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A CHARACTERIZATION OF THE VIETORIS TOPOLOGY

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Dedicated to Helmut Röhrl at the occasion of his seventieth birthday

Abstract

The Vietoris topology on the hyperspace VX of closed subsets of a topological space X is shown to represent the closed-valued, clopen relations with codomain X. Hence, it is characterized by a (co)universal property which can be formulated in concrete categories equipped with a closure operator. Functorial properties of the Vietoris topology are derived, and some examples of closure-structured categories with Vietoris objects are exhibited.

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0. Introduction

A basic open set with respect to the *Vietoris topology* on the set $V_\circ X$ of non-empty closed subsets of a topological space X is given by

$$< U_1, \cdots, U_n >= \{B \in V_\circ X \mid B \subseteq U_1 \cup \cdots \cup U_n \ \& \ \forall i : B \cap U_i \neq \emptyset\},$$

where $U_1, \cdots, U_n$ is any finite sequence of open sets of X. Vietoris [11] showed that with X also $V_\circ X$ is compact Hausdorff; if, moreover, X is metrizable, also $V_\circ X$ is, with the Vietoris topology induced by the Hausdorff metric

$$h_d(A, B) = \sup_{x \in X} |d(x, A) - d(x, B)|,$$

for any admissible metric d on X. Without the compactness assumption (but still in the metrizable case), Beer, Lechicki, Levi and Naimpally [1] characterized the Vietoris topology as

- the coarsest topology which makes the distance functions

  $$d(-, -) : V_\circ X \times V_\circ X \longrightarrow \mathbb{R}$$

  continuous, where d runs through the admissible metrics on X

- the coarsest topology which makes the distance functions

  $$d(x, -) : V_\circ X \longrightarrow \mathbb{R}$$

  continuous where x runs through the points of X and d runs as before.

In this paper we characterize the space $V_\circ X$, for any topological space X. In fact, like the authors of [7] we find it convenient to consider the so-called upper and lower Vietoris topology on the whole power set PX, with the supremum of the two topologies giving the Vietoris topology. At the set level, it is well known (and obvious) that PX represents the relations with codomain X, i.e., that there is a natural correspondence between the relations $R \subseteq Y \times X$ and the functions $Y \to PX$, for every set Y. In this paper, we shall emphasize the inclusion map $r : R \hookrightarrow Y \times X$, rather than the set R, and write $x\ r\ y$ instead of $(y, x) \in R$; the function corresponding to r is denoted by $r^\sharp : Y \to PX$ with $r^\sharp(y) = \{x \in X \mid y\ r\ x\}$. In the context of spaces, one would then ask: which are the relations that correspond to continuous functions $Y \to PX$, with respect to the upper and lower Vietoris topology? These are exactly the *closed* and *open relations*, respectively, as defined in this paper. (The terminology coincides with the one used for equivalence relations.) When restricting PX to the subspace VX of all closed subsets of X, or even to $V_\circ X$, one has to require the relations r to be, in addition, *closed-valued* (so that $r^\sharp(y)$ is a closed set in X for every $y \in Y$), or even non-empty valued. Hence, VX, together with the (inverted) element relation $\epsilon_X : \{(B, x) \mid x \in B\} \hookrightarrow VX \times X$, is characterized by the following couniversal property:

1. ϵ_X is clopen and closed-valued;

2. any clopen and closed-valued relation $r : R \hookrightarrow Y \times X$ factors as $\epsilon_X \circ G(h) = r$, with a uniquely determined continuous function $h : Y \to VX$; here $G(h)$ is the graph of

h, and $\circ$ denotes the usual relational composition.

Our point is that the notions of closed and open relations are strictly categorical, i.e., expressible in any finitely complete category with an axiomatically given subobject- and closure structure, as in [2] and [3]; for closed-valuedness, the category should also be concrete, with singleton sets forming subobjects. Hence, in such a category, the couniversal property gives a notion of *Vietoris object*, for which we give an additional example in the category of preordered sets (see Section 6). While we plan to investigate the categorical aspects in a later paper, our main focus here is on exploiting the categorical property of the (lower/upper) Vietoris topology on PX and VX, in particular on how to describe P and V as functors, and on the properties of these functors. The main justification for considering PX rather than VX stems from the fact that closed and open relations are closed under relational composition, while closed-valuedness is not. Hence P (as a covariant functor) is simply the right adjoint to the graph functor G, and via the induced monad P becomes simply an endofunctor of a category with objects all topological spaces and with morphisms the respective relations; trading G for G° (the opposite graph functor), P becomes contravariant (see Sections 3 and 4). Although V cannot be presented as a right adjoint, there are still reasonable covariant and contravariant functorial descriptions of the upper and lower Vietoris topology. The contravariant functor has good continuity properties, as indicated by the (probably known and obvious) formula $V(X+Y) \cong VX \times VY$ and the new fact that, for certain quotient spaces Y of X, VY can be canonically embedded into VX (see Section 5).

