$ for each $i = k+1, \dots, n$. Consequently, $\boldsymbol{d}_\lambda = (x^\lambda_{m+1}, \dots, x^\lambda_{m+k}, y^\lambda_{m+k+1}, \dots, y^\lambda_{m+n}) \in \prod_{i=1}^n V_i$.

Now, since X satisfies the axiom of first countability and so X^n does, take a countable neighborhood base $\{V_s : s \in \mathbb{N}\}$ of $(a_1, \dots, a_n)$ in X^n such that $V_1 = X^n$ and $V_{s+1} \subsetneq V_s$ for each $s \in \mathbb{N}$. For each $s \in \mathbb{N}$, let $\Lambda_s = \{\lambda \in \Lambda : \boldsymbol{d}_\lambda \in V_s \setminus V_{s+1}\}$, $E_s = \{\boldsymbol{e}_\lambda : \lambda \in \Lambda_s\}$, $B_s = \{\boldsymbol{b}_\lambda : \lambda \in \Lambda_s\}$, $C_s = \{\boldsymbol{c}_\lambda : \lambda \in \Lambda_s\}$ and $D_s = \{\boldsymbol{d}_\lambda : \lambda \in \Lambda_s\}$. Here, our proof is in two cases.

Case 1: *There is a subsequence* $\{t_s : s \in \mathbb{N}\}$ *of* $\mathbb{N}$ *such that* $B_{t_s} \cap U^m \neq \emptyset$ *for each* $s \in \mathbb{N}$ *and* $U \in \mathcal{U}_X$.

By Corollary 1.3, for each $s \in \mathbb{N}$, we have that $0 \in \overline{j_m(B_{t_s})}$ in $A_{2m}(X)$. Thus we can take a subset Λ'_s of Λ_{t_s} such that $0 \in \overline{\{j_m(\boldsymbol{b}_\lambda) : \lambda \in \Lambda'_s\}}$ and $|\Lambda'_s| \leq \tau$. Put

$$
\begin{aligned}
E_\lambda = \{ (\, & (x_1, \dots, x_m, x^\lambda_{m+1}, \dots, x^\lambda_{m+k}, a_{k+1}, \dots, a_n), \\
& (y_1, \dots, y_m, a_1, \dots, a_k, y^\lambda_{m+k+1}, \dots, y^\lambda_{m+n})) \in X^{2(m+n)} : \\
& (\,(x_1, \dots, x_m), (y_1, \dots, y_m)) \in j_m^{-1} j_m(\boldsymbol{b}_\lambda)\}
\end{aligned}
$$

for each $\lambda \in \bigcup_{s=1}^\infty \Lambda'_s$, and $E_1 = \bigcup\{E_\lambda : \lambda \in \bigcup_{s=1}^\infty \Lambda'_s\}$. Then we can see that $|E_1| \leq \omega \cdot \tau = \tau$ and $E_1 \cap U^{m+n} \neq \emptyset$ for each $U \in \mathcal{U}_X$. Let $H_2 = j_{m+n}(E_1)$; then we have that $0 \in \overline{H_2}$ and $|H_2| \leq \tau$. Furthermore it is easy to see that $H_2 \subset H_1$. Consequently the set $H_0 = H_2 - g$ is desired one.

Case 2: *There is* $s_0 \in \mathbb{N}$ *such that for each* $s \geq s_0$ $B_s \cap U^m = \emptyset$ *for some* $U \in \mathcal{U}_X$.

Let $B_1 = \bigcup_{s \geq s_0} B_s$, and assume that there is a $U_1 \in \mathcal{U}_X$ such that $B_1 \cap U_1^m = \emptyset$. Then, we have that

$$\bigcup_{s \geq s_0} E_s \cap U_1^{m+n} = \emptyset. \tag{5}$$

On the other hand, since $(\bigcup_{s<s_0} D_s) \cap V_{s_0} = \emptyset$, we can select $U_2 \in \mathcal{U}_X$ such that $\bigcup_{s<s_0} C_s \cap U_2^n = \emptyset$. This follows that

