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SHRINKINGS OF OPEN NEIGHBOURHOODS AND A TOPOLOGICAL DESCRIPTION OF METRIZABILITY

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Abstract

A purely topological description of metrizability should be as removed as possible from notions of size and the like. In the quest for such, we define weak subproperties of stratifiable spaces, β-, γ- and θ-spaces in terms of shrinkings of open neighbourhoods rather than g-functions. Our result strengthens the celebrated theorem of Hodel. Dugundji Extension Property is similarly accounted for.

The problem of metrization asks for a *topological* description of metrizability. The more interesting and the more *purely* topological the solution, the more it should be removed from notions of local quasi-uniformity and its cousin stratifiability, and from the notions of *size* (as opposed to *containment*) and in particular its transitivity, all obvious and preponderant legacies of the metric. Indeed it was upon such grounds that Alexandroff and Urysohn (Theorem VI.1 of [16]) and A.H. Frink (Cor. to Theorem VI.2 of [16]) were deemed unsatisfactory *topological* solutions to the problem of metrization (see e.g. § 14 of Chapter 6 of Kelley [15]). Bing-Nagata-Smirnov, (Theorem

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VI.3 of [16]) seemingly above criticisms of this sort at its inception, was shown shortly after to be just one step removed from the Double Sequence Theorem (Theorem VI.2 of [16]). These metrization theorems, from Alexandroff-Urysohn to the Double Sequence Theorem, *classical* according to Nagata [17] and *uniform* according to me [11], are all particular cases of Theorem 2.1 of [10]. Of late, Stares [19] equated metrizability with what he called PBN (see also [13]). But then, in the formulation of PBN, the natural number attached to every neighbourhood of every point is very much indicative of the notion of *size*, and, in any case, it leads readily to the fulfillment of the hypothesis of Corollary 2.3 of [10] and Stares' is thus also *classical* according to Nagata and *uniform* according to me.

There are essentially two non-uniform or non-classical metrization theorems, Balogh-Collins-Reed-Roscoe-Rudin [1], [3] and mine [11]. The former, formulated on the insistence of the *openness* of the neighbourhoods, is thus, unlike the latter, a class of its own and difficult to apply. In any case, a powerful and interesting topological description of metrizability is one that involves only factors of weak hereditary properties, individually devoid of obvious and preponderant legacies of the metric, particularly those mentioned above.

In the quest of such, we define five properties in terms of what I call the *shrinkings of open neighbourhoods*, and not the Heath-Hodel *g-function.*[1]They are thus *extra*-Heath-Hodelian sub-properties of semi-stratifiable spaces, stratifiable spaces and θ-spaces. Four of these are *hereditary* and the fifth, $(MP4)$,

[1] g-functions, originated with Heath, were extensively used by Hodel [8], [9] in a unified approach to the study of generalizations of metric spaces, with the advantages that accompany unified approaches, leading to the discovery of new avenues of generalizations, amongst which are concepts of β, γ and θ-spaces which we seek here to further generalize.

can be strengthened to $(MP4*)$ which is also hereditary while still a property of semi-stratifiable spaces. Furthermore, these properties are *monotone* in the sense that more severe shrinkings would not spoil these properties (cf. [2]).

We have thus a necessary and sufficient condition for a BCO (Theorem 2.1), a substantial improvement upon Hodel's celebrated theorem that T_1 semi-stratifiable θ-spaces are developable (Remark 4.8 of [8]), and a new result on the metrization of stratifiable spaces (Cor. 2.2) (cf. 5.12 and 8.3 of [5] and Theorem 3.2 of [12]), with an application on the question of the metrizability of squares (Cor. 2.4) (cf. [14]).

0. Notations and Terminology

Given a topological space $(X, \mathcal{T})$. Let there be $A : \{(x, U) : x \in U \in \mathcal{T}\} \to \mathcal{T}$. A is said to be a *shrinking of open neighbourhoods* on X, if $x \in A(x, U) \subset U$, whenever $x \in U \in \mathcal{T}$. Given two shrinkings, A and B, of open neighbourhoods on X, if $B(x, U) \subset A(x, U)$, whenever $x \in U \in \mathcal{T}$, we write $B < A$. A property $\mathcal{P}$ on the shrinking A of open neighbourhoods on X is said to be *monotone* if, A has property $\mathcal{P} \Rightarrow B$ has property $\mathcal{P}$ whenever $B < A$. In the following, we define five monotone properties on the shrinking A of open neighbourhoods on X.