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1. Hit&Miss Topologies

1.1 For a set X, let $\mathcal{F}$ be a subset of the power set PX. The *upper $\mathcal{F}$-topology* on PX has as generating closed sets the sets

$$F^+ = \{B \subseteq X \,|\, B \cap F \neq \emptyset\} \quad (\text{'}B \textit{ hits } F\text{'}),$$

$F \in \mathcal{F}$; a generating system of closed sets of the *lower $\mathcal{F}$-topology* on PX is formed by the sets

$$\begin{aligned} F^- &= \{B \subseteq X \,|\, B \cap (X \setminus F) = \emptyset\} \quad (\text{'}B \textit{ misses } X \setminus F\text{'}) \\ &= \{B \subseteq X \,|\, B \subseteq F\}, \end{aligned}$$

$F \in \mathcal{F}$. We denote these topologies by $\tau^+(\mathcal{F})$ and $\tau^-(\mathcal{F})$, respectively. For $\mathcal{G} \subseteq PX$, we call the supremum of $\tau^+(\mathcal{F})$ and $\tau^-(\mathcal{G})$ the *$(\mathcal{F}, \mathcal{G})$-hit&miss topology* on PX and denote it by $\tau(\mathcal{F}, \mathcal{G})$.

1.2 Let X be a topological space, and let $\mathcal{F}$ be the set of closed subsets of X. We then call $\tau^+(\mathcal{F})$ and $\tau^-(\mathcal{F})$ the *upper* and *lower Vietoris topology* on PX, respectively, and $\tau(\mathcal{F}, \mathcal{F})$ is the *Vietoris topology* on PX; the resulting topological spaces are denoted by P^+X, P^-X and PX, respectively.

1.3 Note that the sets

$$U^- = PX \setminus (X \setminus U)^+ \quad (U \subseteq X \text{ open})$$

form a subbase of open sets of P^+X, and the sets

$$U^+ = PX \setminus (X \setminus U)^- \quad (U \subseteq X \text{ open})$$

generate the open sets of P^-X. A base of open sets for (the Vietoris topology of) PX is given by the sets

$$\begin{aligned} <\mathcal{U}> &= \{B \in PX \mid B \subseteq \bigcup \mathcal{U} \text{ and } (\forall U \in \mathcal{U} \,:\, B \cap U \neq \emptyset)\} \\ &= (\bigcup \mathcal{U})^- \cap \bigcap \{U^+ \mid U \in \mathcal{U}\}, \end{aligned}$$

where $\mathcal{U}$ is any finite set of open sets in X.

1.4 Although in this paper we are exclusively dealing with the (upper and lower) Vietoris topology, we mention the fact that hit&miss topologies were discussed in fair generality already in the sixties, see especially [9]. Other than the Vietoris topology, of particular importance is the *Fell topology* (see [4]) $\tau(\mathcal{K}, \mathcal{F})$ where $\mathcal{K}$ is the set of compact sets and $\mathcal{F}$ the set of closed sets in X. Wyler [12] discusses this topology when X is locally compact (but not necessarily Hausdorff), although he calls it Vietoris topology; of course, when X is compact Hausdorff, one can't tell the difference.

2. Relations

2.1 For a relation $r : R \hookrightarrow Y \times X$ from a set Y to a set X, we denote by r_1, r_2 the restrictions of the projections of $Y \times X$

to R:

$$\begin{tikzcd} & R \arrow[dl, "r_1"'] \arrow[dr, "r_2"] & \\ Y & & X \end{tikzcd} \tag{1}$$

Taking images and preimages along r_i gives two functions

$$\begin{tikzcd} N \arrow[r, mapsto] & r_2(r_1^{-1}(N)) \\ PY \arrow[r, shift left, "r^*"] & PX \arrow[l, shift left, "r_*"] \\ r_1(r_2^{-1}(M)) & M. \arrow[l, mapsto] \end{tikzcd} \tag{2}$$

For functions $f : X \to Y$ and $g : Y \to X$, we are particularly interested in the relations

$$G^\circ(f) = \{(y,x) \mid y = f(x)\} \text{ and } G(g) = \{(y,x) \mid g(y) = x\}$$

which, when replacing bijections by identity maps, we may depict by

$$\begin{tikzcd} & X \arrow[dl, "f"'] \arrow[dr, equal] & \\ Y & & X \end{tikzcd} \qquad \begin{tikzcd} & Y \arrow[dl, equal] \arrow[dr, "g"] & \\ Y & & X \end{tikzcd} \tag{3}$$

In these cases, (2) is simply the adjunction given by direct image and preimage along the maps in question.

2.2 The singleton map $s_X : X \to PX$ embeds the topological space X into PX (with the Vietoris topology). For another topological space Y and a relation $r : R \hookrightarrow Y \times X$, we denote by $r^\sharp$ the composite map

$$Y \xrightarrow{s_Y} PY \xrightarrow{r^*} PX$$

and prove (with $c_X(M) = \overline{M}$ the usual closure of $M \subseteq X$):

Proposition.

(1) *Equivalent are:*

i. $r^* : P^+Y \to P^+X$ *is continuous,*

ii. $r^\sharp : Y \to P^+X$ *is continuous,*

iii. $r_*(F)$ *is closed in* Y *for every closed set* F *in* X,

iv. $r_*(c_X(M)) \supseteq c_Y(r_*(M))$ *for every subset* $M \subseteq X$.

(2) *Equivalent are:*

i. $r^* : P^-Y \to P^-X$ *is continuous,*

ii. $r^\sharp : Y \to P^-X$ *is continuous,*

iii. $r_*(U)$ *is open in* Y *for every open set* U *in* X,

iv. $r^*(c_Y(N)) \subseteq c_X(r^*(N))$ *for every subset* $N \subseteq Y$.

Proof. (1) For the equivalence of i, ii, iii, it suffices to note the identities

$$(r^\sharp)^{-1}(F^+) = r_*(F), \quad (r^*)^{-1}(F^+) = r_*(F)^+,$$

and iii $\Leftrightarrow$ iv is trivial.

