Topology Proceedings

Web: http://topology.auburn.edu/tp/
Mail: Topology Proceedings
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Auburn University, Alabama 36849, USA
E-mail: topolog@auburn.edu
ISSN: 0146-4124

Topology Proceedings
Volume 22, Summer 1997, 1–13

CYCLICLY CONNECTED OPEN SUBSETS OF HOMOGENEOUS ARCWISE CONNECTED CONTINUA

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Abstract

Let X be a homogeneous arcwise connected compact metric space which is not topologically a circle. Then X contains cyclicly connected open subsets of at least two types: There is a non-empty, non-dense such set, and the complement of each single point of X is such a set.

1. The Terminology and Setting

Let S be a topological space and let $A \subseteq S$. A is arcwise connected if and only if whenever $p, q \in A$, there exists an arc $J \subseteq A$ with $p, q \in J$. A is cyclicly connected if and only if whenever $p, q \in A$, there is a simple closed curve $J \subseteq A$ with $p, q \in J$.

In a compact metric space X, the set A has the Baire property [6, p. 87, 92] if and only if there exists an open set $O \subseteq S$ such that the symmetric difference $(O - A) \cup (A - O)$ is first category. An analytic set $A \subseteq S$ is a set which is the continuous image of some complete, separable metric space. A continuum will mean a compact connected metric space.

A topological space S is homogeneous if and only if, for each $p, q \in S$ there exists a homeomorphism $h : S \to S$ such that $h(p) = q$.

Mathematics Subject Classification: AMS 1991 subject classification:
Primary: 54F15
Secondary:54H15, 54H05

In 1975 K. Kuperberg asked in conversation whether every arcwise connected homogeneous continuum is locally connected. This question first appeared in print in [5]. It is the primary motivation for my continuing work on these continua. For previous results, see my joint paper with L. Lum [3] and the later papers [1] and [2].

2. The Tools

These first few lemmas and theorems, some proven elsewhere, provide the tools needed for the arguments here. Most of these are special cases of stronger known results.

Lemma 2.1. *Every analytic set in a completely metrizable separable space has the Baire property.*

This result is just a special case of Corollary 1 of [6, p. 482]. (What Kuratowski calls the Baire property "in the restricted sense" is actually stronger than the Baire property.)

Lemma 2.2. *Let X be a complete separable metric space and let $p \in X$. Then $\cup\{J \subseteq X | p \in J$ and J is a simple closed curve$\}$ is an analytic set.*

Proof. Let M denote the specified union of simple closed curves. With $S^1 = \left\{x \in \mathbb{R}^2 \middle| \|x\| = 1\right\}$, let E denote the set of embeddings of S^1 into X with $h(1,0) = p$. It is a routine exercise to verify that E, with the compact-open topology, is a separable, completely metrizable space. Define $e : E \to X$ by $e(h) = h(-1,0)$. Then the image of E under e is $M - \{p\}$, so $M - \{p\}$ is an analytic set. Since $\{p\}$ is clearly analytic, the lemma follows from the easy observation that the union of two analytic sets is analytic. □

Lemma 2.3. *Let X be a homogeneous compact metric space and let $\mathcal{H}(X)$ denote the group of all self-homeomorphisms of*

X endowed with the compact-open topology. Then for each $p \in X$, the evaluation map $e : \mathcal{H}(X) \to X$ defined by $e(h) = h(p)$ is an open mapping.

Proof. This is just a special case of the celebrated Effros Theorem [4]. □

Corollary to Lemma 2.3: *If X is a homogeneous compact metric space, $\mathcal{H}$ is its group of homeomorphisms, $\mathcal{U}$ is open in $\mathcal{H}$ and $A \subseteq X$, then*
$\mathcal{U}\ (A) = \{h(a) | h \in \mathcal{U}$ and $a \in A\}$ is open in X.

The next three lemmas involve ways of finding simple closed curves in unions of other simple closed curves. They are true in any Hausdorff space and turn out to be quite useful. The proofs, while mildly technical and tedious, are straightforward and are omitted.