$(MP1)$ $x \in A(y, V)$ and $y \in A(x, U) \Rightarrow$ either $A(y, V) \subset U$ or $A(x, U) \subset V$;

$(MP2)$ $A(x, U) \cap A(y, V) \neq \emptyset \Rightarrow$ either $y \in U$ or $x \in V$ (Monotone Normality, 5.19 of [5]);

$(MP3)$ $x \in U_{n+1} \subset A(x, U_n)$ for all $n \in \omega \Rightarrow \bigcap\{U_n : n \in \omega\} = \{x\}$;

$(MP4)$ $x_{n+1} \in U_{n+1} \subset A(x_n, U_n)\setminus\{x_n\}$ for all $n \in \omega$ and $\xi \in \bigcap\{U_n : n \in \omega\} \Rightarrow \xi$ is a cluster point of $\langle x_n \rangle$; and

$(MP5)$ $y_n \in V_n \in \mathcal{T}$ for all $n \in \omega$ and $y_i \in V_j, y_j \notin V_i$ when $i < j \Rightarrow \bigcap\{A(y_n, V_n) : n \in \omega\} = \emptyset$.

These properties are quite simple. Indeed, one can easily see how the shrinking in Stares' PBN [19], for example, has all five of them. One can also easily see how a shrinking of open neighbourhoods can be constructed on the Sorgenfrey line and the Michael line that has $(MP1)$, $(MP2)$ and $(MP3)$, and, on the Michael line, also one that has $(MP5)$. In contrast, we show below that no shrinking of open neighbourhoods can be constructed, on the Sorgenfrey line, to have $(MP4)$ or $(MP5)$ (Proposition 1.7) or, on the Michael line, to have $(MP4)$ (Proposition 1.8).

For any $1 \leq i \leq 5$, we say X is an MPi-space (or MPi) if on X is a shrinking of open neighbourhoods with (MPi). These properties are obviously *monotone.* Because they are monotone, we can insist that $A(x, U) = \{x\}$ whenever x is an *isolated point.* Clearly, if X is MPi, MPj, MPk, MPl and MPm, $1 \leq i, j, k, l, m \leq 5$, there is on X a shrinking of open neighbourhoods with (MPi), (MPj), (MPk), (MPl) and (MPm), *all at once.*

A point x on $(X, \mathcal{T})$ is said to be of *countable pseudocharacter* if there is a decreasing sequence $\langle h(n, x)\rangle$ of open neighbourhoods such that $\bigcap\{h(n, x) : n \in \omega\} = \{x\}$. We say that $(X, \mathcal{T})$ is of *countable pseudocharacter* if every point on $(X, \mathcal{T})$ is of countable pseudocharacter. $(X, \mathcal{T})$ is said to be *semi-stratifiable* (respectively, *stratifiable*) if, for every $x \in X$, there is a decreasing sequence $\langle g(n, x)\rangle$ of open neighbourhoods such that, given $x \in X$ and an open neighbourhood U, there is such an $n \in \omega$ that $x \notin \bigcup\{g(n, y) : y \notin U\}$ (respectively, $x \notin \mathrm{Cl}\bigcup\{g(n, y) : y \notin U\}$) (Theorem VI.25 of [16]). Clearly, on T_1-spaces, stratifiability $\Longrightarrow$ semi-stratifiability $\Longrightarrow$ being of countable pseudocharacter. $MP3$-spaces are of course

T_1. $(X, \mathcal{T})$ is said to be a *γ-space* (respectively, a *θ-space*) if, for every $x \in X$, there is a decreasing sequence $\langle g(n,x)\rangle$ of open neighbourhoods such that, given $x \in X$ and an open neighbourhood U, there are such $n, m \in \omega$ that $\bigcup\{g(m,y) : y \in g(m,x)\} \subset g(n,x) \subset U$ (respectively, $\bigcup\{g(m,y) : y \in g(m,x), x \in g(m,y)\} \subset g(n,x) \subset U$) (10.5 of [5], 2 of [4]).

