Topology Proceedings

Web:	http://topology.auburn.edu/tp/
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E-mail:	topolog@auburn.edu
ISSN:	0146-4124

Topology Proceedings
Volume 22, Summer 1997, 111–123

ON SOME SEMIGROUP COMPACTIFICATIONS

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Abstract

The LUC−compactification UG of a locally compact group is a semigroup with an operation which extends that of G and which is continuous (only) in one variable. When G is discrete, UG and the Stone-Čech compactification βG are identical. Some algebraic properties, such as the number of left ideals and cancellation, are known to hold in the semigroup $\beta\mathbb{N}$ where $\mathbb{N}$ is the additive semigroup of the integers. We show that these properties are also true in UG for a large class of locally compact groups. The method used is to transfer the information from $\beta\mathbb{N}$ to βG where G is an infinite discrete group (or a cancellative commutative semigroup), and then to UG where G is not necessarily discrete.

1. Introduction and Preliminaries

The Stone-Čech compactification of a discrete semigroup is a semigroup compactification, which has attracted a special attention in the last twenty years. More recently, it is the

* The author wishes to thank Professor J. W. Baker and Professor J. S. Pym for the unlimited help during the preparation of this paper: sending preprints, encouragements, helpful comments and corrections. Thanks are also due to the referee for the very careful reading the paper was given.

Mathematics Subject Classification: 1991: 22A15, 54D35, 43A60.

Key words: locally compact group, compactification, uniformly continuous function, sparse set, right cancellation, left ideal, centre.

$LUC-$compactification of a locally compact group which has become the aim of many mathematicians. One way to produce these compactifications is as follows. Let G be a locally compact, Hausdorff group written additively, and let $LUC(G)$ be the space of bounded, complex-valued functions which are uniformly continuous with the respect to the right uniformity on G. This means that for every $\epsilon > 0$, there is a neighbourhood U of the identity in G such that $|f(s) - f(t)| < \epsilon$ whenever $t - s \in U$ (here $-s$ is the inverse of s in the group G). One can also characterize $LUC(G)$ in the following way. Let $C(G)$ be the space of bounded, continuous, complex-valued functions on G, and for each function f on G let f_s be the left translate of f by s, i.e., $f_s(t) = f(s+t)$. Then $LUC(G)$ is the space of functions f in $C(G)$ which are left norm continuous, i.e., $s \mapsto f_s : G \to C(G)$ is continuous when $C(G)$ has the supremum norm. The $LUC-$compactification of G, which we will denote by UG, may be regarded as the spectrum of $LUC(G)$, i.e.,

$$UG = \{x \in LUC(G)^* : x \neq 0 \text{ and } x(fg) = x(f)x(g) \\ \text{for all} \quad f,\ g \in LUC(G)\},$$

along with the mapping ϕ from G to UG defined by

$$\phi(s)(f) = f(s) \quad \text{for all} \quad s \in G \quad \text{and} \quad f \in LUC(G).$$

The binary operation defined in UG by

$$x + y(f) = x(f_y) \quad \text{for all} \quad x,\ y \in UG \quad \text{and} \quad f \in LUC(G),$$

where

$$f_y(s) = y(f_s) \quad \text{for all} \quad f \in LUC(G),\ y \in UG \quad \text{and} \quad s \in G,$$

turns UG into a semigroup. When equipped with the relative weak*-topology inherited from the Banach conjugate $LUC(G)^*$,

UG becomes a compact, Hausdorff, right topological semigroup (i.e., the mapping $x \rightarrow x + y \ : \ UG \mapsto UG$ is continuous for each $y \in UG$), ϕ a continuous homomorphism, $\phi(G)$ dense in UG, and $x \mapsto \phi(s) + x \ : \ UG \mapsto UG$ is continuous for each $y \in UG$ and $s \in G$. Accordingly, UG is a *semigroup compactification* in the sense of [1, Definition 3.1.1]. The mapping ϕ is a homeomorphism from G into UG, and so we may identify G with $\phi(G)$. For more information, the reader is directed to [1]. The closure in UG of a subset A of UG will be denoted by $\overline{A}$. If A is a subset of G, then A^* will denote $\overline{A} \setminus A$. In particular, $G^* = UG \setminus G$. Finally, we may also recall that when G is not compact, then G^* is a closed two sided ideal of UG (see [3, Lemma 2.1]).

