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A CHARACTERIZATION OF 1-DIMENSIONAL NÖBELING SPACES

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Abstract

For a non-negative integer n the Nöbeling space N_n^{2n+1} is the n-dimensional analogue of Hilbert space. It is a topologically complete, separable, n-dimensional, absolute extensor in dimension n with the property that any mapping of an at most n-dimensional, topologically complete, separable, metric space into N_n^{2n+1} can be approximated arbitrarily closely by a closed embedding. It has been widely conjectured that these properties topologically characterize N_n^{2n+1}. The conjecture is well-known to be true for $n = 0$. In this paper we prove it for $n = 1$. We also prove a Z-set unknotting theorem for N_1^3.

1. Introduction

All spaces considered in this paper are separable and metrizable.

A Hilbert cube is a space homeomorphic to I^ω ($I = [0, 1]$), the Cartesian product of countably many closed intervals. Torunczyk (see [10]) showed that a metric space X is a Hilbert cube if and only if it has the following properties:

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(1_{HC}) compact,

(2_{HC}) absolute retract,

(3_{HC}) any map of any metric compactum into X can be approximated arbitrarily closely by an embedding.

The finite dimensional analogues of the Hilbert space are the Menger cubes M_n^{2n+1}, $n \geq 0$. Bestvina [3] has proved that a metric space X is homeomorphic to M_n^{2n+1} if and only if X has the following properties:

(1_M) compact,

(2_M) dimension n,

(3_M) $AE(n) = LC^{n-1} \cap C^{n-1}$, i.e. absolute extensor in dimension n (see [5]),

(4_M) any map of any at most n-dimensional metric compactum into X can be approximated arbitrarily closely by an embedding.

The topologically complete analogue of the Hilbert cube is Hilbert space. Torunczyk [9] proved that a separable metric space X is homeomorphic to Hilbert space if and only if it has the properties:

(1_{HS}) Polish,

(2_{HS}) AE,

(3_{HS}) every map of a Polish space into X can be approximated arbitrarily closely by a closed embedding.

The finite dimensional analogues of the Hilbert space are the Nöbeling spaces N_n^{2n+1}. The space N_n^{2n+1} is defined to be the set of points in Euclidean $(2n+1)$-space R^{2n+1} at most n of whose coordinates are rational. It is well-known that N_n^{2n+1} has the properties:

(1_N) Polish,

(2_N) dimension n,

(3_N) $AE(n)$,

(4_N) any map of any at most n-dimensional Polish space into X can be approximated arbitrarily closely by a closed embedding.

It has long been known [1] that the space of irrational numbers N_0^1 is topologically characterized by (1_N)-(4_N). Until very recently virtually nothing more was known about positive dimensional Nöbeling spaces.

It is well-known that the Cartesian product $(0,1)^\omega$ is homeomorphic to Hilbert space and is a pseudo-interior in the Hilbert cube I^ω. In [6] it was proved that the n-dimensional Nöbeling space is embeddable as a pseudo-interior of the n-dimensional Menger cube M_n^{2n+1}. It follows for example that Nöbeling space is homogeneous. Hence, the Nöbeling space should pick up most of the good properties of the Menger cube.

It has long been conjectured (see for example [11]) that the Nöbeling space is characterized topologically by (1_N)-(4_N). In this paper we prove that this conjecture is true in the case $n = 1$. We also prove a Z-set unknotting theorem for the 1-dimensional Nöbeling space (see Section 5). We follow [10] in most notations and our approach uses Bing partitioning [4] as did Anderson [2] (see also [8]) in the characterization of the Menger curve. Since we are dealing with non-compact spaces it is more convenient to use infinite partitions rather that the finite partitions which are used in the compact case. The reader may consult [7] for a summary of results on Menger and Nöbeling spaces.

2. Partitions

Definition 2.1. *We say that a space X is an n-dimensional Nobeling-type space or an $\mathcal{N}_n$-space or $X \in \mathcal{N}_n$ if it satisfies the properties: (1_N)-(4_N).*

The main result of this paper is:

Theorem 2.2. *Every pair of $\mathcal{N}_1$-spaces are homeomorphic.*

In this section we will introduce the notion of brick partitions (which is the main tool for proving Theorem 2.2) and give a proof of Theorem 2.2 based on some propositions which will be proved in the following sections.

We recall that a subspace Z of a space X is said to be a Z-set (a strong Z-set) in X if Z is closed and the identity map can be approximated arbitrarily closely by a map $f : X \longrightarrow X$ such that $f(X) \cap Z = \emptyset$ ($\mathrm{cl} f(X) \cap Z = \emptyset$). In Nöbeling -type spaces every Z-set is also a strong Z-set (see [5]). We will write $Z \in Z(X)$ to indicate that Z is a Z-set in X. Clearly if $X \in \mathcal{N}_n$ then $Z \in Z(X)$ if and only if every map of every at most n-dimensional Polish space into X can be arbitrarily closely approximated by closed embeddings not intersecting Z. It is well-known ([5]) that if F is a closed subset of an at most n-dimensional Polish space Y, $X \in \mathcal{N}_n$ and $f : F \longrightarrow X$ is a Z-embedding (= a closed embedding with Z-set image in X) then f can be extended to a Z-embedding of Y. Moreover, any continuous extension of f can be approximated arbitrarily closely by an extension which is a Z-embedding.

Definition 2.3. *A closed subset A of an $\mathcal{N}_1$-space X is called a brick if A is a regular closed set (i.e.* $\mathrm{cl}(\mathrm{int} A) = A$*), A is an $\mathcal{N}_1$-space with ∂A homeomorphic to the set of irrationals and $\partial A \in Z(A)$.*

A family $\mathcal{A}$ of bricks is called a brick partition of X if $\mathcal{A}$ is a locally finite cover of X of order 2 and distinct elements of $\mathcal{A}$ have pairwise disjoint interiors.

We say that a brick partition $\mathcal{A}$ is in general position with a subset Z of X if for every $A \in \mathcal{A}$, $Z \cap \partial A = \emptyset$.