(2) The equivalence of i, ii, iii follows from the identities

$$(r^\sharp)^{-1}(F^-) = Y \setminus r_*(X \setminus F), \quad (r^*)^{-1}(F^-) = (Y \setminus r_*(X \setminus F))^-.$$

For iii $\Rightarrow$ iv, let $N \subseteq Y$ and consider the open set $U := X \setminus c_X(r^*(N))$; then $N \subseteq Y \setminus r_*(U)$, with $r_*(U)$ open by hypothesis, hence $c_Y(N) \subseteq Y \setminus r_*(U)$ and therefore $r^*(c_Y(N)) \subseteq c_X(r^*(N))$ by definition of U. For iv $\Rightarrow$ iii, let $U \subseteq X$ be open and consider

$N := Y \setminus r_*(U)$; then $r^*(N) \subseteq X \setminus U$, hence $c_X(r^*(N)) \subseteq X \setminus U$ and then $r^*(c_Y(N)) \subseteq X \setminus U$, by hypothesis. This implies that $r_*(U) \cap c_Y(Y \setminus r_*(U))$ must be empty, so that $r_*(U)$ is indeed open. □

2.3 We call r *closed* (or *upper semi-continuous*) if the equivalent conditions of 2.2(1) hold, and *open* (or *lower semi-continuous*) if the conditions of 2.2(2) hold. This terminology coincides with the one used for equivalence relations. More importantly for our purposes, if $r : R \hookrightarrow Y \times X$ is the opposite of the graph of a map $X \to Y$, then these relational notions coincide with the map notions. Furthermore, the graph of a continuous function $Y \to X$ is always *clopen*, i.e. closed and open. Hence we have:

Corollaries.

(1) *Equivalent are for a relation $r : R \hookrightarrow Y \times X$:*

i. $r^* : PY \to PX$ *is continuous,*

ii. $r^\sharp : Y \to PX$ *is continuous,*

iii. r *is clopen.*

(2) *Equivalent are for an equivalence relation $r : R \hookrightarrow X \times X$:*

i. r *is closed (open),*

ii. *the natural quotient map $X \to X/r$ is closed (open),*

iii. *the inclusion map $X/r \to P^+X$ (P^-X) is continuous.*

(3) *Equivalent are for a map $f : X \to Y$:*

i. $f^{-1}(-) : P^+Y \to P^+X$ ($P^-Y \to P^-X$) *is continuous,*

ii. $f^{-1}(-) : Y \to P^+X$ (P^-X) *is continuous,*

iii. f *is closed (open),*

iv. $G^\circ(f)$ *is closed (open).*

(4) *If* $g : Y \to X$ *is continuous, so are*

$$g(-) : P^+Y \to P^+X \ \ \textit{and} \ \ g(-) : P^-Y \to P^-X.$$

2.4 Recall that the composite of the relations $s : S \hookrightarrow Z \times Y$ and $r : R \hookrightarrow Y \times X$ is given by

$$r \circ s : \{(z,x) \mid \exists y \ : \ (z,y) \in S \text{ and } (y,x) \in R\} \hookrightarrow Z \times X.$$

Since

$$(r \circ s)^* = r^* \circ s^* \text{ and } (r \circ s)_* = s_* \circ r_*,$$

Proposition 2.2 gives immediately that the composite of closed (open) relations is closed (open). We obtain the categories

$$\mathrm{Rel}_{\mathrm{c}}(\mathcal{T}op), \ \ \mathrm{Rel}_{\mathrm{o}}(\mathcal{T}op), \ \ \mathrm{Rel}_{\mathrm{co}}(\mathcal{T}op)$$

whose objects are topological spaces and whose morphisms are closed, open and clopen relations, respectively; instead of $r : R \hookrightarrow Y \times X$, we shall now write $r : Y \to\!\!\!\!\!\!\!+\ \, X$.

3. Universal Property of the Vietoris Topology

3.1 Recall that in the category $\mathcal{S}et$ of sets, the power set PX represents the relations with codomain X. Thanks to Proposition 2.2 the Vietoris topology lifts this property to the category $\mathcal{T}op$ of topological spaces:

Proposition. *Let* X *be a topological space.*

(1) *The element relation* $\epsilon_X : P^+X \relbar\joinrel\mapstochar\joinrel\rightarrow X$ $(P^-X \relbar\joinrel\mapstochar\joinrel\rightarrow X)$ *is closed (open).*

(2) *For every space* Y *and every closed (open) relation* $r : Y \relbar\joinrel\mapstochar\joinrel\rightarrow X$, *there is exactly one continuous map* $h : Y \to P^+X$ $(Y \to P^-X)$ *with* $\epsilon_X \circ G(h) = r$.

$$\begin{tikzcd} P^+X & P^+X \arrow[r, "\epsilon_X", "|" marking] & X \\ Y \arrow[u, dashed, "h"] & Y \arrow[u, "G(h)", "|" marking] \arrow[ur, "r"', "|" marking] & \end{tikzcd} \tag{4}$$

Proof. (1) Since $\epsilon_X^\sharp = \mathrm{id}_{PX}$, condition ii of 2.2 is trivially satisfied.