Note that when G is discrete, $LUC(G)$ is the space of all bounded complex-valued functions on G, and so UG is the Stone-Čech compactification βG of G. In this situation a number of results are known in βG. In this paper, we show that for a large class of non-compact locally compact groups (which includes all abelian ones), some of these results are also true in UG. This is achieved by a method of transferring information from βG where G is an infinite discrete group to UG where G is not necessarily discrete. This method was used earlier in [2] (see also [11]) to study some algebraic properties of UG when G is a locally compact abelian group, and recently by Koçak and Strauss ([9] and [10]) to study $U\mathbb{R}$. It is worthwhile to note that Koçak and Strauss were able to study not only algebraic properties but also topological ones such as the non-homogeneity of $U\mathbb{R}$ and the Rudin-Keisler order in $U\mathbb{R}$. In particular, we show in Theorem 1 that the set of points that are right cancellative in UG has a dense interior in G^*. This result has been proved for example in [7, Corollary 4.4] (see also [8]) for βS when S is a countable, cancellative, discrete semigroup, and in [3] and [5] whithout the assumption that S is countable. In Theorem 2, we see how free subsemigroups may be generated in G^*. This was done in [13] for $\beta(\mathbb{N}, +)$, and in [4] for βS where S is either a discrete group or a commutative, cancellative, discrete semigroup. Theorem 3 shows, under some conditions on G, the very non-commutativity of the semigroup UG. In fact, for any $x \in G^*$, the set $\{y \in G^* : (G^* + y) \cap (G^* + x) \neq \emptyset\}$ is shown to be nowhere dense is G^*. This was proved in [13] for $\beta(\mathbb{N}, \circ)$, where $\circ$ is is a binary operation on $\mathbb{N}$ such that $m \circ n \longrightarrow \infty$ as $n \longrightarrow \infty$. We end with Theorem 4, where we consider the additive semigroup of the positive reals $[0, \infty)$ with its usual topology, and show that the minimal ideal of $U[0, \infty)$ has right cancellative points in its closure. This result was proved in [7, Theorem 4.6] for $\beta(\mathbb{N}, +)$.

2. On the Semigroup UG

Let H be a locally compact group with identity e and with a compact, open, normal subgroup K, and let $G = \mathbb{R}^n \times H$. Let H/K be the quotient group, the elements of H/K are the right cosets $K + s$. Note that H/K is discrete since K is open. Let $q : H \longrightarrow H/K$ be the quotient mapping. Let $\psi : \mathbb{Z}^n \times H \longrightarrow \mathbb{Z}^n \times H/K$ be the mapping defined by $\psi(m, h) = (m, q(h))$. In fact, $\{0\} \times K$ is a compact, open, normal subgroup of $\mathbb{Z}^n \times H$ and $(\mathbb{Z}^n \times H)/(\{0\} \times K) = \mathbb{Z}^n \times (H/K)$, so ψ is just the quotient mapping. Then, by [1, Theorem 4.4.4], ψ extends to a continuous homomorphism (denoted also by the same letter) $\psi : U(\mathbb{Z}^n \times H) \longrightarrow \beta(\mathbb{Z}^n \times H/K)$. The following diagram indicates how the lift up shall be done.