Lemma 2.4. *Let S be a Hausdorff space and let J, K be simple closed curves contained in S. Assume $x \in J$ and $y \in K$ and that $J \cap K$ contains at least two points. Then there exists a simple closed curve $L \subseteq J \cup K$ with $x, y \in L$.*

Lemma 2.5. *Let S be a Hausdorff space and let $x, y \in S$ and assume that J_1, J_2, and K are simple closed curves in S and that $x \in J_1 \cap J_2$; $y \in K$; $K \cap J_1 \neq \phi$, and $K \cap (J_2 - J_1) \neq \phi$. Then $J_1 \cup J_2 \cup K$ contains a simple closed curve L with $x, y \in L$.*

Lemma 2.6. *Let S be a Hausdorff space and let $x, y \in S$. Suppose there exist simple closed curves J_1, J_2, K_1, and $K_2 \subseteq S$ with $x \in J_1 \cap J_2$; $y \in K_1 \cap K_2$; $K_1 \cap J_1 \neq \phi$; and $K_2 \cap (J_2 - J_1) \neq \phi$. Then there exists a simple closed curve L such that $x, y \in L \subseteq J_1 \cup J_2 \cup K_1 \cup K_2$.*

Cyclic connectivity has been well-understood for decades in the setting of locally connected metric spaces [7]. The preceding Lemmas, along with the next one and the work in [3] permit the elimination of the hypothesis of local connectedness.

Lemma 2.7. *Let S be a Hausdorff space and let K and L be cyclicly connected subsets of S such that $K \cap L$ contains at least two points. Then $K \cup L$ is cyclicly connected also.*

Proof. This can be verified using the previous three lemmas, and considering several cases. □

<u>**Corollary to Lemma 2.7:**</u> *Let X be a homogeneous arcwise connected continuum and let $S \subseteq X$ be a cyclicly connected subset. Suppose $V \subseteq X$ is a non-empty open set such that $V - S$ is first category. Let $\mathcal{H}$ be any set of self-homeomorphisms of X containing the identity, and such that for each $h \in \mathcal{H}$, $h(V) \cap V \neq \phi$. Then $\mathcal{H}(S)$ is cyclicly connected also.*

Proof. Let $p, q \in \mathcal{H}(S)$, and let $h, k \in \mathcal{H}$ be such that $p \in h(S)$ and $q \in k(S)$. Since $h(V) \cap V \neq \phi$, $h(S) \cap S$ contains at least two points so $h(S) \cup S$ is cyclicly connected by Lemma 2.7. Applying Lemma 2.7 again, $h(S) \cup S \cup k(S)$ is cyclicly connected also. Thus there is a simple closed curve J with $p, q \in J \subseteq h(S) \cup S \cup k(S) \subseteq \mathcal{H}(S)$, and the Corollary is proven.

Unfortunately, this Corollary cannot be used directly to prove the first theorem, although the flavor is similar. □

Lemma 2.8. *Let X be a homogeneous continuum which is not a simple closed curve. Then every arc contained in X is nowhere dense. (Hence, so is every simple closed curve or other finite union of arcs in X.)*

Proof. If any arc contains an open subset of X, the homogeneity of X easily implies that X is a one dimensional manifold. By compactness, X is a simple closed curve, contrary to hypothesis. □

3. Main Result

The first theorem resembles a more technical version of the Corollary to Lemma 2.7.

Theorem 3.1. *Let X be a homogeneous arcwise connected continuum which is not a simple closed curve. Then X contains an open set U which is neither empty nor dense such that each two points of U lie on a simple closed curve in U (i.e., U is cyclicly connected.)*

Proof. Assume X is arcwise connected and let $a \in X$. Let d be a metric for X. For every open set $B \subseteq X$ with $a \notin \overline{B}$, define $S(B)$ to be the union of all simple closed curves $J \subset X$ with $a \in J$ and $J \cap \overline{B} = \phi$. Let $\mathcal{B}_0$ be a countable base for X, and let $\mathcal{B} = \left\{B \in \mathcal{B}_0 | a \notin \overline{B}\right\}$. Then $\cup \{S(B) | B \in \mathcal{B}\} = X$, since X is cyclicly connected [3] and so for every $x \in X$ there exists a simple closed curve J such that $a, x \in J$. Since X is not a simple closed curve, $J \neq X$, so $J \cap \overline{B} = \phi$ for some $B \in \mathcal{B}_0$. Since $a \in J$, $a \notin \overline{B}$, so $B \in \mathcal{B}$. Thus, $x \in S(B)$.

