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A STONE TYPE DUALITY AND ITS APPLICATIONS TO PROBABILITY

Roman Frič*

Abstract

We show that the following are dual: the category whose objects are Boolean algebras carrying the initial sequential convergence with respect to the sequentially continuous homomorphisms into the two-element Boolean algebra and whose morphisms are sequentially continuous homomorphisms and the category whose objects are reduced s-perfect fields of sets (ultrafilters having the countable intersection property are fixed) and whose morphisms are measurable maps. The motivation comes from the foundations of probability: s-perfect fields of sets have good categorical properties and yield a suitable model for the field of events. The duality covers the non-topological Stone duality between Boolean algebras and reduced perfect fields of sets as a special case. Indeed, the category of Boolean algebras is isomorphic to the category of Boolean algebras carrying the initial sequential convergence with respect to all homomorphisms into the two-element Boolean algebra.

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1.

Usually, events in probability form a Boolean algebra. Each Boolean algebra is isomorphic to a reduced field of sets and each field of sets can be considered as a Boolean algebra. Further, each measurable map induces a Boolean homomorphism (going the opposite direction) and each Boolean homomorphism of the Borel subsets of the real line R into a field of sets is induced by a measurable function; in probability such function is called a random variable. These facts lead to a duality for fields of events. Homomorphisms preserve the structure of events, but are less useful when calculations are needed. The random variables preserve the structure of events only indirectly (via preimage), but provide a freeway to calculus.

The usual nontopological Stone duality is not exactly what is needed in probability. On the one hand, the perfectness of the domain guarantees that each Boolean homomorphism is induced by a measurable map but, on the other hand, the perfectness amounts to the compactness and hence each additive probability measure is countably additive. Since in the probability theory some natural fields of sets are not perfect and some additive measures are not countably additive, it is natural to seek a duality where the perfectness is replaced by some weaker property. In the present paper we show that s-perfectness of fields of sets is exactly what is needed.

In Section 2 we present arguments supporting our claim that s-perfect fields of sets yield a natural model for the fields of events. Section 3 deals with the nontopological Stone duality. Now, we give the basic definitions and recall some sequential convergence notions.

Not to destroy completeness and cocompleteness of the categories to be dealt with, we do not exclude from our considerations the Boolean algebra for which $0 = 1$ and likewise the field of subsets for which the carrier set is empty. Denote by MM the category whose objects are reduced fields of sets (each two points can be separated by a measurable set; in what follows, all fields of sets will be reduced) and whose morphisms are measurable maps (MM stands for measurable sets and measurable maps, while the category of fields of subsets and sequentially continuous Boolean homomorphisms - as arrows going the opposite direction - will be denoted by FS). The nontopological version of the Stone duality asserts that the subcategory PMM of perfect fields and the category BA of all Boolean algebras and Boolean homomorphisms are dual [SIK],[JO].

Let X be a set. By a sequential convergence on X we understand a subset $\mathbb{L}$ of $X^N \times X$; here $(\langle x_n \rangle, x) \in \mathbb{L}$ means that the sequence $\langle x_n \rangle$ converges under $\mathbb{L}$ to the point $x \in X$. We always assume that: each constant sequence $\langle x \rangle$ converges to x, each subsequence of a convergent sequence converges, and the limits are unique. Sequential continuity of a map is defined in the obvious way.

If X carries an algebraic structure, then a convergence is compatible provided that the algebraic operations are sequentially continuous. More information about the sequential convergence in Boolean algebras can be found e.g. in [JAK]. Categorical approach to measure and integration theory via sequential convergence structures and Boolean algebras has been developed in [BOE], see also [BOER]. We will deal with a special sequential convergence on Boolean algebras and our results do not overlap with those by Jakubík and Boerger. Observe that categorical approach to sequential convergence structures appeared for the first time in [DOL].