$$
\begin{array}{ccc}
UG = U(\mathbb{R}^n \times H) & & \\
\big\uparrow p & & \\
U(\mathbb{Z}^n \times H) \times ([0,1)^n \times \{e\}) & \xrightarrow{\ pr\ } & U(\mathbb{Z}^n \times H) \\
& & \big\downarrow \psi \\
& & \beta(\mathbb{Z}^n \times H/K)
\end{array}
$$

where $p(x, u) = x + u$ for $x \in U(\mathbb{Z}^n \times H)$ and $u \in [0,1)^n \times \{e\}$, and pr is simply the projection mapping. Lemma 2 enables us to pass from $\beta(\mathbb{Z}^n \times H/K)$ to $U(\mathbb{Z}^n \times H)$, and Lemma 1 from $U(\mathbb{Z}^n \times H)$ to UG. We let

$$\tau = \psi \circ pr \circ p^{-1} \; : \; UG = U(\mathbb{R}^n \times H) \to \beta(\mathbb{Z}^n \times (H/K)).$$

It will be deduced from Lemmas 1 and 2 that τ is open, and is continuous on $UG \setminus U(\mathbb{Z}^n \times H)$.

Lemma 1. *Let $G = \mathbb{R}^n \times H$, where H is a locally compact group with identity e. Then $\overline{\mathbb{Z}^n \times H} = U(\mathbb{Z}^n \times H)$, and every $x \in UG$ can be written uniquely as $x = \bar{x} + (s, e)$, where $\bar{x} \in \overline{\mathbb{Z}^n \times H}$ and $s \in [0,1)^n$.*

Proof. For the first part of this lemma, and for the representation of each x in UG as $x = \bar{x} + (s, e)$, where $\bar{x} \in \overline{\mathbb{Z}^n \times H}$ and $s \in [0,1)^n$, see [2].

We show that this representation of x is unique. The case of $n = 0$ is trivial, so we start with $n = 1$. Suppose that $x = \bar{y} + (t, e) = \bar{x} + (s, e)$, where $\bar{x}, \bar{y} \in \overline{\mathbb{Z} \times H}$ and $s, t \in [0, 1)$. With no loss of generality, we may assume that $s \geq t$. Then $\bar{y} = \bar{x} + (s - t, e)$ with $s - t \in [0, 1)$. If $\bar{y} = \bar{x} + (m, e)$ for some $m \in \mathbb{Z}$, then $\bar{x} + (m + t - s, e) = \bar{x}$, an identity which holds if and only if $m + s - t = 0$. For, otherwise, let χ be a continuous character on $\mathbb{R}$ such that $\chi(m + t - s) \neq 1$ (for instance, one may take $\chi(u) = \exp(i\frac{u}{m+s-t}))$ and extend, by [1, Theorem 3.1.7], χ continuously to $U\mathbb{R}$. Let $p : U(\mathbb{R} \times H) \to U\mathbb{R}$ be the extension of the projection mapping. Then

$$\chi(p(\bar{x}) + m + t - s) = \chi(p(\bar{x}))\chi(m + t - s) \neq \chi(p(\bar{x})),$$

and so $p(\bar{x}) + m + t - s \neq p(\bar{x})$, which implies that $\bar{x} + (m + t - s, e) \neq \bar{x}$. Therefore $m + t - s = 0$, which is clearly possible if and only if $m = 0$ and $s = t$. It is then easy to deduce that $\bar{x} = \bar{y}$.

Suppose now that $\bar{y} \neq \bar{x}+(m,e)$ for all $m \in \mathbb{Z}$. In particular, $\bar{y} \neq \bar{x}$ and $\bar{y} \neq \bar{x}+(1,e)$. We pick a function $f \in LUC(\mathbb{Z} \times H)$ such that

$$f(\bar{x}) = f(\bar{x}+(1,e)) = 0 \quad \text{and} \quad f(\bar{y}) = 1.$$

We extend f to a function g which is defined on $\mathbb{R} \times H$ in the following way. We write each $u \in \mathbb{R} \times H$ as $u = \bar{u}+(r,e)$ where $\bar{u} \in \mathbb{Z} \times H$ and $r \in [0,1)$, and let

$$g(u) = g(\bar{u}+(r,e)) = \Big(f(\bar{u}+(1,e)) - f(\bar{u})\Big)r + f(\bar{u}).$$