Clearly, $MP1$-, $MP2$- and $MP5$-spaces are *hereditary*. Noting Proposition 1.1 below, we can also see that $MP3$-spaces are *hereditary*. If we strengthen $(MP4)$ to

$$(MP4*) \quad \begin{aligned} & x_n \in U_n \in \mathcal{T} \text{ for all } n \in \omega, x_j \in A(x_i, U_i) \text{ and} \\ & x_i \notin U_j \text{ when } i < j, \text{ and } \xi \in \bigcap\{A(x_n, U_n) : n \in \omega\} \\ & \Longrightarrow \xi \text{ is a cluster point of } \langle x_n\rangle, \end{aligned}$$

and say X is an $MP4*$-space (or $MP4*$) if on X is a shrinking of open neighbourhoods with $(MP4*)$, then we can see that $MP4*$-spaces are also *hereditary*.

1. Preliminary Results

Proposition 1.1. *Spaces $(X, \mathcal{T})$ of countable pseudocharacter are $MP3$. (And conversely.)*

Proof. If, whenever $x \in U \in \mathcal{T}$ for some non-isolated x, we let $\nu(x, U)$ be the first ordinal n such that $U \backslash h(n,x) \neq \emptyset$ and let $A(x,U)$ be defined to be $h(\nu(x,U), x) \cap U$, then, when given x and U_n's satisfying the hypothesis in $(MP3)$, we have $\nu(x, U_n) < \nu(x, U_{n+1})$ and therefore $\bigcap\{U_n : n \in \omega\} = \{x\}$. $\square$

Proposition 1.2. *Semi-stratifiable spaces $(X, \mathcal{T})$ are $MP4*$ (and of course $MP4$ and, if T_1, also $MP3$).*

Proof. If, whenever $x \in U \in \mathcal{T}$, we let $\nu(x, U)$ be the first ordinal n such that $x \notin g(n,y)\ \forall y \notin U$, and $A(x,U)$ be defined to be $g(\nu(x,U),x)\cap U$, then, when given the x_n's, U_n's and ξ satisfying the hypothesis in $(MP4*)$, we have (because $\xi, x_{n+1} \in g(\nu(x_n,U_n),x_n)$) $\nu(x_n,U_n) < \nu(x_{n+1},U_{n+1})$ and therefore $x_n \to \xi$. □

Remarks. 1. $MP3+MP4^*$ does not imply semi-stratifiability, even in the presence of $MP1$ and $MP2$. The space ω_1 of the countable ordinals (with the order topology) is clearly first countable and therefore $MP3$. It is a LOTS and therefore $MP2$ (§ 5.21 of [5]). It is *not* paracompact and therefore cannot be semi-stratifiable or subparacompact in view of its being $MP2$ (see § 1.8 and § 5.11 of [5]). In contrast, it is $MP4^*$ and $MP1$. For, if we let $A(x,U) = (y,x]$ for some $y < x$, given any x and any open neighbourhood U, neither the hypothesis of $(MP4^*)$ nor that of $(MP1)$ can be fulfilled and both $(MP4^*)$ and $(MP1)$ are (vacuously) satisfied.

2. Is it possible that *paracompact spaces* that are $MP3+MP4^*$ are semi-stratifiable?

Proposition 1.3. *Stratifiable spaces* $(X,\mathcal{T})$ *are* $MP2$ *and* $MP5$ *(and of course* $MP4$, $MP4*$ *and, if* T_1, *also* $MP3$*).*

Proof. If, whenever $x \in U \in \mathcal{T}$, we let $\nu(x,U)$ be the first ordinal n such that $x \notin \mathrm{Cl} \bigcup\{g(n,y) : y \notin U\}$ and $A(x,U)$ be defined to be $g(\nu(x,U),x)\setminus \mathrm{Cl} \bigcup\{g(\nu(x,U),y) : y \notin U\}$, then,

a) when given x,U,y,V satisfying the hypothesis of $(MP2)$, we have $y \in U$ if $\nu(x,U) \le \nu(y,V)$ and $x \in V$ if $\nu(y,V) \le \nu(x,U)$;

b) when given y_n's and V_n's satisfying the hypothesis of $(MP5)$, we have, because $A(y_{n+1},V_{n+1}) \cap A(y_n,V_n) \ne \emptyset \Rightarrow \nu(y_{n+1},V_{n+1}) < \nu(y_n,V_n)$, $\bigcap\{A(y_n,V_n) : n \in \omega\} = \emptyset$. □

Remark. The proof that stratifiable spaces are $MP2$ is of course Borges' and we repeat it here only to show that the same construction works for both cases in the same way and for the same reason.