Denote by $\mathbf{2}$ the two-element Boolean algebra carrying the convergence under which $\langle x_n \rangle$ converges to x iff $x_n = x$ for all but finitely many $n \in N$. Let $\mathcal{A}$ be a Boolean algebra. Denote by $\mathrm{Hom}(\mathcal{A}, \mathbf{2})$ the set of all Boolean homomorphisms of $\mathcal{A}$ into $\mathbf{2}$. By a Stone family of $\mathcal{A}$ we understand a subset H of $\mathrm{Hom}(\mathcal{A}, \mathbf{2})$ such that if $a, b \in \mathcal{A}$ and $a \neq b$, then there exists $h \in H$ such that $h(a) \neq h(b)$.

Let $\mathbb{A}$ be a field of subsets of X. Each $A \in \mathbb{A}$ is represented by its characteristic function $\chi_A : X \longrightarrow \{0,1\}$, $\chi_A(x) = 1$ if $x \in A$ and $\chi_A(x) = 0$ otherwise. The pointwise sequential convergence of characteristic functions is compatible with the Boolean and field structure of $\mathbb{A}$. Each point $x \in X$ represents a Boolean homomorphism $x : \mathbb{A} \longrightarrow \mathbf{2}$, $x(A) = 1$ if $x \in A$ and $x(A) = 0$ otherwise. Clearly, the homomorphism x is sequentially continuous and X is a Stone family of $\mathbb{A}$.

Definition 1.1. *Let $\mathcal{A}$ be a Boolean algebra carrying a sequential convergence $\mathbb{L}$ such that $\langle x_n \rangle$ converges under $\mathbb{L}$ to x iff for each sequentially continuous homomorphism h of $\mathcal{A}$ into $\mathbf{2}$ the sequence $\langle h(x_n) \rangle$ converges in $\mathbf{2}$ to $h(x)$. Then $\mathcal{A}$ carrying $\mathbb{L}$ is said to be $\mathbf{2}$-generated.*

Denote by $\mathcal{B}(\mathbf{2})$ the category whose objects are $\mathbf{2}$-generated Boolean algebras and whose morphisms are sequentially continuous Boolean homomorphisms. Some of the properties of $\mathcal{B}(\mathbf{2})$ are described in [FR] and in [FRIC].

Definition 1.2. *Let $\mathbb{A}$ be a field of subsets of X. If each ultrafilter $\mathcal{F}$ of elements of $\mathbb{A}$ having the countable intersection property (CIP) is fixed (i.e. there exists $x \in X$ such that $\mathcal{F} = \{\mathcal{A} \in \mathbb{A};\ x \in A\}$), then $\mathbb{A}$ is said to be s-perfect.*

Denote by SPMM the full subcategory of MM consisting of all s-perfect fields (s-perfect fields have been considered in [FRI] and presented at the BBFEST in 1996 in Cape Town). Some of the properties of SPMM are described in [FR].

2.

Let $\mathcal{A}$ be a Boolean algebra nad let H be a Stone family of $\mathcal{A}$. For $a \in \mathcal{A}$, denote by $a_H = \{h \in H; h(a) = 1\}$ and put $A_H = \{a_H; a \in \mathcal{A}\}$. Then $\mathcal{A}_H$ is a reduced field of subsets of H and $\mathcal{A}$ and $\mathcal{A}_H$ are isomorphic. If $H = \mathrm{Hom}(\mathcal{A}, \mathbf{2})$, then $\mathcal{A}_H$

is perfect. We shall use the following convention. Let m be an additive measure on $\mathcal{A}$. Then m_H will denote the additive measure on $\mathcal{A}_H$ defined by $m_H(a_H) = m(a)$. Analogously, let h be a homomorphism of $\mathcal{A}$ into $\mathbf{2}$. Then h_H denotes the momomorphism of $\mathcal{A}_H$ into $\mathbf{2}$ defined by $h_H(a_H) = h(a)$. It is known that there is a one-to-one correspondence between the ultrafilters on $\mathcal{A}$, $\{0,1\}$-valued additive measures on $\mathcal{A}$, and homomorphisms of $\mathcal{A}$ into $\mathbf{2}$. The correspondence extends in the natural way to $\mathcal{A}_H$.