Proposition 2.4. *Let $\mathcal{A}$ be a brick partition of an $\mathcal{N}_1$-space X and let for every $A \in \mathcal{A}$, $\mathcal{A}_A$ be a brick partition of A which is in general position with ∂A. Then $\cup\{\mathcal{A}_A : A \in \mathcal{A}\}$ is a brick partition of X.*

Proof. Obvious. □

Proposition 2.5. *Let X be an $\mathcal{N}_1$-space, let Z be a 0-dimensional Z-set in X and let $\mathcal{U}$ be an open cover of X. Then there exists a brick partition $\mathcal{A}$ of X which is in general position with Z and such that $\mathcal{A}$ refines $\mathcal{U}$.*

Proof. The proof is given in Section 3. □

Proposition 2.6. *Let X_1 and X_2 be $\mathcal{N}_1$-spaces, let $Z_1 \in Z(X_1)$ and $Z_2 \in Z(X_2)$ be 0-dimensional and homeomorphic, and let $\mathcal{A}_1$ be a brick partition of X_1 which is in general position with Z_1. Then for every homeomorphism $f : Z_1 \longrightarrow Z_2$ there exist a brick partition partition $\mathcal{A}_2$ of X_2 and a $1-to-1$ correspondence $\mathcal{F} : \mathcal{A}_1 \longrightarrow \mathcal{A}_2$ such that*

(i) $\mathcal{A}_2$ is in general position with Z_2,

(ii) $\mathcal{F}$ preserves the combinatorial structure of $\mathcal{A}_1$ and $\mathcal{A}_2$, i.e. for $A, B \in \mathcal{A}_1$, $\mathcal{F}(A) \cap \mathcal{F}(B) = \emptyset$ if and only if $A \cap B = \emptyset$,

(iii) $\mathcal{F}$ agrees with f, i.e. for $A \in \mathcal{A}_1$, $\mathcal{F}(A) \cap Z_2 = f(A \cap Z_1)$.

Proof. The proof is given in Section 4. □

Proof of Theorem 2.2. Let X_1 and X_2 be $\mathcal{N}_1$-spaces. We will prove that a homeomorphism $f : Z_1 \longrightarrow Z_2$ of 0-dimensional Z-subsets of X_1 and X_2 respectively can be extended to a homeomorphism of X_1 and X_2. Fix complete bounded metrics on X_1 and X_2 such that diam $X_1 < 1$ and diam $X_2 < 1$, and by Proposition 2.5 take a brick partition $\mathcal{A}_1^1$ of X_1 with mesh $\mathcal{A}_1^1 < 1/2$ and such that $\mathcal{A}_1^1$ is in general position with Z_1. Then by Proposition 2.6 there exist a brick partition $\mathcal{A}_2^1$ of X_2 and a correspondence $\mathcal{F}_1 : \mathcal{A}_1^1 \longrightarrow \mathcal{A}_2^1$ such that $\mathcal{F}_1$ preserves the combinatorial structure of $\mathcal{A}_1^1$ and $\mathcal{A}_2^1$, $\mathcal{A}_2^1$ is in general position with Z_2 and $\mathcal{F}_1$ agrees with f. Extend f to a homeomorphism $f_1 : Z_1^1 = Z_1 \cup (\cup\{\partial A : A \in \mathcal{A}_1^1\}) \longrightarrow Z_2^1 = Z_2 \cup (\cup\{\partial A : A \in \mathcal{A}_2^1\})$ such that $f_1(A \cap B) = \mathcal{F}_1(A) \cap \mathcal{F}_1(B)$ for $A \neq B \in \mathcal{A}_1^1$.

Let $B \in \mathcal{A}_2^1$ and $A = (\mathcal{F}_1)^{-1}(\{B\})$. Applying Proposition 2.5 take a brick partition $\mathcal{A}_B$ of B which is in general position with $Z_2^1 \cap B$ and such that mesh $\mathcal{A}_B < 1/3$. By Proposition 2.6 there exist a brick partition $\mathcal{A}_A$ of A and a 1-to-1 correspondence $\mathcal{F}_A : \mathcal{A}_A \longrightarrow \mathcal{A}_B$ such that $\mathcal{F}_A$ preserves the combinatorial structure of $\mathcal{A}_A$ and $\mathcal{A}_B$, $\mathcal{A}_A$ is in general position with $Z_1^1 \cap A$ and $\mathcal{F}_A$ agrees with $f_1|_{Z_1^1 \cap A}$. Define $\mathcal{A}_1^2 = \cup\{\mathcal{A}_A : A \in \mathcal{A}_1^1\}$, $\mathcal{A}_2^2 = \cup\{\mathcal{A}_B : B \in \mathcal{A}_2^1\}$ and let $\mathcal{F}_2 : \mathcal{A}_1^2 \longrightarrow \mathcal{A}_2^2$ be defined by $\mathcal{F}_A$ on every $\mathcal{A}_A$. By Proposition 2.4, $\mathcal{A}_1^2$ and $\mathcal{A}_2^2$ are brick partitions of X_1 and X_2 respectively. Clearly $\mathcal{F}_2$ is 1-to-1, $\mathcal{F}_2$ preserves the combinatorial structure of $\mathcal{A}_1^2$ and $\mathcal{A}_2^2$, $\mathcal{A}_1^2$ and $\mathcal{A}_2^2$ are in general position with Z_1 and Z_2 respectively and $\mathcal{F}_2$ agrees with f.

Proceed by induction and construct for every n a brick partition $\mathcal{A}_i^n$, $i = 1, 2$ of X_i and a 1-to-1 correspondence $\mathcal{F}_n$: $\mathcal{A}_1^n \longrightarrow \mathcal{A}_2^n$ such that $\mathcal{A}_i^{n+1}$ refines $\mathcal{A}_i^n$, mesh $\mathcal{A}_i^n < 2/n$, $\mathcal{F}_n$ preserves the combinatorial structure of $\mathcal{A}_1^n$ and $\mathcal{A}_2^n$, $\mathcal{F}_n$ agrees with f and $\mathcal{F}_{n+1}(A) \subset \mathcal{F}_n(B)$ if and only if $A \subset B$.

Define $f' : X_1 \longrightarrow X_2$ by $f'(x) = \cap\{\mathcal{F}_n(A) : x \in A, A \in \mathcal{A}_1^n, n \in \mathbb{N}\}$. Then f' is well-defined and f' is a homeomorphism which extends f. □

3. Proof of Proposition 2.5

First we prove some auxiliary propositions and lemmas, and in the end of this section we prove Proposition 2.5.