Then it is not difficult to verify that the function g is in $LUC(\mathbb{R} \times H)$, and so we may extend it, by [1, Theorem 3.1.7], continuously to $U(\mathbb{R} \times H)$. We obtain

$$g(\bar{x}+(s-t,e)) = \Big(f(\bar{x}+(1,e)) - f(\bar{x})\Big)(s-t) + f(\bar{x}) = 0,$$
$$\text{whereas} \quad g(\bar{y}) = f(\bar{y}) = 1.$$

Thus $\bar{x}+(s-t,e) \neq \bar{y}$, and so $\bar{x}+(s,e) \neq \bar{y}+(t,e)$, as required.

We deal now with the general case, and let $x \in U(\mathbb{R}^n \times H)$. Let $H_1 = \mathbb{R}^{n-1} \times H$, and $e_1 = (0,0,...,e)$ be the identity in H_1. Then, as in the case $n=1$, x decomposes uniquely into $\bar{x}_1 \in U(\mathbb{Z} \times H_1)$ and (s_1, e_1) with $s_1 \in [0,1)$. In turn, $\bar{x}_1$ decomposes uniquely into $\bar{x}_2 \in U(\mathbb{Z} \times \mathbb{Z} \times \mathbb{R}^{n-2} \times H)$ and $(0, s_2, e_2)$ where $s_2 \in [0,1)$ and e_2 is the identity in $\mathbb{R}^{n-2} \times H$. Inductively, this leads to the desired result. □

Remarks. (1) In the proof above we have chosen a continuous character χ of $\mathbb{R}$ such that $\chi(m+s-t) \neq 1$ to prove that $p(\bar{x}) + (m+s-t) \neq p(\bar{x})$. In fact, this can be used to show that any locally compact abelian group has the property that $x+s \neq x$ for all $x \in UG$ and $s \in G \setminus \{e\}$ (see [2, Proposition 5.3]). With a more complicated proof, this result is known to hold for any locally compact group, see for example [1, Lemma 4.8.9].

(2) As already noted in [9] for $U\mathbb{R}$, the decomposition produced in Lemma 1 defines a homeomorphism from $U(\mathbb{Z}^n \times H) \times ((0,1)^n \times \{e\})$ to $UG \setminus U(\mathbb{Z}^n \times H)$. For p is continuous, and bijective by Lemma 1. Furthermore, if a and b are chosen in $(0,1)$ with $a < b$ then the set $U(\mathbb{Z}^n \times H) \times ([a,b]^n \times \{e\})$ is compact and so the restriction of p to this set is a homeomorphism. Thus the restriction of p to $U(\mathbb{Z}^n \times H) \times ((0,1)^n \times \{e\})$ is open, i.e., $U + (a,b)^n \times \{e\}$ is open in UG whenever U is an open subset of $U(\mathbb{Z}^n \times H)$.

Lemma 2. *Let H be a locally compact group with a normal, compact, open subgroup K. Let $q : UH \to \beta(H/K)$ be the extension of the quotient mapping. Then*

(1) $q(x) = q(y)$ for x and y in UH if and only if $x = k + y$ for some $k \in K$;

(2) q is an open, continuous homomorphism of UH onto $\beta(H/K)$.

Proof. For Statement (1), see [2]. For the proof of the second statement, note first that $q : UH \to \beta(H/K)$ is a closed mapping. In fact, a closed subset C of UH is also compact. Since q is continuous, $q(C)$ is compact, and so it is closed in $\beta(H/K)$. Let now O be an open subset of UH. Then Statement (1) implies that

$$q(UH \setminus (K + O)) = \beta(H/K) \setminus q(O).$$

Since $K + O$ is also open in UH, this yields the end of the proof. □

Recall that an element x is *right cancellative* in UG if the identity $y + x = z + x$ holds if and only if $y = z$.