Observe that if $H = \mathrm{Hom}(\mathcal{A}, \mathbf{2})$, then each additive measure m_H on $\mathcal{A}_H$ is (due to the perfectness of $\mathcal{A}_H$) countably additive and each homomorphism h_H of $\mathcal{A}_H$ into $\mathbf{2}$ (being the point evaluation at some $g \in H$) is sequentially continuous (in the pointwise sequential convergence on $\mathcal{A}_H$).

If G is another Stone family of $\mathcal{A}$, then for some additive measure m the measure m_H on $\mathcal{A}_H$ can be countably additive and the measure m_G on $\mathcal{A}_G$ can fail to be countably additive. From the point of view of probability theory, it is natural to consider H and G to be equivalent if, for each additive probability measure p on $\mathcal{A}$, p_H is countably additive iff p_G is countably additive. As we shall see, s-perfectness is the essence of this equivalence.

Lemma 2.1. *Let $\mathcal{F}$ be an ultrafilter on $\mathcal{A}_H$. Then the following are equivalent:*

(i) $\mathcal{F}$ has the CIP;

(ii) The $\{0,1\}$-valued measure $m^{(\mathcal{F})}$ corresponding to $\mathcal{F}$ is countably additive;

(iii) The homomorphism $h^{(\mathcal{F})}$ corresponding to $\mathcal{F}$ is sequentially continuous.

Proof. Recall that $a_H \in \mathcal{F}$ iff $m^{(\mathcal{F})}(a_H) = h^{(\mathcal{F})}(a_H) = 1$.

(i) implies (ii). Let $\langle a_H(n) \rangle$ be a decreasing sequence in $\mathcal{A}_H$ such that $\bigcap_{n=1}^{\infty} a_H(n) = \emptyset$. We have to prove that

$\lim m^{(\mathcal{F})}(a_H(n)) = 0$. Since $\mathcal{F}$ has the CIP, $a_H(n) \notin \mathcal{F}$ and hence $m^{(\mathcal{F})}(a_H(n)) = 0$ for all but finitely many $n \in N$. Thus $m^{(\mathcal{F})}$ is countably additive.

(ii) implies (i). If $\mathcal{F}$ does not have the CIP, then there exists a decreasing sequence $\langle b_H(n)\rangle$ in $\mathcal{F}$ such that $\bigcap_{n=1}^{\infty} b_H(n) = \emptyset$. Put $a_H(n) = \bigcap_{k=1}^{n} b_H(k), n \in N$. Then $\langle a_H(n)\rangle$ is decreasing and $\bigcap_{n=1}^{\infty} a_H(n) = \emptyset$. Since $a_H(n) \in \mathcal{F}$ for all $n \in N$, $\lim m^{(\mathcal{F})}(a_H(n)) = 1$. Thus $m^{(\mathcal{F})}$ fails to be countably additive.

(ii) and (iii) are equivalent. It is known that each countably additive bounded measure on a field of sets is sequentially continuous [NOV]. On the other hand, each sequentially continuous additive measure is countably additive. The equivalence of (ii) and (iii) follows from the equivalence of (i) and (ii) and the fact that $m^{(\mathcal{F})}(a_H) = h^{(\mathcal{F})}(a_H)$ for all $a_H \in \mathcal{A}_H$. □

Let H be a Stone family of $\mathcal{A}$. Denote by H^* the set of all homomorphisms h of $\mathcal{A}$ into $\mathbf{2}$ such that h_H is sequentially continuous on $\mathcal{A}_H$. Clearly $H \subset H^*$. The proof of the next lemma is straightforward and it is omitted.

Lemma 2.2.

(i) $(H^*)^* = H^*$.

(ii) $H = H^*$ *iff each ultrafilter on* $\mathcal{A}_H$ *having the CIP is fixed, i.e., iff* $\mathcal{A}_H$ *is s-perfect.*

(iii) If G *is a Stone family of* $\mathcal{A}$ *such that* $H \subset G \subset H^*$, *then* $G^* = H^*$.