$$\bigcup_{s<s_0} E_s \cap U_2^{m+n} = \emptyset. \tag{6}$$

Thus, by (5) and (6), $E \cap (U_1 \cap U_2)^{m+n} = \emptyset$, but this contradicts (2). Therefore, we can see that $B_1 \cap U^m \neq \emptyset$ for each $U \in \mathcal{U}_X$. By Corollary 1.3, this means that $0 \in \overline{j_m(B_1)}$, and hence we can take $\Lambda_1 \subset \bigcup_{s \geq s_0} \Lambda_s$ such that $0 \in \overline{\{j_m(\boldsymbol{b}_\lambda) : \lambda \in \Lambda_1\}}$ and $|\Lambda_1| \leq \tau$. Now, by the hypothesis of Case 2, $0 \notin \overline{\{j_m(\boldsymbol{b}_\lambda) : \lambda \in \Lambda_s\}}$ for each $s \geq s_0$. Then we can see that $\Lambda_1 \cap \bigcup_{t \geq s} \Lambda_t \neq \emptyset$ for each $s \geq s_0$, and this means that $(a_1, \ldots, a_n) \in \overline{\{\boldsymbol{d}_\lambda : \lambda \in \Lambda_1\}}$. Put

$$\begin{aligned} E_\lambda = \{ ((x_1, \ldots, x_m, x_{m+1}^\lambda, \ldots, x_{m+k}^\lambda, a_{k+1}, \ldots, a_n), \\ (y_1, \ldots, y_m, a_1, \ldots, a_k, y_{m+k+1}^\lambda, \ldots, y_{m+n}^\lambda)) \in X^{2(m+n)} : \\ ((x_1, \ldots, x_m), (y_1, \ldots, y_m)) \in j_m^{-1} j_m(\boldsymbol{b}_\lambda) \} \end{aligned}$$

for each $\lambda \in \Lambda_1$, and $E_1 = \bigcup \{E_\lambda : \lambda \in \Lambda_1\}$. Then $E_1 \cap U^{m+n} \neq \emptyset$ for each $U \in \mathcal{U}_X$. Finally, put $H_2 = j_{m+n}(E_1)$ and $H_0 = H_2 + g$. Therefore, we can see that $g \in \overline{H_0}$, $|H_0| \leq \tau$ and also $H_0 \subset H$. $\square$

Using the above fundamental results, we show the following main result in this paper. Let κ be an infinite cardinal and $C(\kappa) = \bigoplus_{\alpha < \kappa} C_\alpha$, where each C_α is the set of convergent sequence $\{a_{n,\alpha} : n \in \mathbb{N}\}$ with its limit $\{a_\alpha\}$. Then, the sequential fan $S(\kappa)$ is the quotient image of $C(\kappa)$ identifying the set

$\{a_\alpha : \alpha < \kappa\}$ to a point $\mathbf{0}$. Let Φ be the set of all mappings from κ to $\mathbb{N}$, and for each $\varphi \in \Phi$ and $\alpha < \kappa$, put

$$\begin{aligned} O(\varphi, \alpha) &= \{a_{n,\alpha} : n > \varphi(\alpha)\} \cup \{a_\alpha\} \\ W(\varphi) &= \{a_{n,\alpha} : n > \varphi(\alpha), \alpha < \kappa\} \cup \{\mathbf{0}\}. \end{aligned}$$

Then, $\bigcup_{\alpha<\kappa} O(\varphi, \alpha)$ is a canonical open neighborhood of the set $\{a_\alpha : \alpha < \kappa\}$ in $C(\kappa)$, and $W(\varphi)$ is a canonical open neighborhood of $\mathbf{0}$ in $S(\kappa)$.

Theorem 3.2 *Let κ be an infinite cardinal and $n \in \mathbb{N}$. Then $t(A_{2n}(C(\kappa))) = t(S(\kappa)^n)$.*

Proof: We prove the Theorem by induction with respect to n. It is clear that $t(S(\kappa)) = \omega$ and, by Theorem 2.1, we have that $t(A_2(C(\kappa))) = t(C(\kappa)^2) = \omega$. Then, let $n \geq 2$ and suppose that for each $k < n$, $t(A_{2k}(C(\kappa))) = t(S(\kappa)^k)$.

Claim 1. $t(A_{2n}(C(\kappa))) \leq t(S(\kappa)^n)$.

Let $t(S(\kappa)^n) = \tau, (\omega \leq \tau \leq \kappa)$, and take a subset H of $A_{2n}(C(\kappa))$ and a word $g \in A_{2n}(C(\kappa))$ such that $g \in \overline{H}$. By our inductive assumption, we may assume that $H \subset A_{2n}(C(\kappa)) \setminus A_{2n-2}(C(\kappa))$. Furthermore, if $g \in A_{2n}(C(\kappa)) \setminus A_{2n-2}(C(\kappa))$, by the properties (3) and (5) in Lemma 1.1, we can take a countable subset H_0 of H such that $g \in \overline{H_0}$. Thus, it suffices to show in the following two cases.

Case 1: $g \in A_k(C(\kappa)) \setminus A_{k-1}(C(\kappa))$ for some k such that $1 \leq k \leq 2n-1$.