Theorem 1. *Let $G = \mathbb{R}^n \times H$, where H is a locally compact group which contains a compact, open, normal subgroup K. Suppose that G is not compact. Then the set of points in G^* which are right cancellative in UG has a dense interior in G^*.*

Proof. Recall that $\tau = \psi \circ pr \circ p^{-1}$. Let O be an open subset of G^*. Then Lemma 1 and the continuity of p imply that $p^{-1}(O)$ is a non-empty open subset of $(\mathbb{Z}^n \times H)^* \times ([0,1)^n \times \{e\})$, so we may take an open subset U of $(\mathbb{Z}^n \times H)^*$ and an open subset I of $[0,1)^n$ such that $U \times (I \times \{e\}) \subseteq p^{-1}(O)$. If we denote $(I \times \{e\})$ by I_e, this means that $U + I_e \subseteq O$. Now, Lemma 2 implies that $\psi(U)$ is a non-empty open subset in $(\mathbb{Z}^n \times H/K)^*$. By [5, pages 135-136], we can pick a countably infinite subset V of $\mathbb{Z}^n \times H/K$ which has the property that $(s+V) \cap V$ is finite whenever s is different than the identity and such that $V^* \subseteq \psi(U) \subseteq \tau(O)$ (these were called then sparse subsets). By [5], every point of V^* is right cancellative in $\beta(\mathbb{Z}^n \times H/K)$. Let $x \in \overline{\mathbb{Z}^n \times H}$ such that $\psi(x) \in V^*$, and let y and z be different elements in UG. Write y as $(s,e) + \bar{y}$ and z as $(t,e) + \bar{z}$. Suppose first that $\bar{z} = (0,k) + \bar{y}$ for some $k \in K$. Then $z = (t,e) + \bar{z} = (t,k) + \bar{y}$, and so (t,k) and (s,e) must be two different elements of G. Therefore

$$y + x = (s,e) + (\bar{y} + x) \neq (t,k) + (\bar{y} + x) = z + x$$

by [1, Lemma 4.8.9]. If $\bar{z} \neq (0,k) + \bar{y}$ for all $k \in K$, then by Lemma 2, $\psi(\bar{y}) \neq \psi(\bar{z})$, and so

$$\psi(\bar{y} + x) = \psi(\bar{y}) + \psi(x) \neq \psi(\bar{z}) + \psi(x) = \psi(\bar{z} + x).$$

Hence $\bar{y} + x \neq \bar{z} + x$, and Lemma 1 implies that

$$y + x = (s,e) + (\bar{y} + x) \neq (t,e) + (\bar{z} + x) = z + x.$$

This shows that each point of $\psi^{-1}(V^*)$ is right cancellative in UG. It is then easy to deduce that each point of $\psi^{-1}(V^*) + I_e$ is right cancellative in UG. Since $\psi^{-1}(V^*) + I_e$ is an open subset of UG and $\psi^{-1}(V^*) + I_e \subseteq U + I_e \subseteq O$, the proof is complete. □

Theorem 2. *Let G be as in Theorem 1. Let V be a sparse subset of $\mathbb{Z}^n \times H/K$. Let F be the subset of G^* formed by taking, for each $a \in V^*$, one element from $\psi^{-1}(a)$. Then the elements of F generate a free subsemigroup of G^*.*

Proof. Let x_1 and x_2 be two distinct elements of F. Then $\tau(x_1) = \psi(x_1) \neq \psi(x_2) = \tau(x_2)$. We claim that

$$((UG) + x_1) \cap ((UG) + x_2) = \emptyset.$$

Since $\psi(x_1)$ and $\psi(x_2)$ are two distinct elements in V^*, we have by [4],

$$(\beta(\mathbb{Z}^n \times H/K) + \psi(x_1)) \cap (\beta(\mathbb{Z}^n \times H/K) + \psi(x_2)) = \emptyset.$$