Proposition 3.1. *Let X be an $\mathcal{N}_n$-space and let Z be a Z-set in X. Then a map f of a Polish space Y of* dim $\leq n$ *into X which is a closed embedding on a closed subset F of Y can be arbitrarily closely approximated by a closed map f' from Y to X such that f' coincides with f on F, $f'(Y) \cap Z = f(F) \cap Z$ and for every closed $A \subset Y \setminus F$, $f'|_A$ is a Z-embedding. Moreover, if $f(F)$ is a Z-set then f' can be chosen to be a Z-embedding.*

Proof. If $f(F)$ is a Z-set then denote by $Y_1 \supset Y$ the space obtained from Y by attaching Z to Y and identifying the points of $Z \cap f(F)$ and $F \cap f^{-1}(Z)$ by f. Let $f_1 : Y_1 \longrightarrow X$ be the extension of f sending $Z \subset Y_1$ to $Z \subset X$ by the identity. Now f_1 can be arbitrarily closely approximated by a Z-embedding f_1' such that f_1' coincides with f_1 on $F \cup Z$ (see the remark following Theorem 2.2). Denote $f' = f_1'|_Y$ and we are done.

If $f(F)$ is not a Z-set then fix complete metrics on X and Y for which we use the same notation d. Since $f|_F$ is a closed embedding we may assume that $d(f(x), f(y)) = d(x, y)$ for $x, y \in F$.

Then there exists an open neighborhood V_1 of F such that such that $d(f(x), f(y)) > 1/2$ for every $x, y \in V_1$ with $d(x, y) > 1$. Indeed, let $\mathcal{V}$ be an open cover of X with mesh$\mathcal{V} < 1/8$. Define

$$V_1 = \cup\{f^{-1}(V) \cap O(f^{-1}(V) \cap F, 1/8) : V \cap f(F) \neq \emptyset, V \in \mathcal{V}\}$$

where $O(A, \epsilon)$ stands for the open ϵ-neighborhood of A. Then V_1 has the required property.

Set $f_1 = f$, $F_1 = Y \setminus V_1$ and approximate $f_1|_{F_1}$ by a Z-embedding $g_1 : F_1 \longrightarrow X$ such that $g_1(F_1) \cap Z = \emptyset$ and g_1 is so close to $f_1|_{F_1}$ that g_1 can be extended to a map $f_2 : Y \longrightarrow X$ with $f_2|_F = f|_F$ $(= f_1|_F)$ and such that $\text{dist}(f_2, f_1) \leq 1/2$.

Then there exists an open neighborhood $V_2 \subset V_1$ of F such that $d(f_2(x), f_2(y)) > 1/4$ for every $x, y \in V_2$ with $d(x, y) > 1/2$. Set $F_2 = Y \setminus V_2$. Extend $g_1 : F_1 \longrightarrow X$ to a Z-embedding $g_2 : F_2 \longrightarrow X$ such that $g_2(F_2) \cap Z = \emptyset$ and g_2 is so close to $f_2|_{F_2}$ that g_2 can be extended to a map $f_3 : Y \longrightarrow X$ with $f_3|_F = f|_F$ and such that $\text{dist}(f_3, f_2) \leq 1/2^2$.

Proceed by induction and construct for every m a map $f_m : Y \longrightarrow X$ and an open neighborhood V_m of F such that for $F_m = Y \setminus V_m$ we have $f_m|_F = f|_F$, $f_{m+1}|_{F_m} = f_m|_{F_m}$, $f_{m+1}|_{F_m}$ is a Z-embedding, $d(f_m(x), f_m(y)) > 1/(2m)$ for every $x, y \in V_m$ with $d(x, y) > 1/m$ and $\text{dist}(f_{m+1}, f_m) \leq 1/2^m$. We may also assume that $V_{m+1} \subset V_m$ and $\cap V_m = F$.

Define $f' = \lim f_m$. Then $f'|_F = f|_F$, $f'|_{F_m}$ is a Z-embedding and $d(f'(x), f'(y)) > 1/(2m) - 1/2^{m-2}$ for every $x, y \in V_m$ with $d(x, y) > 1/m$. It is not difficult to check that the last property implies that f' is a closed map. Since for every m, f_{m+1} can be taken arbitrarily close to f_m, f' can also be constructed to be arbitrarily close to f. □

Proposition 3.2. *Let X be an $\mathcal{N}_1$-space and let $\mathcal{U}$ be an open cover of X. Then there exist two discrete families $\mathcal{V}_1$ and $\mathcal{V}_2$ of open connected sets such that $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2$ covers X and refines $\mathcal{U}$.*

Proof. Take discrete families of closed sets $\mathcal{U}_1$ and $\mathcal{U}_2$ such that $\mathcal{U}_1 \cup \mathcal{U}_2$ covers X and refines $\mathcal{U}$. Let U' and U'' be open enlargements of $U \in \mathcal{U}_i$ such that cl$U'' \subset U'$, $\mathcal{U}_i^{'} = \{U' : U \in \mathcal{U}_i\}$, $i = 1, 2$ are discrete and $\mathcal{U}_1^{'} \cup \mathcal{U}_2^{'}$ refines $\mathcal{U}$.

Let $e_U : U'' \longrightarrow X$ be the inclusion. By Proposition 3.1 e_U can be approximated by a closed map $e_U^{'} : U'' \longrightarrow X$ such that $e_U|_U = e_U^{'}|_U$ and $e_U^{'}(U'') \subset U'$. Then the components of $e_U^{'}(U'')$ form a discrete family of closed sets and hence

$\mathcal{V}_i^{'} = \{A : A$ is a component of $e_U^{'}(U''), U \in \mathcal{U}_i\}$, $i = 1, 2$

are discrete families of closed connected sets such that $\mathcal{V}_1^{'} \cup \mathcal{V}_2^{'}$ refines $\mathcal{U}$ and covers X. Take a connected open discrete enlargement $\mathcal{V}_i$ of $\mathcal{V}_i^{'}$, i=1,2 which refines $\mathcal{U}$ and we are done. □

Lemma 3.3. *Let X be an $\mathcal{N}_1$-space and let F_1 and F_2 be disjoint connected closed subsets of X. Then for every discrete family $\mathcal{U}$ of closed connected subsets of X such that every $U \in \mathcal{U}$ intersects at most one of the sets F_1 and F_2 there exist closed, connected and disjoint enlargements $F_1^{'}$ and $F_2^{'}$ of F_1 and F_2 respectively such that for every $U \in \mathcal{U}$, U either is contained in $F_1^{'}$ or in $F_2^{'}$.*