If k is odd, $g \in \overline{A_{2n-1}(C(\kappa)) \setminus A_{2n-2}(C(\kappa))}$. Thus, by Lemma 3.1, we can take a subset H_0 of H such that $g \in \overline{H_0}$ and $|H_0| \leq t(A_{2n-1-k}(C(\kappa)))$. Similarly, if k is even, we can take a subset H_0 of H such that $g \in \overline{H_0}$ and $|H_0| \leq t(A_{2n-k}(C(\kappa)))$. In any case, by our inductive assumption, we have that $|H_0| \leq t(A_{2n-2}(C(\kappa))) = t(S(\kappa)^{n-1}) \leq t(S(\kappa)^n) = \tau$.

Case 2: $g = 0$.

In this case, we may assume that

$$H \subset A_{2n}(C(\kappa)) \setminus A_{2n-1}(C(\kappa)). \tag{1}$$

Let d be a natural metric on $C(\kappa)$ such that $d(x,y)=1$ if $x \in C_\alpha$, $y \in C_\beta$ and $\alpha \neq \beta$, $d(a_\alpha, a_{n,\alpha}) = 1/n$, and $d(a_{m,\alpha}, a_{n,\alpha}) = |1/m - 1/n|$. Since $0 \in \overline{H}$, we have that $0 \in \overline{H_1}$, where $H_1 = H \cap V_d$ and $V_d = \{g \in A(C(\kappa)) : \overline{d}(0,g) < 1\}$. On the other hand, we can put $H_1 = \{\boldsymbol{h} = x_1^\lambda - y_1^\lambda + \cdots + x_n^\lambda - y_n^\lambda : x_i^\lambda, y_i^\lambda \in C(\kappa), \lambda \in \Lambda\}$ such that

$$\begin{aligned} &\text{for each } \lambda \in \Lambda, \\ &\text{there are } \alpha_1^\lambda, \ldots, \alpha_n^\lambda < \kappa \text{ such that } x_i^\lambda, y_i^\lambda \in C_{\alpha_i^\lambda}. \end{aligned} \tag{2}$$

For each $\lambda \in \Lambda$ and $i = 1, \ldots, n$, choose a point $z_i^\lambda \in \{x_i^\lambda, y_i^\lambda\}$ such that the distance between z_i^λ and $a_{\alpha_i^\lambda}$ is larger than the one between another point and $a_{\alpha_i^\lambda}$. Now, we show that $\boldsymbol{\Theta} = (\boldsymbol{0}, \ldots, \boldsymbol{0}) \in \overline{E_1}$, where $E_1 = \{\boldsymbol{e}_\lambda = (z_1^\lambda, \ldots, z_n^\lambda) \in S(\kappa)^n : \lambda \in \Lambda\}$. Take an arbitrary $\varphi \in \Phi$, and put $U_\varphi = \bigcup\{O(\varphi,\alpha) \times O(\varphi,\alpha) : \alpha < \kappa\} \cup \Delta_{C(\kappa)}$. Then $U_\varphi \in \mathcal{U}_{C(\kappa)}$, and by Corollary 1.3, we can see that $\{((x_1^\lambda, \ldots, x_n^\lambda), (y_1^\lambda, \ldots, y_n^\lambda)) : \lambda \in \Lambda\} \cap U_\varphi^n \neq \emptyset$. Thus, there is a $\lambda \in \Lambda$ such that $((x_1^\lambda, \ldots, x_n^\lambda), (y_1^\lambda, \ldots, y_n^\lambda)) \in U_\varphi^n$, and so that each $(x_i^\lambda, y_i^\lambda) \in U_\varphi$. By (1) and (2), we have that $x_i^\lambda, y_i^\lambda \in O(\varphi, \alpha_i^\lambda)$ for each $i = 1, \ldots, n$, and this means that each $z_i^\lambda \in O(\varphi, \alpha_i^\lambda)$. Therefore, $\boldsymbol{e}_\lambda \in W(\varphi)$ and hence we can see that $E_1 \cap W(\varphi) \neq \emptyset$. Since λ is arbitrary, this follows that $\boldsymbol{\Theta} \in \overline{E_1}$.

Now, since $t(S(\kappa)^n) = \tau$, let Λ_0 be a subset of Λ such that $\boldsymbol{\Theta} \in \overline{\{\boldsymbol{e}_\lambda : \lambda \in \Lambda_0\}}$ and $|\Lambda_0| \leq \tau$. On the other hand, from the definition of z_i^λ, we can see that for each $\varphi \in \Phi$, $\{((x_1^\lambda, \ldots, x_n^\lambda), (y_1^\lambda, \ldots, y_n^\lambda)) : \lambda \in \Lambda_0\} \cap U_\varphi^n \neq \emptyset$. Hence, by Corollary 1.3, this means that $0 \in \overline{H_0}$, where $H_0 = \{\boldsymbol{h}_\lambda : \lambda \in \Lambda_0\}$.