Let y and z be arbitrary elements in UG, and write by Lemma 1,

$$y = \bar{y} + (s, e) \quad \text{and} \quad z = \bar{z} + (t, e),$$

where s, $t \in [0, 1)^n$ and $\bar{y}$, $\bar{z} \in U(\mathbb{Z}^n \times H)$. Since ψ is a homomorphism, we have

$$\psi(\bar{y} + x_1) = \psi(\bar{y}) + \psi(x_1) \neq \psi(\bar{z}) + \psi(x_2) = \psi(\bar{z} + x_2),$$

and so $\bar{y} + x_1 \neq \bar{z} + x_2$. Lemma 1 implies then that

$$y + x_1 = (s, e) + (\bar{y} + x_1) \neq (t, e) + (\bar{z} + x_2) = z + x_2,$$

as required.

Let now x_1, x_2, ..., x_n and y_1, y_2, ..., y_m be in F, and suppose that

$$x_1 + x_2 + ... + x_n = y_1 + y_2 + ... + y_m.$$

Then it follows from what we have just proved that $x_n = y_m$. Theorem 1 implies then that $x_1 + x_2 + ... + x_{n-1} = y_1 + y_2 + ... + y_{m-1}$, which in turn implies that $x_{n-1} = y_{m-1}$. The proof is completed by induction on n. □

Remark. The first part of the proof given above shows also that there are at least 2^c distinct left ideals in UG, where c is the cardinality of the continuum. However, Lau, Milnes and Pym have obtained this result recently in [12] with a better cardinality and for any locally compact group.

Theorem 3. *Let G be as in Theorem 1, and suppose in addition that H/K is countable. Then, for each $x \in G^*$, the set*

$$C_x = \{y \in G^* \ : \ (G^* + y) \cap (G^* + x) \neq \emptyset\}$$

is nowhere dense in G^*.

Proof. Recall that $\tau = \psi \circ pr \circ p^{-1}$. Let $x \in G^*$, and write $x = \bar{x} + (s_1, e)$. We claim that

$$\tau(C_x) = C_{\tau(x)} = C_{\psi(\bar{x})}.$$

Let

$$C_{\bar{x}} = \{y \in (\mathbb{Z}^n \times H)^* \ : \ ((\mathbb{Z}^n \times H)^* + y) \cap ((\mathbb{Z}^n \times H)^* + \bar{x}) \neq \emptyset\}.$$

We prove first that $pr \circ p^{-1}(C_x) = C_{pr \circ p^{-1}(x)} = C_{\bar{x}}$. Let $y \in C_x$, then $a + y = b + x$ for some a and b in $(\mathbb{R}^n \times H)^*$. Write $y = \bar{y} + (s_2, e)$, $a = \bar{a} + (s_3, e)$ and $b = \bar{b} + (s_4, e)$. Then

$$(\bar{a} + \bar{y}) + (s_2 + s_4, e) = (\bar{b} + \bar{x}) + (s_1 + s_3, e).$$

Since $s_2 + s_4 = l + s$ and $s_1 + s_3 = m + t$ for some l, $m \in \{0,1\}^n$ and s, $t \in [0,1)^n$, this leads to the identity

$$(\bar{a} + l + \bar{y}) + (s, e) = (\bar{b} + m + \bar{x}) + (t, e),$$

which, by Lemma 1, holds if and only if $s = t$ and $(\bar{a} + l) + \bar{y} = (\bar{b} + m) + \bar{x}$. Therefore $\bar{y} = pr \circ p^{-1}(y) \in C_{\bar{x}}$. Conversely, let $y \in C_{\bar{x}}$. Then $a + y = b + \bar{x}$ for some a and b in $(\mathbb{Z}^n \times H)^*$, and so

$$a + (y + (s_1, e)) = b + (\bar{x} + (s_1, e)) = b + x.$$

This means that $y + (s_1, e) \in C_x$, and so $y \in pr \circ p^{-1}(C_x)$.