(2) Given r, at the $\mathcal{S}et$-level, $h = r^\sharp$ is the only fitting map which, by 2.2, is continuous for the respective topologies. □

3.2 According to 2.3, G can be considered a functor

$$\mathcal{T}op \longrightarrow \mathrm{Rel_c}(\mathcal{T}op), \quad \mathcal{T}op \longrightarrow \mathrm{Rel_o}(\mathcal{T}op), \quad \mathcal{T}op \longrightarrow \mathrm{Rel_{co}}(\mathcal{T}op)$$

which maps objects identically. Proposition 3.1 says:

Theorem. *There are functors*

$$P^+ : \mathrm{Rel_c}(\mathcal{T}op) \longrightarrow \mathcal{T}op, \quad P^- : \mathrm{Rel_o}(\mathcal{T}op) \longrightarrow \mathcal{T}op,$$
$$P : \mathrm{Rel_{co}}(\mathcal{T}op) \longrightarrow \mathcal{T}op$$

which are right adjoint to the respective graph functor G.

3.3 The three adjunctions of 3.2 induce monads on $\mathcal{T}op$ whose functor parts we again denote by P^+, P^- and P, respectively (since G maps objects identically); on morphisms they act by taking images (see 2.3(4)). The units of these monads are given by the singleton maps s_X (see 2.2), while 'multiplication' maps are given by set-theoretic union; hence the union map

$$u_X = P\epsilon_X = \epsilon_X^* : PPX \to PX$$

is continuous (also when P is traded for P^+ or P^-). We leave it for later work to describe the algebras of these monads. Here we mention only that, by 2.2(2), continuity of u_X implies the inequality

$$\bigcup c_{PX}(\mathcal{N}) \subseteq c_X(\bigcup \mathcal{N}) \tag{5}$$

for every $\mathcal{N} \subseteq PX$. For later use we give an example showing that this inequality is in general strict, even for good spaces X.

3.4 Example. For $X = \mathbb{R}$ the real line, consider

$$\mathcal{N} = \{\, [a,1] \cup [\tfrac{1}{a}, \infty[\,;\ 0 < a < 1\}.$$

We claim that for all $B \in PX$ with $0 \in B$, $B \notin c_{PX}(\mathcal{N})$. In fact, if B has an upper bound L, then (in the notation of 1.3)

$$< \{\,]-\infty, L+1[\, \} >$$

is a neighbourhood of B in PX which does not meet $\mathcal{N}$. Hence, we may assume that B is not bounded from above. If $B \cap [0,1] = \{0\}$, then B lies in

$$< \{\,]-\infty, \tfrac{1}{2}[,\ ]1, \infty[\, \} >$$

although this open subset of PX does not meet $\mathcal{N}$. Otherwise, we fix an element $b \in B$ with $b > 1$ and consider

$$\mathcal{U} = \{\,]-\infty, \tfrac{1}{b+1}[,\ ]0,2[,\ ]1,b+1[,\ ]b,\infty[\, \};$$

then $<\mathcal{U}>$ contains B since $\bigcup\mathcal{U} = X$ and each interval in $\mathcal{U}$ intersects B, by our hypotheses. However, $<\mathcal{U}>$ does not intersect $\mathcal{N}$; assuming

$$([a,1] \cup [\tfrac{1}{a}, \infty[) \cap]-\infty, \tfrac{1}{b+1}[\neq \emptyset$$

for some $a < 1$ we would obtain $a < \frac{1}{b+1}$, hence $\frac{1}{a} > b+1$, so that

$$([a,1] \cup [\tfrac{1}{a}, \infty[) \cap]1, b+1[= \emptyset$$

would be empty.

Hence, the point $0 \in c_{\mathbf{R}}(\bigcup\mathcal{N})$ does not lie in $\bigcup c_{P\mathbf{R}}(\mathcal{N})$.

4. P as a Contravariant Functor

4.1 We saw that P^+, P^- and P may be considered endofunctors of $\mathcal{T}op$. But by composing the right adjoints of 3.2 with G° (rather than with G), which may be considered a functor

$$\mathcal{T}op_{\mathrm{c}}^{\mathrm{op}} \longrightarrow \mathrm{Rel}_{\mathrm{c}}(\mathcal{T}op), \qquad \mathcal{T}op_{\mathrm{o}}^{\mathrm{op}} \longrightarrow \mathrm{Rel}_{\mathrm{o}}(\mathcal{T}op), \qquad \mathcal{T}op_{\mathrm{co}}^{\mathrm{op}} \longrightarrow \mathrm{Rel}_{\mathrm{co}}(\mathcal{T}op),$$

we obtain functors

$$\tilde{P}^+ : \mathcal{T}op_{\mathrm{c}}^{\mathrm{op}} \longrightarrow \mathcal{T}op, \qquad \tilde{P}^- : \mathcal{T}op_{\mathrm{o}}^{\mathrm{op}} \longrightarrow \mathcal{T}op, \qquad \tilde{P} : \mathcal{T}op_{\mathrm{co}}^{\mathrm{op}} \longrightarrow \mathcal{T}op;$$

they map objects as P^+, P^- and P do, respectively, and act on morphisms by taking inverse images (see 2.3(3)). Of course, here $\mathcal{T}op_{\mathrm{c}}$ ($\mathcal{T}op_{\mathrm{o}}$, $\mathcal{T}op_{\mathrm{co}}$) denotes the category of topological spaces with continuous closed (open; clopen) maps as morphisms.