Question: $\sum_{i=2}^{5} MPi =$ Stratifiability?

Proposition 1.4. *γ-spaces $(X, \mathcal{T})$ are $MP1$.*

Proof. If, whenever $x \in U \in \mathcal{T}$, we let $\nu(x, U)$ and $\mu(x, U)$ be any ordinals such that $\bigcup\{g(\mu(x, U), y) : y \in g(\mu(x, U), x)\} \subset g(\nu(x, U), x) \subset U$, and $A(x, U)$ be defined to be $g(\mu(x, U), x)$, then, when given x, U, y, V satisfying the hypothesis of $(MP1)$, we have $A(x, U) \subset V$ if $\mu(y, V) \leq \mu(x, U)$ and $A(y, V) \subset U$ if $\mu(x, U) \leq \mu(y, V)$. $\square$

Remarks. 1. Note that the space ω_1 is $MP1$ but *not* a γ-space (Ex. 4.12 of [8] and Remark 1 after Proposition 1.2 above). The converse of Proposition 1.4 is not true, even in the presence of compactness: The space $\omega_1 + 1$ is clearly not first countable and therefore cannot be a γ-space.

2. Indeed any LOTS with a Sorgenfrey modification on its topology is necessarily $MP1$ and $MP2$.

3. Indeed, θ-spaces are $MP1$. The question is whether first countable $MP1$-spaces are θ-spaces.

Proposition 1.5. *Spaces $(X, \mathcal{T})$ that are $MP2$ and $MP5$ are (hereditarily) paracompact.*

Proof. Let there be a shrinking A of open neighbourhoods on X with $(MP2)$ and $(MP5)$. Let there be a well-ordered open cover $\mathcal{U}$ of X. For every $U \in \mathcal{U}$, we write $\tilde{U}$ for $U \setminus \bigcup\{V : V \prec U\}$ and $A(\tilde{U}, U)$ for $\bigcup\{A(x, U) : x \in \tilde{U}\}$. We are to show that the *open refinement* $\{A(\tilde{U}, U) : U \in \mathcal{U}\}$ of $\mathcal{U}$ is *point-finite* and the assertion follows. Suppose otherwise. Suppose there are an $\xi \in X$ and a sequence $\langle V_n \rangle$ of elements of $\mathcal{U}$, increasing according to the well order $\prec$ on $\mathcal{U}$, such that $\xi \in A(\tilde{V}_n, V_n)$ for every $n \in \omega$. There is then $y_n \in \tilde{V}_n$ for every $n \in \omega$ such that $\xi \in A(y_n, V_n)$. Clearly $y_i \in V_j$ (and $y_j \notin V_i$) when $i < j$ (because of $(MP2)$), and the hypothesis of $(MP5)$ is satisfied, and the conclusion of $(MP5)$ delivers the contradiction. □

Remarks. Indeed, we have, with the same proof: Spaces $(X, \mathcal{T})$ that are $MP2^-$ and $MP5^-$ are (hereditary) paracompact, if we bring in the more restricted concept of a shrinking *A of an open neighbourhood assignment* $\{U_x : x \in X\}$, so that $x \in A(U_x) \subset U_x$ for each $x \in X$, and define two monotone (and hereditary) properties.

$(MP2^-)$ $A(U_x) \cap A(U_y) \neq \emptyset \Rightarrow$ either $y \in U_x$ or $x \in U_y$; and

$(MP5^-)$ $x_i \in U_{x_j}, x_j \notin U_{x_i}$ when $i < j, i, j \in \omega \Rightarrow$
$\bigcap\{A(U_{x_n}) : n \in \omega\} = \emptyset$.

We say X is $MP2^-$, and respectively $MP5^-$, if, for every open neighbourhood assignment, there is a shrinking A of it with $(MP2^-)$, and respectively $(MP5^-)$.