Denote by $P(H)$ the set of all additive probability measures p on $\mathcal{A}$ such that p_H is countably additive on $\mathcal{A}_H$.

Proposition 2.3. *Let* H, G *be Stone families of* $\mathcal{A}$. *Then* $P(H) = P(G)$ *iff* $H^* = G^*$.

Proof. 1. Assume that $P(H) = P(G)$. Since each sequentially continuous homomorphism of $\mathcal{A}_H$ and of $\mathcal{A}_G$ into $\mathbf{2}$, respectively, is a $\{0,1\}$-valued countably additive measure on $\mathcal{A}_H$ and on $\mathcal{A}_G$, respectively, necessarily $H^* = G^*$.

2. Assume $H^* = G^*$. Observe that the natural isomorphisms between $\mathcal{A}_H$ and $\mathcal{A}_{H^*}$ and between $\mathcal{A}_G$ and $\mathcal{A}_{G^*}$, respectively, are sequentially continuous. From $H^* = G^*$ it follows that the isomorphism sending a_H to a_G, $a \in \mathcal{A}$, is sequentially continuous, too. Hence if p is an additive probability measure on $\mathcal{A}$, then the measure p_H is sequentially continuous on $\mathcal{A}_H$ iff the measure p_G is sequentially continuous on $\mathcal{A}_G$. But since for bounded additive measures the sequential continuity is equivalent to the countable additivity [NOV], we have $P(H) = P(G)$. □

Let $\mathbb{A}$ be a field of subsets of $X \neq 0$. Then each $x \in X$ represents a sequentially continuous homomorphism of $\mathbb{A}$ into $\mathbf{2}$ and X becomes a Stone family of $\mathbb{A}$. Clearly, we can identify $\mathbb{A}$ and $\mathbb{A}_X$. Further, $\mathbb{A}$ and $\mathbb{A}_X$ are isomorphic and both the natural isomorphism h sending A to A_{X^*}, $A \in \mathbb{A}$, and its inverse are sequentially continuous.

Lemma 2.4. *If the natural isomorphism h of $\mathbb{A}$ onto $\mathbb{A}_{X^*}$ is induced by a measurable map f of X^* into X, then $\mathbb{A}$ is s-perfect.*

Proof. Let $\mathcal{F}$ be an ultrafilter on $\mathbb{A}$ having the CIP. Then $h(\mathcal{F}) = \{(\mathcal{A}); \mathcal{A} \in \mathcal{F}\}$ is an ultrafilter on $\mathbb{A}_{X^*}$ having the CIP. Since $\mathbb{A}_{X^*}$ is s-perfect, there exists $x \in X^*$ such that $h(\mathcal{F}) = \{B \in \mathbb{A}_{X^*}; x \in B\}$. Then $\mathcal{F} = \{A \in \mathbb{A}; f(x) \in \mathbb{A}\}$. This is possible only if $x \in X$ and $f(x) = x$. □

The proof of the next lemma is straightforward and it is omitted.

Lemma 2.5. *Let h be a sequentially continuous homomorphism of a field $\mathbb{A}$ of subsets of $X \neq \emptyset$ into a field $\mathbb{B}$ of subsets of $Y \neq \emptyset$. Let $\mathcal{F}$ be an ultrafilter on $\mathbb{B}$ having the CIP. Then $h^{\leftarrow}(\mathcal{F}) = \{A \in \mathbb{A}; h(A) \in \mathcal{F}\}$ is an ultrafilter on $\mathbb{A}$ having the CIP.*

Proposition 2.6. *Let $\mathbb{A}$ be a field of subsets of $X \neq \emptyset$ and let $\mathbb{B}$ be a field of subsets of $Y \neq \emptyset$*

(i) Let f be a measurable map of Y into X. Then the induced homomorphism $f^{\leftarrow}$ of $\mathbb{A}$ into $\mathbb{B}$ is sequentially continuous.

(ii) Let h be a sequentially continuous Boolean homomorphism of $\mathbb{A}$ into $\mathbb{B}$. If $\mathbb{A}$ is s-perfect, then h is induced by a uniquely determined measurable map f of Y into X.