4.2 Proposition. *The category $\mathcal{T}op_{\mathrm{o}}$ has all small colimits, and $\widetilde{P}^{-}$ transforms them into limits in $\mathcal{T}op$.*

Proof. Sums in $\mathcal{T}op_{\mathrm{o}}$ are formed as in $\mathcal{T}op$, and the same is true for coequalizers. In fact, for open maps $f, g : X \to Y$ consider the equivalence relation e on Y generated by $\{(f(x), g(x)) \mid x \in X\}$. Then the image of an open set $U \subseteq Y$ under the quotient map $p : Y \to Y/e$ must be open since $p^{-1}(p(U))$ is a union of sets of type

$$g(f^{-1}(g(\cdots f^{-1}(U)\cdots))),\ f(g^{-1}(f(\cdots g^{-1}(U)\cdots))).$$

Since the right adjoints of 3.2 preserve all existing limits, it now suffices to show that the functor $G^{\mathrm{o}} : \mathcal{T}op_{\mathrm{o}}^{\mathrm{op}} \to \mathrm{Rel}_{\mathrm{o}}(\mathcal{T}op)$ is continuous. In fact, for a family $(r_i : Y \not\to X_i)_{i\in I}$ of open relations, also the induced relation $r : Y \not\to \sum\limits_{i\in I} X_i$ is open, as the formula

$$r_*(U) = \bigcup_{i\in I} (r_i)_*(U)$$

for all $U \subseteq Y$ shows. Furthermore, for a coequalizer as above we must show that there is an equalizer diagram

$$Y/e \xrightarrow{G^{\mathrm{o}}(p)} Y \overset{G^{\mathrm{o}}(f)}{\underset{G^{\mathrm{o}}(g)}{\rightrightarrows}} X$$

in $\mathrm{Rel}_{\mathrm{o}}(\mathcal{T}op)$. In fact, if the open relation $r : Z \not\to Y$ satisfies $G^{\mathrm{o}}(f) \circ r = G^{\mathrm{o}}(g) \circ r$, one has $(z\ r\ f(x) \Leftrightarrow z\ r\ g(x))$ for all $z \in Z$ and $x \in X$. This condition makes the relation $s : Z \not\to Y/e$ with $(z\ s\ p(y) \Leftrightarrow z\ r\ y)$ for all $z \in Z$ and

$y \in Y$ well defined; it is, of course, the only relation s with $G^{\circ}(p) \circ s = r$, and the formula

$$s_*(V) = p(r_*(V))$$

for all $V \subseteq Z$ shows that it is open. □

4.3 An inspection of the proof of 4.2 shows that $\mathcal{T}op_{\mathrm{c}}$ and $\mathcal{T}op_{\mathrm{co}}$ have finite sums which are transformed into products by $\widetilde{P}^+$ and $\widetilde{P}$, respectively. In order to be able to trade 'open' for 'closed' when considering coequalizers, we should consider only *bounded pairs* $f, g : X \to Y$ of closed maps; this means, by definition, that for every closed set $F \subseteq Y$, there is only a finite set of sets of type

$$g(f^{-1}(g(\cdots f^{-1}(F)\cdots))),\ f(g^{-1}(f(\cdots g^{-1}(F)\cdots))).$$

Under this restriction we obtain:

Proposition. *The categories $\mathcal{T}op_{\mathrm{c}}$ and $\mathcal{T}op_{\mathrm{co}}$ have finite sums and coequalizers of bounded pairs. These colimits are transformed into limits in $\mathcal{T}op$ by the functors $\widetilde{P}^+$ and $\widetilde{P}$, respectively.*

4.4 The coequalizer of an unbounded pair in $\mathcal{T}op_{\mathrm{c}}$ or $\mathcal{T}op_{\mathrm{co}}$ may not exist or, if it does exist, may fail to look like its coequalizer in $\mathcal{T}op$ and to be transformed into an equalizer in $\mathcal{T}op$ by $\widetilde{P}^+$ and $\widetilde{P}$, respectively. Let us consider, for example, the clopen maps $f, g : \mathbb{R} \to \mathbb{R}$ with $f(x) = x$ and $g(x) = x + 1$. Their coequalizer $p : \mathbb{R} \to S^1$ in $\mathcal{T}op$ fails to be closed (since for the closed set $A = \{n + \frac{1}{n} \,|\, n \in \mathbb{Z},\ n \geq 2\}$ in $\mathbb{R}$, the set $p(A) = \{p(\frac{1}{n}) \,|\, n \geq 2\}$ is not closed in the 1-sphere). In fact, one can show that any closed map $h : \mathbb{R} \to Y$ with $h \cdot f = h \cdot g$ must be constant. Hence, the coequalizer of f, g in $\mathcal{T}op_{\mathsf{c}}$ and $\mathcal{T}op_{\mathsf{co}}$ is $\mathbb{R} \to \{*\}$, which is obviously not transformed into an equalizer diagram by $\widetilde{P}^+$ and $\widetilde{P}$.

Let us now look at the subspace $X = \{0\} \cup \{n + \frac{1}{k} \mid n, k \in \mathbb{Z},\ n \geq 0, k \geq 2\}$ of $\mathbb{R}$ and assume that the restrictions $f', g' : X \to X$ of f, g had a coequalizer $q : X \to Y$ in $\mathcal{T}op_{\mathsf{c}}$ or $\mathcal{T}op_{\mathsf{co}}$. Then with A as above, also the set $q^{-1}(q(A))$ would be closed in X, and the limit point 0 would have to lie in $q^{-1}(q(A))$, so that $q(0) = q(m + \frac{1}{m}) = q(\frac{1}{m})$ for some $m \geq 2$. Now, the (clopen) map $h : X \to \{0, 1\}$ (discrete) with $h(x) = 1$ if and only if x is of the form $x = n + \frac{1}{m}$ for some $n \geq 0$ satisfies $h \cdot f = h \cdot g$, but h cannot factor through q, since $h(0) \neq h(\frac{1}{m})$: contradiction.