Proposition 1.6. *Proto-metrizable spaces* [18] *are $MP1$ and $MP2$.*

(Result is obvious if one notes the famous *rank 1 pair-base* characterization due to Gruenhage and Zenor [6] of proto-metrizability.)

Proposition 1.7. *The Sorgenfrey line S (Example III 5 of* [16]*) is not $MP4$ or $MP5$.*

Proof. Suppose otherwise. 1) Suppose there is a shrinking A of open neighbourhoods on S that has $(MP4)$. Because of the monotonicity of $(MP4)$, we can assume that, for every $x \in S$ and every open neighbourhood U of x, $A(x, U) = (u, x]$ for some $u < x$. Consequently, for any sequence $\langle x_n \rangle$ and any point ξ as in the hypothesis of $(MP4)$ above (and there exist some such) we have $\xi < \ldots x_2 < x_1 < x_0$ and ξ can never be a cluster point of $\langle x_n \rangle$.

2) Suppose there is a shrinking A of open neighbourhoods on S that has $(MP5)$. For every $x \in S$ and the open neighbourhood $(-\infty, x]$ of x, let $A(x, (-\infty, x])$ be $(x', x]$. For every *rational* q on S, we let $S_q \equiv \{x \in S : x' < q < x\}$. With respect to the *usual* topology on the underlying set of S, there is such a rational ρ that S_ρ is of the *second category*. There is therefore a sequence $\langle x_n \rangle$ in S_ρ, increasing according to the usual order on the underlying set of S, so that $\rho \in \bigcap\{(x_n', x_n] : n \in \omega\}$, an arrangement satisfying the hypothesis but not the conclusion of $(MP5)$, and therefore a contradiction. □

Proposition 1.8. *The Michael line M (Example V.2 of* [16]*) is not $MP4$.*

Proof. Suppose otherwise. Suppose there is a shrinking A of open neighbourhoods on M that has $(MP4)$. Because of the monotonicity of $(MP4)$, for all *rational* x, we can assume $A(x, U)$ to be an open *interval* containing x, its closure (with respect to the order topology) contained in U, whatever U. We are going to build a tree of open intervals of height ω so that each given element has exactly two immediate successors in the form of subintervals, their closures (with respect to the order topology) disjoint and contained individually in the given interval. Pick $x \in M \cap \mathbb{Q}$. Pick $x_0, x_1 \in A(x, M) \cap \mathbb{Q}$ so that $x_0 < x < x_1$. Pick $x_{00}, x_{01} \in A(x_0, A(x, M) \setminus \{x\}) \cap \mathbb{Q}$ and $x_{10}, x_{11} \in A(x_1, A(x, M) \setminus \{x\}) \cap \mathbb{Q}$, so that $x_{00} < x_0 < x_{01} < x < x_{10} < x_1 < x_{11}$, ad infinitum. Clearly, this tree has 2^ω branches and there is a branch $\mathcal{B}$ such that $\bigcap \mathcal{B}$ contains an *irrational* ξ, a situation that allows an arrangement that satisfies the hypothesis of $(MP4)$ but not its conclusion. A contradiction is entailed and the assertion proved. □

2. Main Results

Theorem 2.1. *Spaces $(X, \mathcal{T})$ that are $MP1, MP3$ and $MP4$ have Bases of Countable Order (6.3 of* [5]*) and are therefore metrizable if they are paracompact and Hausdorff. Consequently, Metrizability $= \Sigma_{i=1}^{5} MPi$.*