Proof. Adding to F_i the set $\cup\{U : U \in \mathcal{U}, U \cap F_i \neq \emptyset\}$, $i = 1, 2$ we may assume that no element of $\mathcal{U}$ intersects $F_1 \cup F_2$. Let for every $U \in \mathcal{U}$, U' be a closed connected neighbourhood of U that such $\mathcal{U}' = \{U' : U \in \mathcal{U}\}$ is discrete and no element of $\mathcal{U}'$ intersects $F_1 \cup F_2$. Take disjoint closed connected neighborhoods G_1 and G_2 of F_1 and F_2 which also do not intersect the elements of $\mathcal{U}'$. Let C_1 be the component of $X \setminus G_2$ which contains G_1 and let $C_2 = X \setminus C_1$. Then C_2 is connected, $G_2 \subset C_2$ and for every element $U' \in \mathcal{U}'$, U' either is contained in C_1 or in C_2. Let $F_i^* = F_i \cup (\cup\{U : U \in \mathcal{U}, U \subset C_i\}$, $i = 1, 2$. Then $F_i^* \subset \mathrm{int} C_i$ and for every element $U \in \mathcal{U}$, U is contained in either F_1^* or in F_2^*.

Apply Proposition 3.1 to approximate the inclusion of C_1 into X by a closed map $f_1 : C_1 \longrightarrow X$ such that f_1 is the identity on F_1^*, $f_1(C_1) \setminus F_1^*$ is a sigma Z-set and $f_1(C_1) \cap F_2^* = \emptyset$. Applying Proposition 3.1 again approximate the inclusion of C_2 into X by a closed map $f_2 : C_2 \longrightarrow X$ such that f_2 is the identity on F_2^* and $f_1(C_1) \cap f_2(C_2) = \emptyset$, and set $F_i^{'} = f_i(C_i)$. □

Lemma 3.4. *Let X be an $\mathcal{N}_1$-space and let F_1 and F_2 be disjoint, connected closed subsets of X. Then for every $\epsilon > 0$ there exist closed disjoint connected sets $F_1^{'}$ and $F_2^{'}$ and a discrete family $\mathcal{U}$ of closed connected sets with* mesh$\mathcal{U} < \epsilon$ *such that $F_1 \subset F_1^{'}$, $F_2 \subset F_2^{'}$, the elements of $\mathcal{U}$ do not meet $F_1 \cup F_2$, $\mathcal{U}$ covers $X \setminus (F_1^{'} \cup F_2^{'})$ and every $U \in \mathcal{U}$ intersects both $F_1^{'}$ and $F_2^{'}$.*

Proof. Let $\mathcal{V}_1$ and $\mathcal{V}_2$ be discrete families of closed connected subsets of X such that $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2$ covers X and mesh $\mathcal{V} < \epsilon$. We may assume that $\mathcal{V}$ is so small that $F_1^* = F_1 \cup (\cup\{V : V \in \mathcal{V}, V \cap F_1 \neq \emptyset\})$ and $F_2^* = F_2 \cup (\cup\{V : V \in \mathcal{V}, V \cap F_2 \neq \emptyset\})$ are disjoint and every $V \in \mathcal{V}$ intersects at most one of the sets F_1^* and F_2^*.

By Lemma 3.3 let $F_1^{'}$ and $F_2^{'}$ be disjoint closed and connected enlargements of F_1^* and F_2^* such that $F_1^{'} \cup F_2^{'}$ contains the elements of $\mathcal{V}_1$. We may also assume that $F_1^{'}$ contains the elements of $\mathcal{V}_2$ which do not intersect $F_2^{'}$ and $F_2^{'}$ contains the elements of $\mathcal{V}_2$ which do not intersect $F_1^{'}$. Let $\mathcal{U} = \{V : V \in \mathcal{V}_2, V$ intersects both $F_1^{'}$ and $F_2^{'}\}$ and we are done. □

Lemma 3.5. *Let X, $\mathcal{U}$,F_i, $F_i^{'}$, $i = 1, 2$ be as in Lemma 3.4. Let for each $U \in \mathcal{U}$, U' be a connected closed set not intersecting $F_1 \cup F_2$ and such that $U \subset$* intU' *and $\mathcal{U}' = \{U' : U \in \mathcal{U}\}$ is discrete and of mesh $< \epsilon$. Then for every $U' \in \mathcal{U}'$ there*

exist two closed maps $f_1^{U'}, f_2^{U'} : U' \longrightarrow U'$ *such that* $f_i^{U'}|_{F_i' \cap \partial U'}$, $i = 1, 2$ *is the identity and* $f_1^{U'}(U') \cap f_2^{U'}(U') = \emptyset$.
Denote $F_i'' = (F_i' \setminus (\cup\{U' : U' \in \mathcal{U}'\})) \cup (\cup\{f_i^{U'}(U') : U' \in \mathcal{U}'\})$, $i = 1, 2$.
Then F_1'' *and* F_2'' *are connected disjoint and closed sets containing* F_1 *and* F_2 *respectively and* $X \setminus (F_1'' \cup F_2'')$ *is covered by* $\mathcal{U}'$.

Proof. Fix $U \in \mathcal{U}$ and take closed sets U''' and U'' such that $U \subset \text{int}U''' \subset U''' \subset \text{int}U'' \subset U'' \subset U'$ and $\text{int}U'''$, $\text{int}U''$ are connected. Set $G_i''' = \text{int}((F_i' \cap U''') \cup U)$, $H_i'' = ((F_i' \cap U'') \cup U)$ and $H_i' = ((F_i' \cap U') \cup U)$, $i = 1, 2$.

Replacing U by a larger set with connected interior we may assume that $\text{int}U$ is connected. Then G_i''' is open and connected. Thus G_i''' is $AE(1)$ (an absolute extensor in dimension 1), and hence the inclusions $e_1''' : G_1''' \setminus (\text{int}H_2') \longrightarrow G_1'''$ and $e_2''' : G_2''' \setminus (\text{int}H_1') \longrightarrow G_2'''$ can be extended to maps $f_1''' : G_1''' \cup H_2' \longrightarrow G_1'''$ and $f_2''' : G_2''' \cup H_1' \longrightarrow G_2'''$.