Now, since $\cup \{S(B) | B \in \mathcal{B}\} = X$ and $\mathcal{B}$ is countable, it follows that some $S(B)$ is second category. Since this $S(B)$ is an analytic set, it has the Baire property. Thus, there exists a nonempty open set W such that the symmetric difference of W and $S(B)$ is first category. Let $s \in W$ and let $D \in \mathcal{B}$ be such that $\overline{D} \subseteq B$. Let $\epsilon > 0$ be such that the ϵ-ball $B(s, \epsilon) \subseteq W$ and such that $\epsilon < d(\overline{D}, X - B)$. Let $\mathcal{H}$ be the set of homeomorphisms of X within $\frac{\epsilon}{2}$ of the identity.

Let $U = \mathcal{H}(S(B))$ and let $V = \mathcal{H}(a)$. Then U and V are both open sets in X by the Corollary to Lemma 2.3, and $a \in V \subseteq U$. The proof now involves proving four claims, the last of which is just a restatement of the theorem.

<u>Claim 1</u>: For every $b \in V$, there exists a simple closed curve $J \subseteq U$ such that $a, b \in J$.

<u>**Proof of Claim 1**</u>: Suppose $b \in V$. Let $h \in \mathcal{H}$ be a homeomorphism with $h(a) = b$. Then $h(S(B)) \cap S(B) \neq \phi$; (in fact $h(S(B)) \cap S(B)$ is second category.) Let $y \in h(S(B)) \cap S(B)$. Then there exist simple closed curves. $J_1 \subseteq h(S(B))$ and $K_1 \subseteq S(B)$ such that $b, y \in J_1$ and $a, y \in K_1$.

If $J_1 \cap K_1$ has more than one point, the Claim follows by Lemma 2.4, so assume $J_1 \cap K_1 = \{y\}$. Since $J_1 \cup K_1$ is nowhere dense in X by Lemma 2.8, it follows that $h(S(B)) \cap S(B) \not\subseteq J_1 \cup K_1$. Let $z \in (h(S(B)) \cap S(B)) - (J_1 \cup K_1)$. Then, again, there exist simple closed curves $J_2 \subseteq h(S(B))$ and $K_2 \subseteq S(B)$ such that $b, z \in J_2$ and $a, z \in K_2$. Then (since $z \notin J_1 \cup K_1$) by Lemma 2.6, $J_1 \cup J_2 \cup K_1 \cup K_2$ contains a simple closed curve J such that $a, b \in J \subseteq h(S(B)) \cup S(B) \subset U$. Thus Claim 1 holds.

<u>**Claim 2**</u>: For every $b, c \in V$, there exists a simple closed curve J such that $b, c \in J \subset U$.

<u>**Proof of Claim 2**</u>: By <u>Claim 1</u>, there exist simple closed curves $J_1, K_1 \subseteq U$ such that $a, b \in J_1$ and $a, c \in K_1$. If $J_1 \cap K_1$ has more than one point, $J_1 \cup K_1$ contains a simple closed curve J with $b, c \in J$ by Lemma 2.4. Thus, assume $J_1 \cap K_1 = \{a\}$. Let $h, k \in \mathcal{H}$ with $h(a) = b$ and $k(a) = c$. Then $h^{-1} \in \mathcal{H}$ also, and so $h^{-1} \circ k$ is within ϵ of the identity on X. Thus $(h^{-1} \circ k)(S(B)) \cap S(B)$ is also second category, and applying h, $k(S(B)) \cap h(S(B))$ is second category. Let z belong to $[k(S(B)) \cap h(S(B))] - (J_1 \cup K_1)$.

Then let $J_0, K_0 \subseteq S(B)$ be simple closed curves such that $h^{-1}(z), a \in J_0$ while $k^{-1}(z), a \in K_0$. Then let $h(J_0) = J_2$; $k(K_0) = K_2$; $z, b \in h(J_0)$; and $z, c \in K_2$ (since $b = h(a)$ and $c = k(a)$).

Thus, J_1, J_2, K_1, K_2 satisfy the hypotheses of Lemma 2.6, and so the union of $J_1, J_2, K_1,$ and K_2 contains a simple closed curve J with $b, c \in J \subseteq h(S(B)) \cup k(S(B)) \subseteq U$.

<u>**Claim 3**</u>: Let $p \in U$ be arbitrary and let $b \in V$ be arbi-

trary. Then there exists a simple closed curve $J \subset U$ such that $p, b \in J$.