Proof. Let there be a shrinking A of open neighbourhoods on X with $(MP1), (MP3)$ and $(MP4)$. We are to construct a tree of open subsets of height ω, each branch $\mathcal{B}$ of which constitutes a base at any $\xi \in \bigcap \mathcal{B}$ and each element of which is covered by the family of its immediate successors. Let the initial level $\mathcal{U}_0$ be $\{X\}$. Let $\mathcal{S}_1(X) \equiv \{A(x, X)\backslash\{x\} : x \in X\}$, $X_1 \equiv \bigcup \mathcal{S}_1(X), X_2 \equiv X\backslash X_1$, and $\mathcal{S}_2(X) \equiv \{A(x, X) : x \in X_2\}$. Clearly, X_2 is discrete and closed in X. Indeed, $y \notin A(x, X)$ if $x, y \in X_2$ and $x \neq y$. Let $\mathcal{S}(X) \equiv \mathcal{S}_1(X) \cup \mathcal{S}_2(X)$ be the family of immediate successors of the only member X of the initial level $\mathcal{U}_0$. For every $U \in \mathcal{S}_1(X)$, let $\mathcal{S}_1(U) \equiv \{A(x, U)\backslash\{x\} : x \in U\}$, $U_1 \equiv \bigcup \mathcal{S}_1(U), U_2 \equiv U\backslash U_1$, and $\mathcal{S}_2(U) \equiv \{A(x, U) : x \in U_2\}$. Let $\mathcal{S}(U) \equiv \mathcal{S}_1(U) \cup \mathcal{S}_2(U)$ be the family of immediate successors of the member U. For every member U of $\mathcal{S}_2(X)$, $U = A(y, X)$ for some $y \in X_2$, let $\mathcal{S}_1(U) \equiv \{A(y, X)\backslash\{y\}\}$, and $\mathcal{S}_2(U) \equiv \{A(y, U)\}$. Let $\mathcal{S}(U) \equiv \mathcal{S}_1(U) \cup \mathcal{S}_2(U)$ be the family of immediate successors of the member U, ad infinitum, taking care to jettison the empty members along the way.

Given any branch $\mathcal{B}$. Either

1) for arbitrarily high levels, there is a $B \in \mathcal{B}$ such that its immediate successor along the branch is a member of $\mathcal{S}_1(B)$, or

2) beyond a certain level, the immediate successor of every member B along the branch is a member of $\mathcal{S}_2(B)$.

If 1) is true, and if $\xi \in \bigcap \mathcal{B}$, then, there are $\langle x_n \rangle$ and $\langle U_n \rangle$ satisfying the hypothesis of $(MP4)$ such that $\{U_n : n \in \omega\} \subset \mathcal{B}$. By $(MP4)$ we have $x_n \to \xi$. To show that $\mathcal{B}$ is a base at ξ, we let W be an open neighbourhood of ξ. Clearly $x_n \in A(\xi, W)$ (for large enough n) and $\xi \in A(x_n, U_n)$, and by $MP1$ either $A(\xi, W) \subset U_n$ or $A(x_n, U_n) \subset W$. We cannot have the first alternative for every n, which $\Rightarrow \xi \notin \mathrm{Cl}\{x_n : n \in \omega\}$. The second alternative, of course, implies that $U_{n+1} \subset W$ and that $\mathcal{B}$ is a base at ξ.

If 2) is true, then, there are x and $\langle U_n \rangle$ satisfying the hypothesis of $(MP3)$ such that $\langle U_n \rangle \subset \mathcal{B}$. To show that $\mathcal{B}$ is a base at x, we let W be an open neighbourhood of x. Clearly $x \in A(x, W)$ and $x \in A(x, U_n)$ and by $MP1$, either $A(x, W) \subset U_n$ or $A(x, U_n) \subset W$. The first alternative for every n entails that x is an isolated point and $U_n = \{x\}$ for every $n > 0$, i.e. $\mathcal{B}$ is a base at x. The second alternative again implies that $U_{n+1} \subset W$ and that $\mathcal{B}$ is a base at x.

We have therefore a BCO on X. Spaces that are $MP2$ and $MP5$ being paracompact (Proposition 1.5 above), we have Metrizability $= \Sigma_{i=1}^{5} MPi$. □

Remarks. 1. Indeed, if we weaken $(MP3)$ and $(MP4)$ to

$(MP3^-)$ $x \in U_{n+1} \subset A(x, U_n)$ for all $n \in \omega \Rightarrow \bigcap\{U_n : n \in \omega\}$ is not a neighbourhood of x unless x is isolated, and

$(MP4^-)$ $x_{n+1} \in U_{n+1} \subset A(x_n, U_n)\setminus\{x_n\}$ for all $n \in \omega$ and $\bigcap\{U_n : n \in \omega\} \neq \emptyset \Rightarrow \langle x_n \rangle$ has a cluster point,

and say X is $MP3^-$ and $MP4^-$ if on X is a shrinking of open neighbourhoods with $(MP3^-)$ and $(MP4^-)$, we have *mutatis mutandis*: Regular T_1-spaces that are $MP1$, $MP3^-$ and $MP4^-$ have BCO's and Metrizability $= MP1 + \sum_{i=2}^{5} MPi^-$ for T_1-spaces.