Let $f_i'' : U' \longrightarrow H_i'$, $i = 1, 2$ be the map which is the identity on $H_i' \setminus G_i'''$ and coincides with f_i''' on $G_1''' \cup H_2'$ and $G_2''' \cup H_1'$ for $i = 1$ and $i = 2$ respectively.

By Proposition 3.1 approximate f_1'' by a closed map $f_1' : U' \longrightarrow U'$ such that f_1' is the identity on $F_1' \cap \partial U'$, $f_1'(U') \cap F_2' \cap \partial U' = \emptyset$ and $f_1'(U') \cap H_2''$ is a Z-set. Applying Proposition 3.1 again approximate f_2'' by a closed map $f_2' : U' \longrightarrow U'$ such that $f_2'(U') \cap f_1'(U') = \emptyset$ and f_2' is the identity on $F_2' \cap \partial U'$. □

Lemma 3.6. *Let an* $\mathcal{N}_1$*-space* X *be the union* $X = A \cup B$ *of two closed regular subsets* A *and* B *such that* $\text{int}A \cap \text{int}B = \emptyset$ *and* $\partial A = \partial B = A \cap B$ *is* 0*-dimensional. Then* $A \cap B$ *is homeomorphic to the set of irrationals.*

Proof. Assume that $A \cap B$ is not homeomorphic to the set of irrationals. Then there exists a point $x \in A \cap B$ such that x has a compact neighborhood in $A \cap B$. Take connected open neighborhoods U and U' of x in X such that $\text{cl}U \subset U'$ and

$A \cap B \cap \mathrm{cl}U'$ is compact. Let $Y = \mathbb{N} \times U$ and let $f : Y \longrightarrow X$ be the map sending $\mathbb{N} \times \{u\}$ to u for $u \in U$. Take $x_A \in (A \setminus B) \cap U$ and $x_B \in (B \setminus A) \cap U$. Approximate f by a closed embedding $f' : Y \longrightarrow X$ such that $f'(\mathbb{N} \times \{x_A\}) \subset A \setminus B$, $f'(\mathbb{N} \times \{x_B\}) \subset B \setminus A$ and $f'(Y)$ does not intersect $X \setminus U'$. Then $X_i = f'(\{i\} \times U) \cap A \cap B \cap \mathrm{cl}U' \neq \emptyset$ for every $i \in \mathbb{N}$ and hence $\{X_i : i \in \mathbb{N}\}$ is an infinite discrete family of closed subsets of the compact set $A \cap B \cap \mathrm{cl}U'$. This contradiction proves the lemma. □

Proposition 3.7. *Let X be an $\mathcal{N}_1$-space, let Z be a 0-dimensional Z-set in X and let F_1 and F_2 be disjoint connected closed subsets of X. Then there exists a 2-element brick partition $\mathcal{A} = \{A_1, A_2\}$ of X which is in general position with Z and such that $F_i \subset \mathrm{int}A_i$, $i = 1, 2$.*

Proof. If $Z \neq \emptyset$ then take disjoint closed connected neighborhoods F_1^* and F_2^* of F_1 and F_2. Decompose Z into a discrete sequence of closed subsets $Z = Z_1 \cup Z_2 \cup \ldots$ such that each Z_i has a connected closed neighborhood V_i which intersects at most one of the sets F_1^* and F_2^*. Let Y be the free union of the sets V_i, $i = 1, 2, \ldots$ and let $f : Y \longrightarrow X$ be the map which sends V_i to V_i by the identity. Approximate f by a closed embedding f' which is the identity on $\cup Z_i$ and such that every $f'(V_i)$ intersects at most one of the sets F_1 and F_2. By Lemma 3.3 we can replace F_1 and F_2 by larger closed connected and disjoint sets whose union contains $\cup f'(V_i)$ and, hence, contains Z. Thus we may assume that $Z = \emptyset$.

It is easy to see that applying Lemmas 3.4 and 3.5 one can construct closed connected sets F_1^n, F_2^n and discrete families $\mathcal{U}_n$ of closed connected sets such that:

(i) $F_i^n \subset \mathrm{int}F_i^{n+1}$ and $F_i \subset F_i^1$;

(ii) $\mathcal{U}_{n+1}$ refines $\mathcal{U}_n$, mesh $\mathcal{U}_n < 1/n$, $\mathcal{U}_n$ covers $X \setminus (F_1^n \cup F_2^n)$ and every element of $\mathcal{U}_n$ meets both F_1^n and F_2^n;

(iii) for every $U \in \mathcal{U}_n$ there exists a closed map $f_i^U : U \longrightarrow U \cap F_i^n$, $i = 1, 2$ which is the identity on $F_i^n \cap \partial U$.

Denote $A_i = \mathrm{cl}(\cup_n F_i^n)$. Then $F_i \subset \mathrm{int}A_i$ and we will show that $\mathcal{A} = \{A_1, A_2\}$ is a brick partition of X. Clearly A_i is a regular set and $\dim(A_1 \cap A_2) = 0$. Hence by Lemma 3.6, ∂A_i is homeomorphic to the set of irrationals.

Fix i. Let $\mathcal{U}$ be an open cover of A_i and let $g : Y \longrightarrow A_i$ be a map of a 1-dimensional Polish space into A_i. Define
$\mathcal{U}_1^* = \{U : U \in \mathcal{U}_1, A_i \cap U$ is contained in an element of $\mathcal{U}\}$
and by induction on n define
$\mathcal{U}_n^* = \{U : U \in \mathcal{U}_n, A_i \cap U$ is contained in an element of $\mathcal{U}$ and U is not contained in the elements of $\mathcal{U}_1^* \cup \mathcal{U}_2^* \cup ... \cup \mathcal{U}_{n-1}^*\}$.