Consequently, in any case, we can take a subset H_0 of H such that $g \in \overline{H_0}$ and $|H_0| \leq \tau$, and hence we get Claim 1.

Claim 2. $t(S(\kappa)^n) \leq t(A_{2n}(C(\kappa)))$.

Let $t(A_{2n}(C(\kappa))) = \tau, (\omega \leq \tau \leq \kappa)$, and take a subset E of $S(\kappa)^n$ and a point $\boldsymbol{x} \in S(\kappa)^n$ such that $\boldsymbol{x} \in \overline{E}$. By our inductive assumption, we may assume that $\boldsymbol{x} = \boldsymbol{\Theta}$ $(= (\mathbf{0}, \ldots, \mathbf{0}))$. Put $E = \{\boldsymbol{e}_\lambda = (x_1^\lambda, \ldots, x_n^\lambda) : \lambda \in \Lambda\}$. Then for each $\lambda \in \Lambda$ and $i = 1, \ldots, n$, there is $\alpha_i^\lambda < \kappa$ such that $x_i^\lambda \in C_{\alpha_i^\lambda}$ (if $x_i^\lambda = \mathbf{0}$ we regard x_i^λ as $a_{\alpha_i^\lambda}$). Now, put $H = \{\boldsymbol{h}_\lambda = x_1^\lambda - a_{\alpha_1^\lambda} + \cdots + x_n^\lambda - a_{\alpha_n^\lambda} : \lambda \in \Lambda\}$. Then, with the similar argument of the proof of Claim 1, it is easy to see that $0 \in \overline{H}$. Since $t(A_{2n}(C(\kappa))) = \tau$, take a subset Λ_0 of Λ such that $0 \in \overline{\{\boldsymbol{h}_\lambda : \lambda \in \Lambda_0\}}$ and $|\Lambda_0| \leq \tau$. Then, by the definition of H and Corollary 1.3, we can see that $\boldsymbol{\Theta} \in \overline{E_0}$, where $E_0 = \{\boldsymbol{e}_\lambda : \lambda \in \Lambda_0\}$. Therefore we have that $t(S(\kappa)^n) \leq \tau$.

From Claim 1 and 2, we can prove that $t(A_{2n}(C(\kappa))) = t(S(\kappa)^n)$. □

Arhangel'skiĭ, Okunev and Pestov [2] proved that for a metrizable space X, $t(A(X)) \leq w(X')$, where X' is the set of all non-isolated points in X. Thus we obtain the following.

Corollary 3.3 *Let κ be an infinite cardinal and $n \in \mathbb{N}$. Then $t(S(\kappa)^n) = t(A_{2n}(C(\kappa))) \leq t(A(C(\kappa)) \leq w(C(\kappa)') = \kappa$.*

Let X be a metrizable space with $w(X') = \kappa$, where $cf(\kappa) > \omega$. Then $C(\kappa)$ can be embedded in X as a closed subset. By this fact, the property (6) of Lemma 1.1 and Corollary 3.3, we obtain the following.

Corollary 3.4 *Let X be a metrizable space with $w(X') = \kappa$, where $cf(\kappa) > \omega$, and $n \in \mathbb{N}$. Then $t(S(\kappa)^n) = t(A_{2n}(C(\kappa))) \leq t(A_{2n}(X)) \leq t(A(X)) \leq w(X') = \kappa$.*

Consequently, we obtain the following partial answer to the question (in [2]) whether $t(A(X))$ is equal to $w(X')$ for metrizable spaces,

Theorem 3.5 *Let X be a metrizable space with $w(X') = \kappa$, where $cf(\kappa) > \omega$. Assume that $t(S(\kappa)^n) = \kappa$ for some $n \in \mathbb{N}$. Then $t(A_{2n}(X)) = t(A(X)) = w(X') = \kappa$.*

Corollary 3.6 *Let X be a metrizable space with $w(X') = \omega_1$. Then $t(A_4(X)) = t(A(X)) = w(X') = \omega_1$.*

Proof: Since Gruenhage and Tanaka [6] proved that $t(S(\omega_1)^2) = \omega_1$, Theorem 3.5 follows the result. □

4 Countable tightness of $A(X)$ for metrizable spaces X

In [2], the main results with respect to the free abelian topological groups on metrizable spaces are the following.