Proof. (i)Let a sequence $\langle B_n \rangle$ converge in $\mathbb{B}$ to B. Fix $a \in X$. If $f(a) \in B$, then $f(a) \in B_n$ for all but finitely many $n \in N$. If $f(a) \notin B$, then $f(a) \notin B_n$ for all but finitely many $n \in N$. Hence the sequence $\langle f^{\leftarrow}(B_n) \rangle$ converges in $\mathbb{A}$ to $f^{\leftarrow}(B)$.

(ii)can be proved virtually in the same way as the first part of Proposition 11.1 in [SIK]. From the assumption that h is sequentially continuous it follows that if $\mathcal{F}_y$ is an ultrafilter determined by a point $y \in Y$, then its preimage $h^{\leftarrow}(\mathcal{F}_y)$ is an ultrafilter on $\mathbb{A}$ having the CIP. Hence, if $\mathbb{A}$ is s-perfect, then $h^{\leftarrow}(\mathcal{F}_y)$ is determined by a point $x \in X$. Since $\mathbb{A}$ is reduced, the point x is uniquely determined and hence we can define $f(y) = x$. Straightforward calculations show that f induces h. □

Proposition 2.7. *Let $\mathbb{A}$ be a field of subsets of $X \neq \emptyset$. If $\mathbb{A}$ is s-perfect, then the generated sigma-field $\sigma(\mathbb{A})$ is s-perfect, too.*

Proof. Let $\mathcal{F}$ be an ultrafilter on $\sigma(\mathbb{A})$ having the CIP. Then its trace $\{F \in \mathcal{F}; F \in \mathbb{A}\}$ is generated by some point $x \in X$. Since there exists a unique $\{0,1\}$-valued countably additive measure m on $\sigma(\mathbb{A})$ such that $\mathcal{F} = \{A \in \sigma(\mathbb{A}); m(A) = 1\}$, necessarily m is the Dirac measure at x. Thus $\mathcal{F}$ is fixed. □

It can be shown [FR] that the minimal field $\mathbb{B}_0$ of subsets of R containing all open intervals of the form $(-\infty, a), a \in R$, is s-perfect. Hence, by Proposition 2.7, the field $\mathbb{B}$ of all Borel measurable subsets of R is s-perfect, too. Further, s-perfectness is preserved by arbitrary products of fields and sigma-fields [FR].

Based on the properties of s-perfect fields of sets we make the following suggestion. In Kolmogorov's theory of probability [LOE] we can assume that all fields of events are s-perfect. Proposition 2.3 guarantees that no probability information is lost. The assumption is not restrictive if we start with events forming a Boolean algebra, represent it as an s-perfect field, and pass to the generated sigma-field. In this case the Boolean homomorphisms are induced by measurable maps. In fact, s-perfectness makes the Boolean model [HAL] and [LO],[LOS] and the Kolmogorov's model equivalent. Further suggestions about the fields of events and random variables can be found in [FR].

3.

The notion of s-perfectness leads to three natural categories. We describe their subcategories and their relationships. In particular, we prove that $\mathcal{B}(\mathbf{2})$ and SPMM are dual.

Denote by FS the category whose objects are fields of sets considered as Boolean algebras and whose morphisms are sequentially continuous Boolean homomorphisms. If $\mathbb{A}$ is a field of subsets of X, then the corresponding object of FS will be denoted by $(X, \mathbb{A})$. Denote by SPFS the full subcategory of FS consisting of s-perfect fields.