Secondly, we show that $\psi(C_x) = C_{\psi(x)}$ for any $x \in U(\mathbb{Z}^n \times H)$. Let $y \in C_x$, and let $a, b \in (\mathbb{Z}^n \times H)^*$ such that $a+y = b+x$. Then, since ψ is a homomorphism,

$$\psi(a) + \psi(y) = \psi(b) + \psi(x).$$

Now Lemma 2 implies that $\psi(a)$ and $\psi(b)$ are in $(\mathbb{Z}^n \times H/K)^*$, and so $\psi(y) \in C_{\psi(x)}$. For the converse, let $z \in C_{\psi(x)}$ and let $y \in (\mathbb{Z}^n \times H)^*$ such that $z = \psi(y)$. Then for some a and b in $(\mathbb{Z}^n \times H)^*$, we have $\psi(a) + \psi(y) = \psi(b) + \psi(x)$, and so $\psi(a + y) = \psi(b + x)$. Lemma 2 implies then that $b + x = (0, k) + a + y$ for some $k \in K$, which means clearly that $y \in C_x$ and so $z \in \psi(C_x)$, as required. Combining the two identities proved above, we obtain $\tau(C_x) = C_{\tau(x)}$.

Suppose now that $\overline{C_x}$ has a non-empty interior, and let O be an open subset of UG contained in $\overline{C_x}$. Then, as in the proof of Theorem 1, we pick an open subset U of $(\mathbb{Z}^n \times H)^*$ and an open subinterval I of $(0,1)^n$ such that $U + I_e$ is open subset of G^* contained in $UG \setminus U(\mathbb{Z}^n \times H)$ and $U + I_e \subseteq O \subseteq \overline{C_x}$. Now, since $p : U(\mathbb{Z}^n \times H) \times ((0,1)^n \times \{e\}) \to UG \setminus U(\mathbb{Z}^n \times H)$ is a homeomorphism, τ is continuous on $UG \setminus U(\mathbb{Z}^n \times H)$, and so we have

$$\tau(U + I_e) = \tau\left((U + I_e) \cap \overline{C_x}\right) \subseteq \overline{\tau(C_x)} = \overline{C_{\tau(x)}}.$$

But from [13, Theorem 6] we deduce that $C_{\tau(x)}$ is nowhere dense in $(\mathbb{Z}^n \times H/K)^*$. Since $\tau(U + I_e)$ is an open subset of $(\mathbb{Z}^n \times H/K)^*$, this yields a contradiction. □

Corollary 1. *Let G be a locally compact abelian group. Then all statements of Theorems 1, 2 and 3 hold.*

Proof. This is due to the structure theorem which says that $G = \mathbb{R}^n \times H$, where H contains a compact open subgroup, see for example [6, Theorem 24.30]. □

Theorem 4. *If M is the minimal ideal of $U[0,\infty)$, then M does not contain any right cancellative point of $U[0,\infty)$, but $\overline{M}$ does.*

Proof. It is not difficult to verify that the points which are not right cancellative in $U[0,\infty)$ (in fact, in any semigroup) form a right ideal, and so it contains M. In other words, none of the points in M is right cancellative in $U[0,\infty)$. By Lemma 1, we write $M = N + I$ where $N \subseteq \beta\mathbb{N}$ and $I \subseteq [0,1)$. Since M is an ideal in $U[0,\infty)$, it follows that $I = [0,1)$ and N is an ideal in $\beta\mathbb{N}$. Moreover, N is the minimal ideal. To see that N is as claimed, suppose that N' is an ideal of $\beta\mathbb{N}$ contained in N. Then $N' + [0,1)$ is an ideal of $U[0,\infty)$ that is contained in M. Thus $N' + [0,1) = N + [0,1) = M$, and so by Lemma 1, $N' = N$. Thus N is the minimal ideal of $\beta\mathbb{N}$. By [7, Theorem 4.6], there exists $x \in \overline{N}$ such that right cancellation holds at x in $\beta\mathbb{N}$. Now, much as in the proof of Theorem 1, the point x (regarded as a point in $U[0,\infty)$) is seen to be right cancellative in $U[0,\infty)$. The proof is complete. □

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