2. We can also replace the property of being a β-space with $MP4^-$ in the Theorem of Chaber (Theorem 8.2 of [5]).

Corollary 2.2. *On T_1 $MP1$-spaces, stratifiability = metrizability and semi-stratifiability = developability.*

Proof. Noting that Stratifiability $\Rightarrow \Sigma_{i=2}^{5} MPi$ (Proposition 1.3 above) and that semi-stratifiability $\Rightarrow MP3 + MP4+$ submetacompactness (Proposition 1.2 above and Theorem 5.11 of [5]), we see the truth of the Corollary. □

Corollary 2.3. (Hodel [8], [9]). *T_1 stratifiable θ-spaces are metrizable. T_1 semi-stratifiable θ-spaces are developable.*

Proof. Noting Proposition 1.4 above, we see this special case of Cor. 2.2. □

Corollary 2.4. *Given a T_1-space X, embedded in which is a copy of $\omega+1$. Metrizability of X^2 is equivalent to $MP1+MP2$.*

Proof. According to a theorem of Zenor (4.1 of [7]), by virtue of X^2 being $MP2$, X and therefore X^2 is stratifiable. In the presence of $MP1$, X^2 is metrizable. □

3. Addendum: Dugundji Extension

Within the framework of shrinkings of open neighbourhoods, we can have a strengthening of the Dugundji Extension theorems of Borges (Theorem 5.23 of [5]) and of Stares [20], the proof being *essentially* the same as that of Theorem 5.23 in [5] and that of Theorem 2.3 in [20].

Theorem 3.1. *Given a T_1-space $(X,\mathcal{T})$ that is $MP2$ and $MP5$. Given a closed subspace Y. There is on $C(Y)$ a linear extender Φ into $C(X)$ so that, for each $f \in C(Y)$, the range of $\Phi(f)$ is contained in the convex hull of that of f, provided there is a shrinking A of open neighbourhoods on X such that*

(*) given $\xi \in X \backslash Y$ so that the collection $\mathcal{F}_\xi \equiv \{(y,V) : y \in \mathrm{bdy} Y, y \in V \in \mathcal{T}, \xi \in A(y,V)\}$ is non-void, there is $\hat{\xi} \in Y$ so that $\hat{\xi} \in W$ for any $(z,W) \in \mathcal{F}_\xi$.

Proof. We can assume that we have on X a shrinking A of open neighbourhoods with properties $(MP2)$ and $(MP5)$, and with the property $(*)$. For any $\xi \in X \backslash Y$ so that $\mathcal{F}_\xi = \emptyset$, we choose an *arbitrary* $\hat{\xi} \in Y$, so that we have a mapping $\hat{}$ of $B = X \backslash Y$ into Y.

B being paracompact, we have a locally finite *partition of unity* $\{\rho_x : x \in B\}$ subordinated to the open cover $\{A(x, B) : x \in B\}$ of B, so that $\rho_x(z) = 0$ if $z \notin A(x, B)$.

Given $f \in \mathcal{C}(Y)$, we define $\overline{f}$ on X as follows:

i) $\overline{f}(z) = f(z)$, for every $z \in Y$,
ii) $\overline{f}(z) = \sum_{x \in B} f(\hat{x})\rho_x(z)$, for every $z \in X \backslash Y$.

$\overline{f}$ evidently continuous on B and on the interior of Y, we need only show that $\overline{f}$ is continuous on *bdy* Y in order that $\overline{f}$ be a member of $\mathcal{C}(X)$. For this purpose, we note that, given $y \in$ *bdy* Y and $\epsilon > 0$, if, for some open neighbourhood V of y, $f(y) - \epsilon < f(z) < f(y) + \epsilon$ for all $z \in Y \cap V$, then $f(y) - \epsilon < \overline{f}(w) < f(y) + \epsilon$ for all $w \in A(y, A(y, V))$. For, given $w \in A(y, A(y, V)) \backslash Y, \rho_x(w) > 0$ only if $x \in A(y, V) \backslash Y$ (because of $(MP2)$) and $\overline{f}(w) = \sum_{x \in A(y,V) \backslash Y} f(\hat{x})\rho_x(w)$. The points $\hat{x}$ being in V by virtue of $(*)$, $f(\hat{x})$ and therefore $\overline{f}(w)$ are within the interval $(f(y) - \epsilon, f(y) + \epsilon)$. Thus $\overline{f} \in \mathcal{C}(X)$ and can be taken to be $\Phi(f)$. Clearly, the Φ so defined is a linear extender of $\mathcal{C}(Y)$ into $\mathcal{C}(X)$ and the convex hull of the range of $\Phi(f)$ is equal to that of f for every $f \in \mathcal{C}(Y)$. □