Proposition 3.1. *SPFS is a bireflective subcategory of FS.*

Proof. For each object $(X, \mathbb{A})$ of FS define $s((X, \mathbb{A})) = (X^*, A_{X^*})$. Define a homomorphisms h of $\mathbb{A}$ into $\mathbb{A}_{X^*}$ by $h(A) = A_{X^*}$, $A \in \mathbb{A}$. Then h is a one-to-one and onto, and both h and $h^{\leftarrow}$ are sequentially continuous. Thus s yields a bireflector. □

Proposition 3.2. *The category SPFS and the dual category SPMM$^{\mathrm{op}}$ of SPMM are isomorphic.*

Proof. All three categories have the same objects. The assertion is a straightforward corollary to Proposition 2.6. □

Proposition 3.3. *The categories $\mathcal{B}(\mathbf{2})$ and SPFS are equivalent.*

Proof. Let $(\mathcal{A}, \mathbb{L})$ be a $\mathbf{2}$-generated Boolean algebra. Let $H \subset \mathrm{Hom}(\mathcal{A}, \mathbf{2})$ be the set of all sequentially continuous homomorphisms of $(\mathcal{A}, \mathbb{L})$ into $\mathbf{2}$. Then $H = H^*$. According to Lemma 2.2, the field $\mathcal{A}_H$ is s-perfect. Denote by $\mathbb{L}_H$ the pointwise sequential convergence on $\mathcal{A}_H$. The map sending a to a_H, $a \in \mathcal{A}$, is a sequentially continuous isomorphism of $(\mathcal{A}, \mathbb{L})$ onto $(\mathcal{A}_H, \mathbb{L}_H)$ and its inverse is also sequentially continuous. Putting $F((\mathcal{A}, \mathbb{L})) = (H, \mathcal{A}_H)$ we get a functor F from $\mathcal{B}(\mathbf{2})$ into SPFS.

Let $(X, \mathbb{A})$ be an s-perfect field of sets. Then $\mathbb{A}$ carrying the pointwise sequential convergence $\mathbb{P}$ belongs to $\mathcal{B}(\mathbf{2})$ and $X = X^* \subset \mathrm{Hom}(\mathbb{A}, \mathbf{2})$ is the set of all sequentially continuous homomorphisms of $\mathbb{A}$ into $\mathbf{2}$. Put $G((X, \mathbb{A})) = (\mathbb{A}, \mathbb{P})$. This yields a functor G from SPFS into $\mathcal{B}(\mathbf{2})$. It follows that F is left adjoint to G and the adjunction is an equivalence (both the unit and the counit of the adjunction are isomorphisms; see [JO]. □

Corollary 3.4. *The categories $\mathcal{B}(\mathbf{2})$ and SPMM are dual.*

Remark 3.5 (i) The bireflector s in Proposition 3.1 yields a contravariant functor M from FS into the subcategory SPMM of MM, assigning to each $(X, \mathbb{A})$ the s-perfect object $s((X, \mathbb{A})) = (X^*, \mathbb{A}_{X^*})$ and to each sequentially continuous

Boolean homomorphism h of $(X, \mathbb{A})$ into $(Y, \mathbb{B})$ the uniquelly determined measurable map f^* of $(Y^*, \mathbb{B}_{Y^*})$ into $(X^*, \mathbb{A}_{X^*})$. Since for each measurable map f of $(Y, \mathbb{B})$ into $(X, \mathbb{A})$ its preimage $f^{\leftarrow}$ is a sequentially continuous Boolean homomorphism of $(X, \mathbb{A})$ into $(Y, \mathbb{B})$, this gives a contravariant functor P from MM into FS. This yields a pair of contravariant functors between FS and MM which induces the duality between SPFS and SPMM.

(ii) Denote by AFS the full subcategory of FS consisting of all fields of subsets $(X, \mathbb{A})$ such that $\mathbb{A}$ is a sigma-field. It was shown in [FR] that the functor σ sending each field $(X, \mathbb{A})$ to the generated sigma-field $(X, \sigma(\mathbb{A}))$ is an epireflector. In [FRIC] a subcategory $\mathcal{AB}(\mathbf{2})$ of absolutely sequentially closed objects (with respect to sequentially continuous homomorphisms into $\mathbf{2}$) has been defined and proved to be epireflective in $\mathcal{B}(\mathbf{2})$. Observe that the equivalence between $\mathcal{B}(\mathbf{2})$ and SPFS yields an equivalence between $\mathcal{AB}(\mathbf{2})$ and the full subcategory ASPFS of SPFS consisting of sigma-fields.