Let $\mathcal{U}^* = \cup_n \mathcal{U}_n^*$ and denote by f the map $f : A_i \longrightarrow A_i$ which is the identity on $A_i^* = A_i \setminus (\cup\{U : U \in \mathcal{U}^*\})$ and for every $U \in \mathcal{U}^*$ and every $x \in U \cap A_i$, $f(x) = f_i^U(x)$. Since $\mathcal{U}^*$ is discrete, f is continuous. Clearly g and $f \circ g$ are $\mathcal{U}$-close in A_i and $\mathrm{cl}f(A_i) \cap \partial A_i = \emptyset$. Then since X is an $\mathcal{N}_1$-space, $f \circ g$ can be arbitrarily closely approximated by a closed embedding into A_i with the range not intersecting ∂A_i. This shows both that A_i is an $\mathcal{N}_1$-space and that ∂A_i is a Z-set in A_i. Thus $\mathcal{A}$ is a brick partition. □

Proof of Proposition 2.5. By Proposition 3.2 take two discrete families $\mathcal{V}_1$ and $\mathcal{V}_2$ of closed connected sets such that $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2$ refines $\mathcal{U}$ and covers X. Also take for every $V \in \mathcal{V}$ an open neighborhood V' such that $\mathcal{V}_i' = \{V' : V \in \mathcal{V}_i\}$, $i = 1, 2$ is discrete and $\mathcal{V}' = \mathcal{V}_1' \cup \mathcal{V}_2'$ refines $\mathcal{U}$.

For $V \in \mathcal{V}$ define $F_V = V \setminus (\cup\{U' : U \in \mathcal{V}, U \neq V\}$. We may assume that $F_V \neq \emptyset$ for every $V \in \mathcal{V}$. Then $\{F_V : V \in \mathcal{V}\}$ is discrete. Let Y be the free union of the sets V, $V \in \mathcal{V}$ and let $f : Y \longrightarrow X$ be the map sending V to V by the identity for each $V \in \mathcal{V}$. Denote $F = \cup\{F_V : V \in \mathcal{V}\}$. It is not difficult to see that applying Proposition 3.1 f can be approximated by a closed map f' which is the identity on F and such that the family $\{f'(V) : V \in \mathcal{V}\}$ is discrete and $f'(V) \subset V'$ for $V \in \mathcal{V}$.

By Proposition 3.7 for every pair $V, U \in \mathcal{V}$, $V \neq U$ take a two element brick partition $\{A_{VU}, A_{UV}\}$ which is in general position with Z and such that $f'(V) \subset A_{VU}$ and $f'(U) \subset A_{UV}$. For $V \in \mathcal{V}$ define

$$A_V = f'(V) \cup (\cup\{(U' \cap V' \cap A_{VU} : U \in \mathcal{V}, U \neq V\}).$$

Since $f'(V) \cup (U' \cap V' \cap A_{VU})$ is connected for every $U \in \mathcal{V}$, $U \neq V$, A_V is also connected. It is easy to check that A_V is a brick and, hence, $\mathcal{A} = \{A_V : V \in \mathcal{V}\}$ is the desired brick partition. □

4. Proof of Proposition 2.6

For proving Proposition 2.6 we need the following propositions and constructions.

Proposition 4.1. *Let $\mathcal{A}$ be a brick partition of an $\mathcal{N}_1$-space X which is in general position with a subset Z of X and let $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \ldots$ be a decomposition of $\mathcal{A}$ into a sequence of disjoint subfamilies $\mathcal{A}_i$ such that $B_i = \cup\{A : A \in \mathcal{A}_i\}$ is connected for every i. Then $\mathcal{B} = \{B_1, B_2, \ldots\}$ is a brick partition of X which is in general position with Z.*

Proof. The proof follows directly from the definition. □

Construction 4.2. *A connectification of a closed set avoiding a Z-set.*

Let X be an $\mathcal{N}_1$-space, let F be closed subset of X and let Z be a Z-set in X such that $F \cap Z = \emptyset$. By Proposition 3.1 approximate the identity on X by a closed map $f : X \longrightarrow X$ which is the identity on F and $f(X) \cap Z = \emptyset$. Then for every connected closed set C containing F, $f(C)$ is a connected closed set which contains F and does not intersect Z. If F is a Z-set then f can be chosen to be a Z-embedding.

Construction 4.3. *A realization of a nerve avoiding a Z-set.*

Let X be a $\mathcal{N}_1$-space and let $\mathcal{A}$ be a brick partition of X which is in general position with a set $Z \in \mathcal{Z}(X)$. For every $A \in \mathcal{A}$ pick a point $x_A \in (\mathrm{int}A)\setminus Z$ and for every pair $A, B \in \mathcal{A}$ with $A \neq B$ and $A \cap B \neq \emptyset$ pick a point $x_{AB} = x_{BA} \in A \cap B$. Set $Z_A = A \cap Z$.

By Proposition 3.1 extend the embedding of $G_A = \{x_{AB} : B \in \mathcal{A}, A \neq B, A \cap B \neq \emptyset\}$ to a Z-embedding of the cone N_A over G_A into A which does not intersect Z_A and such that $\partial A \cap N_A = G_A$ and the vertex of the cone is sent to x_A, i.e. we realize N_A as a subset of A.

Define $N = \cup\{N_A : A \in \mathcal{A}\}$. It is easy to see that N is a Z-set in X. We will refer to N as a (Z-)realization of the nerve of A.

Note that we can construct an infinite discrete family of Z-realizations of the nerve of $\mathcal{A}$. Indeed, take discrete families of points x^n_{AB}, x^n_A, $n \in \mathbb{N}$ such that $x^n_A \in \mathrm{int}A \setminus Z$ and $x^n_{AB} = x^n_{BA} \in A \cap B$ for $A, B \in \mathcal{A}$, $A \neq B$ and $A \cap B \neq \emptyset$. Define $f_A : \mathbb{N} \times G_A \longrightarrow A$ by $f_A(n, x_{AB}) = x^n_{AB}$ and extend f_A to a Z-embedding $g_A : \mathbb{N} \times N_A \longrightarrow A$ avoiding Z_A and such that $g_A(\{n\} \times N_A) \cap \partial A = \{x^n_{AB} : B \in \mathcal{A}, A \neq B\}$ and $g_A(n, x_A) = x^n_A$. Let $g : \mathbb{N} \times N \longrightarrow X$ coincide with g_A on each $\mathbb{N} \times N_A$. Then $\{g(\{n\}\times N) : n \in \mathbb{N}\}$ is a discrete family of Z-realizations of the nerve of $\mathcal{A}$ avoiding Z. □

Construction 4.4. *A connectification of an amalgam.*

Let $\mathcal{A}$ be a brick partition of an $\mathcal{N}_1$-space X which is in general position with a Z-subset Z of X and let $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2 \cup ...$ be a decomposition of $\mathcal{A}$ into a sequence of disjoint subfamilies $\mathcal{A}_k$, $k = 1, 2, ...$ Set $B_k = \cup\{A : A \in \mathcal{A}_k\}$ and $\mathcal{B} = \{B_1, B_2, ...\}$.