5. The Vietoris Space of Closed Subsets

5.1 For a topological space X, let $VX \subseteq PX$ be the set of closed subsets of X. We denote by V^+X, V^-X, VX the corresponding subspaces of P^+X, P^-X, PX and call them the *upper Vietoris space*, the *lower Vietoris space*, the *Vietoris space* of X, respectively.

5.2 A relation $r : Y \,\,-\!\!\!\!\!\mapsto\,\, X$ between topological spaces is said to *have closed values* if $r^\sharp(y) = \{x \in X \mid y \; r \; x\}$ is a closed set in X for every $y \in Y$. The following rules are obvious:

(1) for every topological space X, the (restriction of the) element relation $\epsilon_X : VX \,\,-\!\!\!\!\!\mapsto\,\, X$ has closed values;

(2) $G^\circ(f)$ has closed values if $f : X \to Y$ is continuous and Y

is a $T1$-space;

(3) $G(g)$ has closed values if $g : Y \to X$ is a map into a $T1$-space X;

(4) if $f : X' \to X$ and $g : Y' \to Y$ are continuous and $r : Y \multimap X$ has closed values, also $G^{\circ}(f) \circ r \circ G(g) : Y' \multimap X'$ has closed values;

(5) if $h : X \to X'$ is a closed map and $r : Y \multimap X$ has closed values, so has $G(h) \circ r : Y \multimap X'$.

5.3 Proposition 3.1 says that P^+X (P^-X, PX) together with the element relation *represents the closed (open, clopen) relations with codomain* X. Obviously, the equivalence of statements ii - iv in both parts of the Proposition remains valid if we consider the subspaces V^+X (V^-X, VX) instead and ask the relations to have closed values. Hence:

Theorem. *V^+X (V^-X, VX) is the (up to homeomorphism) uniquely determined space which represents the closed-valued and closed (open, clopen, respectively) relations with codomain X.*

5.4 Unfortunately, V^+, V^- and V may not be considered functors in the same way as P^+, P^- and P were considered functors in 3.2, since we cannot form a category of spaces including $\mathbb{R}$, $V\mathbb{R}$ and $VV\mathbb{R}$ whose morphisms are closed and open relations with closed values; indeed, Example 3.4 implies:

Corollary. *The composite of clopen closed-valued relations between Hausdorff spaces may fail to have closed values.*

Proof. For a space X, consider $\epsilon_X : VX \multimap X$ and ϵ_{VX} :

$VVX \longrightarrow\!\!\!\!\!\!\!+ VX$. For $\mathcal{B} \in VVX$ one has

$$(\epsilon_X \circ \epsilon_{VX})^\sharp(\mathcal{B}) = \epsilon_X^*(\epsilon_{VX}^\sharp(\mathcal{B})) = \bigcup \mathcal{B},$$

which may fail to be closed in X, even for $X = \mathbb{R}$: consider $\mathcal{B} = c_{VX}(\mathcal{N})$ with $\mathcal{N}$ as in 3.4. □

5.5 In light of the fact that for $\mathcal{B} \subseteq VX$ closed the set $\bigcup \mathcal{B}$ may fail to be closed, the following remarks are of interest:

(1) Michael [8] showed that for X Hausdorff and $\mathcal{B} \subseteq VX$ compact, $\bigcup \mathcal{B}$ is closed in X; see also [8] and [9] for related results.

(2) It is easy to prove that for X a $T1$-space and $\mathcal{B} \subseteq VX$ open, $\bigcup \mathcal{B}$ is open in X.

(3) The map $r^* : PY \to PX$ can*not* be restricted to a map $VY \to VX$, in general. However, as already said, statements ii - iv of 2.2 remain equivalent if we trade P^+X, P^-X for V^+X, V^-X respectively; likewise, statements ii, iii of Corollary 2.3(1) remain equivalent with VX instead of PX. Furthermore, Corollaries 2.3(2) - (4) remain valid after the P-V exchange, provided that in (2) X/r and in (3) Y are assumed to be $T1$-spaces, while in (4) g must be a closed map in order to insure closed-valuedness of the relations in question.

5.6 The composition rules 5.2(4), (5) enable us to define 'V-restrictions' of the functors

$$P^+, P^-, P : Top \longrightarrow Top \text{ (see 3.3)},$$

$$\widetilde{P}^{+},\ \widetilde{P}^{-},\ \widetilde{P} : \mathcal{T}op^{\mathrm{op}}_{\mathrm{c,o,co}} \longrightarrow \mathcal{T}op \text{ (see 4.1).}$$

In fact, it follows directly from 3.1 that there are functors

$$V^{+},\ V^{-},\ V : \mathcal{T}op_{\mathrm{c}} \longrightarrow \mathcal{T}op$$

which act on morphisms by taking direct images, and functors

$$\widetilde{V}^{+},\ \widetilde{V}^{-},\ \widetilde{V} : \mathcal{T}op^{\mathrm{op}}_{\mathrm{c,o,co}} \longrightarrow \mathcal{T}op$$

which act on morphisms by taking inverse images.