Let $\mathcal{A}$ be a Boolean algebra. Consider the initial sequential convergence $\mathbb{L}_{\mathcal{A}}$ on $\mathcal{A}$ with respect to $\mathrm{Hom}(\mathcal{A}, \mathbf{2})$: $\langle a(n) \rangle$ converges to a under $\mathbb{L}_{\mathcal{A}}$ iff $\langle h(a(n)) \rangle$ converges to $h(a)$ in $\mathbf{2}$ for all $h \in \mathrm{Hom}(\mathcal{A}, \mathbf{2})$. Clearly, $\mathbb{L}_{\mathcal{A}}$ is $\mathbf{2}$-generated and it is finer than any other $\mathbf{2}$-generated convergence $\mathbb{L}$ on $\mathcal{A}$ (i.e., $\mathbb{L}_{\mathcal{A}} \subseteq \mathbb{L}$).

Definition 3.6. *Let $\mathcal{A}$ be a Boolean algebra. Then the initial sequential convergence $\mathbb{L}_{\mathcal{A}}$ with respect to* $\mathrm{Hom}(\mathcal{A}, \mathbf{2})$ *is said to be fine and $(\mathcal{A}, \mathbb{L}_{\mathcal{A}})$ is said to be a fine $\mathbf{2}$-generated Boolean algebra.*

Denote by $\mathcal{FB}(\mathbf{2})$ the full subcategory of $\mathcal{B}(\mathbf{2})$ consisting of fine objects.

Lemma 3.7. *Let h be a homomorphism of a Boolean algebra $\mathcal{A}$ into a Boolean algebra $\mathcal{A}'$. Then h is sequentially continuous as a homomorphism of $(\mathcal{A}, \mathbb{L}_{\mathcal{A}})$ into $(\mathcal{A}', \mathbb{L}_{\mathcal{A}'})$.*

Proof. Suppose, on the contrary, that h is not sequentially continuous. Then some sequence $\langle a(n) \rangle$ converges to a under $\mathbb{L}_{\mathcal{A}}$ and the sequence $\langle h(a(n)) \rangle$ does not converge to $h(n)$ under $\mathbb{L}_{\mathcal{A}'}$. Since $\mathbb{L}_{\mathcal{A}'}$ is $\mathbf{2}$-generated, there exists $g \in \mathrm{Hom}(\mathcal{A}', \mathbf{2})$ such that the sequence $\langle g(h(a(n))) \rangle$ does not converge to $g(h(a))$ in $\mathbf{2}$. Since $g \circ h \in \mathrm{Hom}(\mathcal{A}, \mathbf{2})$, we have a contradiction. □

As an immediate corollary we have the following.

Proposition 3.8. *The categories BA and* $\mathcal{FB}(\mathbf{2})$ *are isomorphic.*

Let $\mathcal{A}$ be a Boolean algebra. Recall that if $H = \mathrm{Hom}(\mathcal{A}, \mathbf{2})$, then $\mathcal{A}_H$ is a reduced perfect field of subsets of H, for each $h \in H$ the homomorphism h_H is sequentially continuous, and each ultrafilter on $\mathcal{A}_H$ is fixed. Hence, if we start with $\mathcal{A}$, pass to $(\mathcal{A}, \mathbb{L}_{\mathcal{A}})$, apply the functor F from Proposition 3.4, then we end up with a reduced perfect field $(\mathcal{A}_H, \mathbb{L}_H)$. In the opposite direction, if we start with a reduced perfect field $\mathbb{A}$ of subsets of X carrying the pointwise convergence, then $\mathbb{A}$ can be considered as a fine $\mathbf{2}$-generated Boolean algebra and hence we can forget the fine convergence and end up with a Boolean algebra. This yields an equivalence between the categories $\mathcal{FB}(\mathbf{2})$ and PFS.

Corollary 3.9. (The nontopological Stone duality.) *The categories BA and PMM are dual.*

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Matematický Ústav SAV, Grešákova 6, 040 01 Košice, Slovakia
E-mail address:fric@ns.saske.sk