Note that B_i may be not connected. We will construct a brick partition $\mathcal{B}' = \{B_1', B_2', ...\}$ which is in general position with Z and such that $\mathcal{B}'$ has the same combinatorial structure as $\mathcal{B}$ and $B' \cap Z = B \cap Z$ for every $B \in \mathcal{B}$.

Let N be a realization of the nerve of $\mathcal{A}$ avoiding Z. Take two bricks $A, B \in \mathcal{A}_k$ which lie in different components of B_k

and let $A_1, A_2, ..., A_n$ be a finite chain of bricks in $\mathcal{A}$ connecting $A_0 = A$ and $A_{n+1} = B$, i.e. $A_i \cap A_j \neq \emptyset$ if and only if $|i - j| \leq 1$. Let $I = I_{AB}$ be the arc of the nerve N connecting the vertices $x_{A_0}, x_{A_0A_1}, x_{A_1}, x_{A_1A_2}, ..., x_{A_{n+1}}$. Let $G = G_{AB}$ be a small neighborhood of I with closure contained in $\mathrm{int}(A_0 \cup A_2 ... \cup A_{n+1})$ and such that $G \cap Z = \emptyset$.

We will connect A_0 and A_{n+1} by modifying A_i to $A_i^{'}$ such that

(i) $\mathcal{A}' = (\mathcal{A} \setminus \{A_0, A_1, ..., A_{n+1}\}) \cup \{A_0^{'}, A_2^{'}, ..., A_{n+1}^{'}\}$ is a brick partition of X,

(ii) $A_i \setminus G = A_i^{'} \setminus G$,

(iii) $A_i^{'} \cap A_j^{'} \cap G \neq \emptyset$ if and only if either $|i - j| \leq 1$ or $i = 0$ and $j = n + 1$.

Let a set F containing $x_{A_nA_{n+1}}$ be clopen in $A_n \cap A_{n+1}$ and contained in G. Also assume that F is so small that $(G \cap A_n \cap A_{n+1}) \setminus F \neq \emptyset$ and there is a connectification M of F in $A_n \cap G$ such that $M \in Z(A_n)$ and $M \cap \partial A_n = F$ (apply Construction 4.2).

Set $A_{n+1}^{'} = A_{n+1}$ and $A_n^* = A_n$. Then $M_n^{''} = (I \cap A_n^*) \cup M \in Z(A_n^*)$ and $M_n^{''}$ is connected. Clearly $M_n = (A_n^* \setminus G) \cup ((A_{n+1}^{'} \cap A_n^*) \setminus F)$ is closed and does not intersect $M_n^{''}$. Hence there is a closed connectification $M_n^{'} \subset A_n^*$ of M_n which does not intersect $M_n^{''}$. By Proposition 3.7 there exists a two element brick partition $\{A_n^{'}, A_n^{''}\}$ of A_n^* which is in general position with ∂A_n^* and such that $M_n \subset M_n^{'} \subset A_n^{'}$ and $M_n^{''} \subset A_n^{''}$.

Set $A_{n-1}^* = A_{n-1} \cup A_n^{''}$. Then A_{n-1}^* is a brick, $M_{n-1}^{''} = (I \cap A_{n-1}^*) \cup M \in Z(A_{n-1}^*)$, $M_{n-1}^{''}$ is connected, $(\partial A_{n-1}^*) \cap M_{n-1}^{''} = F$ and F is clopen in ∂A_{n-1}^*. Clearly $M_{n-1} = (A_{n-1}^* \setminus G) \cup ((A_n^{'} \cap A_{n-1}^*) \setminus F)$ is closed and does not intersect $M_{n-1}^{''}$. Hence there is a closed connected set $M_{n-1}^{'} \subset A_{n-1}^*$ containing M_{n-1} and not intersecting $M_{n-1}^{''}$. Take a two element brick partition $\{A_{n-1}^{'}, A_{n-1}^{''}\}$ of A_{n-1}^* which is in general position with ∂A_{n-1}^* and such that $M_{n-1} \subset M_{n-1}^{'} \subset A_{n-1}^{'}$ and $M_{n-1}^{''} \subset A_{n-1}^{''}$.

Set $A_{n-2}^* = A_{n-2} \cup A_{n-1}^{''}$ and proceed by induction and finally set $A_0^{'} = A_0 \cup A_1^{''}$. It is not difficult to see that $A_i^{'}$,

$i = 0, n+1$ satisfy the conditions (i)-(iii).

Now we will show that connecting separately the elements of $\mathcal{A}_k$ we will get a brick partition.

Take an infinite discrete family N_{AB}, $A \neq B \in \mathcal{A}$ of realizations of the nerve of $\mathcal{A}$ avoiding Z. We may take the arc I_{AB} used for connecting the elements $A, B \in \mathcal{A}_k$, $k = 1, 2, \ldots$ to be in N_{AB}. Then the family of I_{AB}'s is also discrete. Suppose that the open neighborhood G_{AB} of I_{AB} is taken such that in addition to the properties described above the family of G_{AB}'s is discrete. Let $I_{\mathcal{A}} = \cup I_{AB}$ and $G_{\mathcal{A}} = \cup G_{AB}$. Since $I_{\mathcal{A}}$ is a Z-set we may also assume that $A \setminus G_{\mathcal{A}}$ is connected for each A in $\mathcal{A}$.

Apply the above construction sequentially for every pair of elements of $\mathcal{A}$ that we need to connect. Denote by A' the set obtained from $A \in \mathcal{A}$ after all the modifications. The process of modifying elements of $\mathcal{A}$ is carried out in a locally finite manner and hence in order to show that A' is a brick the only thing that we have to verify is that A' is connected. Since the connected set $A \setminus G_{\mathcal{A}}$ is contained in A' and the construction carried out inside $G_{\mathcal{A}}$ it follows that A' is also connected.