Although we cannot argue as in 4.2 and 4.3, we can still prove:

5.7 Theorem.

(1) $\widetilde{V}^{-} : \mathcal{T}op^{\mathrm{op}}_{\mathrm{o}} \to \mathcal{T}op$ *transforms sums and coequalizers in* $\mathcal{T}op_{\mathrm{o}}$ *into products and equalizers in* $\mathcal{T}op$*, respectively.*

(2) *The functors* $\widetilde{V}^{+} : \mathcal{T}op^{\mathrm{op}}_{\mathrm{c}} \to \mathcal{T}op$, $\widetilde{V} : \mathcal{T}op^{\mathrm{op}}_{\mathrm{co}} \to \mathcal{T}op$ *transform finite sums and coequalizers of bounded pairs of maps in* $\mathcal{T}op_{\mathrm{c}}$, $\mathcal{T}op_{\mathrm{co}}$*, into products and equalizers in* $\mathcal{T}op$*, respectively.*

Proof. (1) We must show that, for a coequalizer diagram

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y \xrightarrow{\ p\ } Y/e$$

as in 4.2, the canonical map

$$\begin{aligned} k : V^{-}(Y/e) &\longrightarrow \{B \in VY \mid f^{-1}(B) = g^{-1}(B)\} \subseteq V^{-}Y \\ C &\longmapsto p^{-1}(C) \end{aligned}$$

is a homeomorphism. It is trivially injective; furthermore, any $B \in VY$ with $f^{-1}(B) = g^{-1}(B)$ is of the form $p^{-1}(C)$ with $C = p(B)$, since the sets

$$\begin{aligned} g(f^{-1}(g(\cdots f^{-1}(B)\cdots))) \\ &= g(f^{-1}(B)), f(g^{-1}(f(\cdots g^{-1}(B)\cdots))) \\ &= f(g^{-1}(B)) \end{aligned}$$

are contained in B. Hence, k is a continuous bijection. Since

$$k(U^+) = (p^{-1}(U))^+ \cap \{B \in VY \mid f^{-1}(B) = g^{-1}(B)\}$$

for every $U \subseteq Y/e$ open, k is also open.

(2) The same argumentation as in (1) applies. □

5.8 Corollary. *Let $e : E \hookrightarrow X \times X$ be an equivalence relation on X. Then:*

(1) *$V^-(X/e)$ is homeomorphic to the subspace of V^-X given by the e-symmetric sets B in VX (i.e, those B which satisfy ($x \in B \Leftrightarrow y \in B$) for all $x, y \in X$ with $x\ e\ y$), provided that E is an open subset of $X \times X$.*

(2) *$V^+(X/e)$ is homeomorphic to the subspace of V^+X given by the e-symmetric set B in VX, provided that E is a compact subset of $X \times X$ and X is Hausdorff.*

5.9 With $V_\circ X$ denoting the subspace $VX \backslash \{\emptyset\}$ of VX, Todorcevic [10] mentions the formula $V_\circ(X + Y) \cong V_\circ X + V_\circ Y + (V_\circ X \times V_\circ Y)$ which, without the removal of the empty set, becomes

$$V(X + Y) \cong VX \times VY.$$

In addition, the (easy) homeomorphism is canonical in the sense that $\widetilde{V} : \mathcal{T}op_{\mathrm{co}}^{\mathrm{op}} \to \mathcal{T}op$ preserves finite products (see 5.7).

6. Vietoris Objects

6.1 Let $\mathcal{X}$ be a finitely complete category with a factorization system $(\mathcal{E}, \mathcal{M})$ for morphisms, with $\mathcal{E}$ stable under pullback and $\mathcal{M} \subseteq \text{Mono}\mathcal{X}$. We take 'subobject' to mean '$\mathcal{M}$-subobject'; hence a *relation* $r : Y \nrightarrow X$ in $\mathcal{X}$ is given by morphisms $r_1 : R \to Y$, $r_2 : R \to X$ with $r =< r_1, r_2 >: R \to Y \times X$ in $\mathcal{M}$. Relational composition is defined, as usual: the composite $r \circ s$ of $s : S \to Z \times Y$ with r is obtained by $(\mathcal{E}, \mathcal{M})$-factorization of $< s_1 \cdot p_1, r_2 \cdot p_2 >: S \times_Y R \to Z \times X$. The category $\text{Rel}(\mathcal{X})$ has as its objects the objects in $\mathcal{X}$, and its morphisms are the relations in $\mathcal{X}$.

6.2 With sub$X$ denoting the preordered class of subobjects of X in $\mathcal{X}$, for every morphism $f : X \to Y$ one has the image-preimage adjunction

$$\text{sub}Y \underset{f(-)}{\overset{f^{-1}(-)}{\underset{\longleftarrow}{\overset{\longrightarrow}{\top}}}} \text{sub}X,$$

which allows us to define, for every relation $r : Y \nrightarrow X$, the functors

$$\text{sub}Y \underset{r_*}{\overset{r^*}{\underset{\longleftarrow}{\longrightarrow}}} \text{sub}X$$

as in Section 2.

6.3 A *closure operator* c on $\mathcal{X}$ w.r.t. $(\mathcal{E}, \mathcal{M})$ is given by a family of extensive, monotone functions $c_X : \text{sub}X \to \text{sub}X$, for all $X \in \mathcal{X}$, such that $f(c_X(m)) \leq c_Y(f(m))$ for all $f : X \to Y$ in $\mathcal{X}$ and $m \in \text{sub}X$ (cf. [2], [3]). We call the relation $r : Y \nrightarrow X$ *c-closed* if $r_*(c_X(m)) \geq c_Y(r_*(m))$ for all $m \in$

sub$X$, and *c-open* if $r^*(c_Y(n)) \leq c_X(r^*(n))$ for all $n \in \text{sub}Y$; in case $r =< f, 1_X >$ is the converse of the graph of a morphism $f : X \to Y$, this terminology coincides with established notions for morphisms.