<u>**Proof of Claim 3**</u>: Let $p_0 \in S(B)$ and let $h \in \mathcal{H}$ be such that $h(p_0) = p$. Then $h(a) \in V$, and so if $K_0 \subseteq S(B)$ is a simple closed curve with $a, p_0 \in K_0$, then $h(a), p \in K_1 = h(K_0)$. Also, there exists a simple closed curve $J_1 \subseteq U$ with $b, h(a) \in J_1$ by Claim 1. If $J_1 \cap K_1 \neq \{h(a)\}$ then Lemma 2.4 guarantees that $J_1 \cup K_1$ contains a simple closed curve joining p and b, so assume $J_1 \cap K_1 = \{h(a)\}$. Then $K_1 \cap V \neq \{h(a)\}$ since $K_1 \cap V$ is open in K_1. Let $d \in K_1 \cap V; d \neq h(a)$. Then there exists a simple closed curve $J_2 \subseteq U$ such that $b, d \in J_2$. Since $d \notin J_1, K_1 \cap (J_2 - J_1) \neq \phi$, and thus $J_1 \cup J_2 \cup K$, contains a simple closed curve J joining p to b by Lemma 2.5, and Claim 3 is established.

<u>**Claim 4:**</u> Let $p, q \in U$. Then there is a simple closed curve $J \subseteq U$ with $p, q \in J$.

<u>**Proof of Claim 4**</u>: Let $p, q \in U$ be arbitrary. Let J_1 and K_1 be simple closed curves contained in U such that $a, p \in J_1$ and $a, q \in K_1$. If $J_1 \cap K_1$ contains a point other than a, the proof is done by Lemma 2.4, so assume $J_1 \cap K_1 = \{a\}$. Then $J_1 \cup K_1$ does not contain all of V by Lemma 2.8. Consequently, there exists $d \in V - (J_1 \cup K_1)$, and by Claim 3, there exist simple closed curves $J_2, K_2 \subseteq U$ such that $p, d \in J_2$ and $q, d \in K_2$. Thus, by Lemma 2.6 $J_1 \cup J_2 \cup K_1 \cup K_2$ contains a simple closed curve J such that $p, q \in J$.

This concludes the proof of Theorem 3.1. □

<u>**Corollary to Proof of Theorem 3.1.**</u> *Let X be a homogeneous arcwise connected continuum, let W be a cyclicly connected open subset of X, and let $p \in W$. Then there exists a cyclicly connected open set U with $p \in U \subseteq \overline{U} \subseteq W$.*

Proof. Let $\{U_n\}_{n=1}^{\infty}$ be a sequence of open sets with $p \in U_1$, and for all $n, \overline{U}_n \subseteq U_{n+1}$, such that $\bigcup_{n=1}^{\infty} U_n = U$. For each n, let $S_n = \cup\{J | J$ is a simple closed curve with $p \in J \subseteq U_n\}$. Then $\bigcup_{n=1}^{\infty} S_n = U$, so some S_k is second category.

Let $V \subseteq X$ be an open set such that $V - S_k$ is first category. Let $\epsilon > 0$ be such that both the following hold i) for some $x \in V$, the ϵ-ball about x is contained in V, and ii) the distance from $\overline{U}_k$ to $X - U_{k+1}$ is greater than ϵ. The remainder of the proof proceeds exactly as in Theorem 3.1. □

The next theorem builds upon the one just proven, but takes some surprising turns in its proof.

Theorem 3.2. *Let X be a homogeneous arcwise connected continuum which is not a simple closed curve. Let $p \in X$. Then $X - \{p\}$ is cyclicly connected.*

Proof. This proof makes use of a set-valued set-function η. For a continuum X and a set $A \subseteq X$, define $\eta(A) = X - \{x|$ there exists a cyclicly connected open set U with $x \in U$ and $\overline{U} \cap A = \phi\}$. Some additional lemmas are incorporated into the proof. □

Lemma 3.3. *The function η is idempotent, that is, $\eta \circ \eta = \eta$ on any homogeneous arcwise connected continuum X.*

Proof. Let $A \subseteq X$ and suppose $x \notin \eta(A)$. Let $U \subseteq X$ be a cyclicly connected open set such that $x \in U$ and $\overline{U} \cap A = \phi$. By the Corollary to the proof of Theorem 3.1, there exists a cyclicly connected open set V such that $x \in V \subseteq \overline{V} \subseteq U$. Since $\overline{V} \subseteq U$, $\overline{V} \cap \eta(A) = \phi$. Consequently, $x \notin \eta(\eta(A))$, so that $\eta \circ \eta(A) \subseteq \eta(A)$. The reverse inclusion is clear, so $\eta \circ \eta(A) = \eta(A)$.