Remark. That decreasing (G) spaces (for definition, see [20]) are $MP2$ and $MP5$ can be seen if we let $A(x, U) \equiv V(x, U)$ and note that

a) given $\xi \in A(x, U) \cap A(y, V), x \in W(s, \xi) \subset U$ and $y \in W(t, \xi) \subset V$, we have $y \in U$ if $s \leq t$ and $x \in V$ if $t \leq s$;

b) given y_n's and V_n's satisfying the hypothesis of $(MP5)$, $\xi \in \bigcap\{A(y_n, V_n) : n \in \omega\}$ and $y_n \in W(s_n, \xi) \subset V_n$, for every $n \in \omega$, we have $s_{n+1} < s_n$ (cf. Proposition 1.3 above).

REFERENCES

[1] Z. Balogh, *Topological spaces with point-networks*, Proc. Amer. Math. Soc., **94** (1985), 497-501.

[2] J. Chaber, M.M. Čoban, and K. Nagami, *On monotone generalizations of Moore spaces, Čech-complete spaces and p-spaces*, Fund. Math., **84** (1974), 107-119.

[3] P.J. Collins, G.M. Reed, A.W. Roscoe and M.E. Rudin, *A lattice of conditions on topological spaces*, Proc. Amer. Math. Soc., **94** (1985), 487-496.

[4] P. Fletcher and W.F. Lindgren, *θ-spaces*, Gen. Top. Appl., **9** (1978), 139-153.

[5] G. Gruenhage, *Generalized metric spaces*, Handbook of Set-theoretic Topology, Ch. 10, Ed. K. Kunen and J.E. Vaughan, North-Holland, Amsterdam, 1983.

[6] G. Gruenhage and P.L. Zenor, *Proto-metrizable spaces*, Houston J. Math., **3** (1977), 47-53.

[7] R.W. Heath, D.J. Lutzer and P.L. Zenor, *Monotonically Normal Spaces*, Trans. Amer. Math. Soc., **178** (1973), 481-493.

[8] R.E. Hodel, *Spaces defined by sequences of open covers which guarantee that certain sequences have cluster points*, Duke Math. J., **39** (1972), 253-263.

[9] ———, *Some results in metrization theory, 1950-1972*, Proc. Top. Conf. VPI and State Univ. 1973. Springer Lect. Notes 375, 120-136.

[10] H.H. Hung, *A contribution to the theory of metrization*, Canad. J. Math., **29** (1977), 1145-1151.

[11] ———, *Another view of metrizability*, Proc. Amer. Math. Soc., **101** (1987), 551-554.

[12] ———, *Factorization of metrizability*, Papers on General Topology and Applications, Annals New York Acad. Sci., **788** (1996), 133-137.

[13] ———, *A note on a recent metrization theorem*, Top. Proc., **21** (1996), 125-128.

[14] ———, *Metrization and Stratification of squares of topological spaces*, Top. Appl., **82** (1998), 205-210.

[15] J.L. Kelley, *General Topology*, Van Nostrand, Princeton, N.J., 1955.

[16] J. Nagata, *Modern General Topology*, 2nd rev'd Edition, North-Holland, Amsterdam, 1985.

[17] ———, *A survey of metrization theory II*, Q & A Gen. Top., **10** (1992), 15-30.

[18] P.J. Nyikos, *Some surprising base properties in topology*, Studies in Topology, ed. Stavrakas and Allen, Academic Press, 1975.

[19] I.S. Stares, *Borges normality and generalised metric spaces*, Top. Proc., **19** (1994), 277-305.

[20] ———, *Concerning the Dugundji Extension property*, Top. Appl., **63** (1995), 165-172.

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