As in the case $\mathcal{X} = \mathcal{T}op$ with its (epi, regular mono)-factorization system and the usual Kuratowski closure, the notions of c-closedness and c-openess are stable under relational composition.

6.4 An object E of $\mathcal{X}$, together with a relation $\varepsilon : E \mathrel{+\!\!\!\to} X$, is said to *represent the c-closed* (*c-open, c-clopen*) *relations* with codomain X if

1. ε is c-closed (c-open, c-clopen),
2. every c-closed (c-open, c-clopen) relation $r : Y \mathrel{+\!\!\!\to} X$ factors as $\varepsilon \circ < 1_Y, g >= r$, with a uniquely determined morphism $g : Y \to E$.

Adapting the terminology used for $\mathcal{T}op$ to the general case, we may say equivalently that the composite functor

$$\mathcal{X}^{\text{op}} \xrightarrow{G^{\text{op}}} (\text{Rel}_{\text{c}}(\mathcal{X}))^{\text{op}} \xrightarrow{\text{Rel}_{\text{c}}(\mathcal{X})(-,X)} \mathcal{S}et$$

is representable by E (and $\text{Rel}_{\text{c}}(\mathcal{X})$ may be traded for $\text{Rel}_{\text{o}}(\mathcal{X})$, $\text{Rel}_{\text{co}}(\mathcal{X})$, respectively).

6.5 In this paper, we do not propose a general categorical notion of c-closed-valuedness, which would have to be less natural than the notions of c-closedness and c-openess, as it would depend on a notion of 'point'. Instead, we resort to a concrete category $\mathcal{X}$ such that for every Y in $\mathcal{X}$ and every $y \in Y$, the function $\{y\} \hookrightarrow Y$ underlies a morphism in $\mathcal{M}$ (as in every concrete topological category $\mathcal{X}$, with $\mathcal{M}$ the class of regular monomorphisms). Now the relation $r : Y \nrightarrow X$ is said to *have c-closed values* if $r^*(\{y\})$ is a c-closed subobject of X for every $y \in Y$. Finally, a *c-Vietoris object* of X is an object E of $\mathcal{X}$ which, together with a relation ε, represents the c-clopen and c-closed-valued relations with codomain X, in an obvious extension of the terminology used in 6.4.

In addition to the classical topological construction, we give another rather natural example of Vietoris object.

6.6 Let $\mathcal{X}$ be the category of preordered sets and monotone maps with its (epi, regular mono)-factorization system, and let c be the up-closure, i.e. $c_X(M) = \{x \in X \mid \exists a \in M : a \leq x\}$ for $M \subseteq X$. It is fairly easy to show that a relation $r : Y \nrightarrow X$ is c-closed if and only if

(+) $\forall x \in X \quad \forall y, y' \in Y \ (y' \geq y \ \& \ y \ r \ x \ \Rightarrow \ \exists x' \in X : y' \ r \ x' \ \& \ x' \geq x)$,

and r is c-open if and only if

(-) $\forall x \in X \quad \forall y, y' \in Y \ (y' \leq y \ \& \ y \ r \ x \ \Rightarrow \ \exists x' \in X : y' \ r \ x' \ \& \ x' \leq x)$.

In other words, $r : Y \nrightarrow X$ is c-open if and only if $r : Y^{\mathrm{op}} \nrightarrow X^{\mathrm{op}}$ is c-closed; here X^{op} is obtained from X by inverting the preorder of X.

Condition (+) tells us how to define a preorder on PX so that ϵ_X becomes c-closed: for $B_1, B_2 \subseteq X$, put

$$B_1 \leq^+ B_2 \ :\Leftrightarrow \ \forall x_1 \in B_1 \ \exists x_2 \in B_2 : x_1 \leq x_2.$$

Similarly, from (-):

$$B_1 \leq^- B_2 \;:\Leftrightarrow\; \forall x_2 \in B_2 \;\exists x_1 \in B_1 \;:\; x_1 \leq x_2;$$

hence $\leq^-$ is simply the 'finer' relation on PX. We denote by P^+X, P^-X, PX the powerset of X with the preorder $\leq^+$, $\leq^-$, $\leq^+ \cap \leq^-$, respectively, and obtain immediately:

$r : Y \relbar\joinrel\mapstochar\joinrel\rightarrow X$ c-closed $\Leftrightarrow r^\sharp : Y \to P^+X$ monotone,

$r : Y \relbar\joinrel\mapstochar\joinrel\rightarrow X$ c-open $\Leftrightarrow r^\sharp : Y \to P^-X$ monotone,

$r : Y \relbar\joinrel\mapstochar\joinrel\rightarrow X$ c-clopen $\Leftrightarrow r^\sharp : Y \to PX$ monotone.

The relation r has c-closed values if and only if

$$\forall x, x' \in X \;\; \forall y \in Y \; (y\, r\, x \;\&\; x \leq x' \;\Rightarrow\; y\, r\, x').$$

Now with VX denoting the set of c-closed (that is: up-closed) subsets of X, provided with the preorder inherited from PX, we obtain a c-Vietoris object of X in $\mathcal{X}$.

Note that $\mathcal{X}$, like $\mathcal{T}op$, is a topological category over $\mathcal{S}et$.

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