Following standard practice for set valued set functions, we write $\eta(x)$ for $\eta(\{x\})$. □

Lemma 3.4. *Let X be a homogeneous arcwise connected continuum. Then the relation "$p \in \eta(q)$" is an equivalence relation on X.*

Proof. Let X be a homogeneous arcwise connected continuum. Suppose $p \in X$. Clearly $p \in \eta(p)$, so the relation is reflexive. Suppose $p \in \eta(q)$ and $q \in \eta(r)$. Then $p \in \eta(q) \subseteq \eta(\eta(r)) = \eta(r)$, so the relation is transitive. To prove symmetry, consider $\mathcal{M} = \{\eta(p)|p \in X\}$. This is a family of closed subsets of X and so is partially ordered by set inclusion. By Hausdorff Maximality there is a maximal linearly ordered $\mathcal{C} \subset \mathcal{M}$. Let $A = \cap\mathcal{C}$. By compactness, $A \neq \phi$. Let $a \in A$ be arbitrary. Then for each p such that $\eta(p) \in \mathcal{C}$, we have $a \in \eta(p)$. Thus $\eta(a) \subset \eta \circ \eta(p) = \eta(p)$; that is, $\eta(a) \subseteq A$. But by maximality of $\mathcal{C}$, $\eta(a) \in \mathcal{C}$, and thus $A \subseteq \eta(a)$. Consequently, for every $a \in A$, $\eta(a) = A$. Let $a \in A$ and suppose $b \in \eta(a)$. Then $b \in A$ and as a result, $\eta(b) = A$ and $a \in \eta(b)$. This means that there exists a point $a \in X$ such that for every $b \in X$, if $b \in \eta(a)$ then $a \in \eta(b)$.

By homogeneity, every $a \in X$ must have this property, so the relation is symmetric, and hence is an equivalence relation. □

<u>**Corollary to Lemma 3.4.**</u> *The collection $\{\eta(p)|p \in X\}$ is a continuous decomposition of X.*

Proof. This follows from Lemma 3.4 and Lemma 2.3. □

Lemma 3.5. *Let X be an arcwise connected homogeneous continuum. Then for each $x \in X$, $\eta(x)$ is the intersection of all the simple closed curves J with $x \in J \subseteq X$.*

Proof. If X is a simple closed curve itself this is clear since both $\eta(x)$ and the intersection of all simple closed curves containing x are equal to X itself; so assume that X is not a simple closed curve. Let $y \notin \eta(x)$. Then by Lemma 3.4, $x \notin \eta(y)$.

Let U be a cyclicly connected open set with $x \in U$ and $y \notin \overline{U}$. Let $p \in U$ with $p \neq x$. Then there exists a simple closed curve $J \subseteq U$ with $p,\ x \in J$. Since $y \notin J$, we have $y \notin \cap\{J \subseteq X | J$ is a simple closed curve and $x \in J\}$. Consequently, by contraposition, $\cap\{J \subseteq X | J$ is a simple closed curve and $x \in J\}$ is a subset of $\eta(x)$.

To prove the reverse inclusion, first observe that $\eta(x) \neq X$, since by Theorem 3.1, there is a cyclicly connected open set U with $\overline{U} \neq X$, and by homogeneity, there is such a set U_0 with $x \notin \overline{U}_0$. Let z be any element of $X - \eta(x)$, and let $J \subseteq X$ be a simple closed curve with $x,\ z \in J$. This is possible since X is cyclicly connected by Theorem 3 of [3, p. 395]. Since $z \notin \eta(x)$, there is a cyclicly connected open set V_0 with $z \in V_0$ and $x \notin \overline{V}_0$.

By the Corollary to the proof of Theorem 3.1, there exists a cyclicly connected open set V with $z \in V \subseteq \overline{V} \subseteq V_0$. Thus $\overline{V} \cap \eta(x) = \phi$. Let q be an arbitrary element of $X - (J \cup \overline{V})$. Let $\epsilon > 0$ be such that the ϵ-ball about z is a subset of V and such that ϵ is smaller than the distance from q to $(\overline{V} \cup J)$. Let $\mathcal{H}$ denote the set of homeomorphisms within ϵ of the identity of X. Then, by Lemma 2.7 and the corollary to it, $M = \mathcal{H}(\overline{V} \cup J)$ is a cyclicly connected open set, $x \in M$ and $q \notin \overline{M}$. Hence $x \notin \eta(q)$, and by symmetry $q \notin \eta(x)$. Since $q \in X - (\overline{V} \cup J)$ was arbitrary, $\eta(x) \subseteq \overline{V} \cup J$. But, since $\eta(x) \cap \overline{V} = \phi$, $\eta(x) \subseteq J$. Since $z \in J$ and $z \notin \eta(x)$, $\eta(x)$ is a proper subset of J.