Theorem 4.1 ([2]) *Let X be a metrizable space and X' the set of all non-isolated points in X. Then;*

(1) *the tightness of $A(X)$ is countable iff X' is separable,*

(2) *$A(X)$ is a k-space iff X is locally compact and X' is separable.*

With respect to the result (2) in Theorem 4.1, the author [11] showed the following.

Theorem 4.2 ([11]) *Let X be a metrizable space.*

(1) *The following are equivalent:*

 (a) *$A_n(X)$ is a k-space for each $n \in \mathbb{N}$,*

 (b) *$A_4(X)$ is a k-space,*

 (c) *the canonical mapping $i_n : (X \oplus -X \oplus \{0\})^n \longrightarrow A_n(X)$ is quotient for each $n \in \mathbb{N}$,*

(d) i_4 *is quotient,*

(e) *either* X *is locally compact and* X' *is separable, or* X' *is compact.*

(2) *The following are equivalent:*

(a) $A_3(X)$ *is a* k*-space,*

(b) i_3 *is quotient,*

(c) X *is locally compact or* X' *is compact.*

(3) $A_2(X)$ *is a* k*-space and* i_2 *is quotient.*

On the other hand, apply the results in the previous section. Then, with respect to the result (1) in Theorem 4.1, we can show the following.

Theorem 4.3 *Let* X *be a metrizable space.*

(1) *The following are equivalent:*

(a) *the tightness of* $A(X)$ *is countable,*

(b) *the tightness of* $A_4(X)$ *is countable,*

(c) X' *is separable.*

(2) *the tightness of* $A_3(X)$ *is countable.*

Proof: To prove the statement 1, we suffice to show the implication (b) $\Rightarrow$ (c). Suppose that X' is not separable, i.e. $w(X') \geq \omega_1$. Then, $C(\omega_1)$ is embedded in X as a closed subspace. Hence, by Corollary 3.6, we can see that $\omega_1 = t(A_4(C(\omega_1))) \leq t(A_4(X))$.

Next, to show that $t(A_3(X)) \leq \omega$, take a subset H of $A_3(X)$ and a word $g \in A_3(X)$ such that $g \in \overline{H}$. By the property (5) in Lemma 1,1 and Theorem 2.1, we may assume that $g \in X \cup -X$ and $H \in A_3(X) \setminus A_2(X)$. Now apply Lemma 3.1 as $m = n = 1$. Then we can take a subset H_0 of H such that $g \in \overline{H_0}$ and $|H_0| \leq t(A_2(X)) = \omega$. This follows that $t(A_3(X)) \leq \omega$. $\square$

References

[1] A. V. Arhangel'skiĭ, Mapping related to topological groups, Soviet Math. Dokl. **9** (1968), 1011-1015.

[2] A. V. Arhangel'skiĭ, O. G. Okunev and V. G. Pestov, Free topological groups over metrizable spaces, Topology Appl. **33** (1989), 63-76.

[3] K. Eda, G. Gruenhage, P. Koszmider, K. Tamano and S. Todorčević, Sequential fans in topology, Topology Appl., **67** (1995), 189-220.

[4] R. Engelking, General Topology, Heldermann, Berlin, 1989.

[5] M. I. Graev, Free topological groups, Izv. Akad. Nauk SSSR Ser. Mat. **12(3)** (1948), 279-324 (in Russian); English transl.: Amer. Math. Soc. transl. **35** (1951); Reprint: Amer. Math. Soc. Transl. **8** (1962), 305-364.

[6] G. Gruenhage and Y. Tanaka, Products of k-spaces and spaces of countable tightness, Trans. Amer. Math. Soc. **273** (1982), 299-308.

[7] C. Joiner, Free topological groups and dimension, Trans. Amer. Math. Soc. **220** (1976), 401-418.

[8] A. A. Markov, On free topological groups, Izv. Akad. Nauk SSSR Ser. Mat. **9** (1945), 3-64 (in Russian); English transl.: Amer. Math. Soc. Transl. 30 (1950) 11-88; Reprint: Amer. Math. Soc. Transl. **8** (1962), 195-272.

[9] V. G. Pestov, On neighbourhoods of unity in free topological groups, Vestnik Moscov. Univ. Ser. 1 Mat. Mech. **3** (1985), 8-10 (in Russian).

[10] M. G. Tkačenko, Completeness of free abelian topological groups, Soviet Math. Dokl. **27** (1983), 341-345.

[11] K. Yamada, Characterizations of a metrizable space X such that every $A_n(X)$ is a k-space, Topology Appl. **49** (1993), 75-94.

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