Now it is straightforward to verify that $\eta(x)$ is a subset of every simple closed curve containing x, for let K be an arbitrary simple closed curve with $x \in K \subseteq X$. Since K cannot be a proper subset of J, $K \nsubseteq \eta(x)$. Let $z_0 \in K - \eta(x)$. Then (with z_0 in place of z) the same argument as used above with the simple closed curve J guarantees that $\eta(x) \subseteq K$. Since K was arbitrary, $\eta(x) \subseteq \cap\{J \subseteq X | J$ is a simple closed curve and $x \in J\}$, so that the two are equal. □

Lemma 3.6. *If X is any homogeneous arcwise connected con-*

tinuum which is not a simple closed curve, then X contains a theta curve; that is, the suspension of a three-point set.

Proof. This is an easy exercise using the facts that X is cyclicly connected and that the complement of each point in X is arcwise connected, proven in [3]. □

Lemma 3.7. *Let X be a homogeneous arcwise connected continuum which is not a simple closed curve. Then for each $x \in X$, $\eta(x) = \{x\}$.*

Proof. Let X be as given with $x \in X$. Since X contains a theta curve, by homogeneity there is a theta curve $M = A \cup B \cup C$ with x as one of its vertices, where A, B, and C are arcs from x to another point $\hat{x} \in X$ such that

$$A \cap B = A \cap C = B \cap C = \{x, \hat{x}\}.$$

Then by Lemma 3.5

$$\eta(x) \subseteq (A \cup B) \cap (A \cup C) \cap (B \cup C) \subseteq \{x, \hat{x}\}.$$

Thus $\eta(x)$ contains at most two points. Since by homogeneity each $\eta(x)$ has the same cardinality, we have either for every $x \in X$, $\eta(x) = \{x\}$ or for every $x \in X$, $\eta(x)$ has two points. We shall rule out the second possibility as follows: If $\eta(x)$ always has cardinality two, define $\theta : X \to X$ by: $\theta(x)$ is that point $\hat{x}$ such that $\{\hat{x}, x\} = \eta(x)$ for each $x \in X$. Then θ is continuous and $\theta \circ \theta$ is the identity on X. Let $T = A \cup B \cup C$ be a θ curve, where as above each of A, B, and C is an arc joining two points a and b. Suppose $q \in A$ is arbitrary. Then $(A \cup B) \cap (A \cup C) = A$, and thus $\eta(q) \subseteq A$. Hence $\theta|A : A \to A$ is a fixed-point-free map of the arc A onto itself, a contradiction. Thus, $\eta(x) = \{x\}$ for all $x \in X$, completing the proof.

To complete the proof of Theorem 3.2 let $x \in X$ and let $\mathcal{U}$ be a covering of $X - \{x\}$ by cyclicly connected open sets U with $x \notin \overline{U}$. Then since $X - \{x\}$ is connected, given any $p, q \in$

, $X - \{x\}$, there is a simple chain $U_1, U_2, \cdots U_n$ of elements of $\mathcal{U}$ with $p \in U_1$; $q \in U_n$. By Lemma 2.7 and an easy induction, $\bigcup_{i=1}^{n} U_i$ is cyclicly connected. Thus there is simple closed curve J such that $p,\ q \in J \subseteq \bigcup_{i=1}^{n} U_i \subseteq X - \{x\}$. □

Since $p, q \in X - \{x\}$ were arbitrary, $X - \{x\}$ is cyclicly connected, and Theorem 3.2 is proven.

Finally, perhaps the following restatement is more intuitive.

Corollary to Theorem 3.2. *Let X be a homogeneous arcwise connected continuum which is not a simple closed curve. Then given any three distinct points p, q, and r in X, there is a simple closed curve $J \subseteq X$ with $p, q \in J$ but $r \notin J$.*

Question. Let X be a homogeneous arcwise connected continuum which is not a simple closed curve, and let F, G be disjoint finite subsets of X. Does there always exist a simple-closed curve J such that $F \subseteq J$ and $G \cap J = \phi$?

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