Since $G_{\mathcal{A}} \cap Z = \emptyset$ we have that $A' \cap Z = A \cap Z$ and $\partial A' \cap Z = \partial A \cap Z = \emptyset$ for every $A \in \mathcal{A}$. Thus $\mathcal{A}' = \{A' : A \in \mathcal{A}\}$ is a brick partition of X which is in general position with Z. From the construction it follows that $B_k' = \cup\{A' : A \in \mathcal{A}_k\}$ is connected and $\mathcal{B}' = \{B_1', B_2', \ldots\}$ has the same combinatorial structure as $\mathcal{B}$ and we are done. $\square$

Proof of Proposition 2.6. Let $N \subset X_1$ be a realization of the nerve of $\mathcal{A}_1$ avoiding Z_1. Denote $C_1 = N \cup Z_1$. Extend f to a closed embedding of C_1 onto some $C_2 \subset X_2$ and now embed X_2 into X_1 identifying C_2 with C_1 by the previous embedding. Thus we realize X_2 as a subset of X_1 with $Z_2 = Z_1$ and $C_2 = C_1$ and let us set $Z = Z_2 = Z_1$ and $C = C_1 = C_2$.

Arrange $\mathcal{A}_1$ into a sequence $\mathcal{A}_1 = \{A_1, A_2, \ldots\}$. Let $\mathcal{A}$ be a brick partition of X_2 which is in general position with Z.

Denote $B_i = \cup\{A : A \in \mathcal{A}, A \cap A_i \neq \emptyset$ and $A \cap A_j = \emptyset$ for $j < i\}$ and $\mathcal{B} = \{B_i : i = 1, 2, ...\}$. By Proposition 2.5 $\mathcal{A}$ can be chosen so small that $A_i \cap Z = B_i \cap Z$ and the condition $B_i \cap B_j \neq \emptyset$ implies $A_i \cap A_j \neq \emptyset$, i.e. we do not create additional intersections in $\mathcal{B}$. Now note that we have on $N \subset C$ all the intersections of $\mathcal{A}_1$ and hence we can take the elements of $\mathcal{A}$ so small that we do not miss intersections of $\mathcal{A}_1$, that is $B_i \cap B_j \neq \emptyset$ if and only if $A_i \cap A_j \neq \emptyset$.

Take a connectification $\mathcal{A}_2 = \mathcal{B}'$ of $\mathcal{B}$ in $X = X_2$ as described in Construction 4.4 and we are done. □

5. Remarks

Remark 5.1. *A Z-set unknotting theorem.*

In the proof of Theorem 2.2 we have shown that every homeomorphism $f : Z_1 \longrightarrow Z_2$ of 0-dimensional Z-subsets of $\mathcal{N}_1$-spaces X_1 and X_2, respectively, can be extended to a homeomorphism of X_1 and X_2. This result remains true with no dimensional restriction on Z_1 and Z_2 and it is called a Z-set unknotting theorem. Here we will present a proof of a Z-set unknotting theorem for $\mathcal{N}_1$-spaces.

Fix a complete bounded metric on X_1 such that $\operatorname{diam} X_1 < 1$. Clearly $X_1 \setminus Z_1 \in \mathcal{N}_1$. Take a brick partition $\mathcal{A}_1 = \{A_1, A_2, ...\}$ of $X_1 \setminus Z_1$ such that for every $A \in \mathcal{A}_1$, $\operatorname{diam} A < \operatorname{dist}(A, Z_1)$. Let N be the nerve of $\mathcal{A}$ realized in $X_1 \setminus Z_1$. Then $C_1 = Z_1 \cup N \in Z(X_1)$. Embed C_1 as a Z-set C_2 into X_2 identifying Z_1 with Z_2 by f and embed X_2 into X_1 identifying C_1 and C_2 by the previous embedding. Now we will regard X_2 as a subset of X_1 and let us denote $Z = Z_1 = Z_2$ and $C = C_1 = C_2$. Take a brick partition $\mathcal{A}_2$ of $X_2 \setminus Z$ and denote $B_i = \cup\{A : A \in \mathcal{A}_2, A \cap A_i \neq \emptyset$ and $A \cap A_j = \emptyset$ for $j < i\}$ and $\mathcal{B} = \{B_i : i = 1, 2, ...\}$. We may assume that the elements of $\mathcal{A}_2$ are so small that $\mathcal{B}$ has the same combinatorial structure as $\mathcal{A}_1$ and $\operatorname{diam} A < \operatorname{dist}(A, Z)$ for every $A \in \mathcal{A}_2$.

We will connectify the B_i's paying attention to some properties which allow us to extend the map f.
For every $A \in \mathcal{A}_2$ pick a point $p_A \in \mathrm{int}_{X_2} A$. For every pair of bricks $A, B \in \mathcal{A}_2$ define $\alpha(A, B)$ =inf{diam$L$: L is an arc connecting p_A and p_B in X_2} and take an arc L_{AB} connecting p_A and p_B in X_2 and such that diam$L_{AB} < 2\alpha(A, B)$. It is easy to check that for
$S_n = \cup\{A : A \in \mathcal{A}_2$ such that there is $B \in \mathcal{A}_2$ with $\alpha(A, B) > 1/n$ and such that A and B are contained in the same element of $\mathcal{B}\}$
we have cl$S_n \cap Z = \emptyset$ for each $n = 1, 2, \ldots$ Then since Z is a Z-set in X_2 we may assume using Proposition 3.1 that the L_{AB}'s are chosen such that for
$L_n = \cup\{L_{AB} : \alpha(A, B) > 1/n$ and A and B are contained in an element of $\mathcal{B}\}$
we have cl$L_n \cap Z = \emptyset$ for each $n = 1, 2, \ldots$
Connectify $\mathcal{B}$ applying Construction 4.4 and assuming that each arc I_{AB} is chosen to be contained in the bricks of $\mathcal{A}_2$ intersecting L_{AB}. Let $\mathcal{B}'$ be a connectification of $\mathcal{B}$ obtained in this way (note that the procedure of connectification is carried out in $X_2 \setminus Z$).

An extension $f : X_1 \longrightarrow X_2$ of f will be constructed as follows. Define f on ∂A_i, $A_i \in \mathcal{A}_1$ such that $f|_{\partial A_i} : \partial A_i \longrightarrow \partial B_i'$ is homeomorphism and $f(\partial A_i \cap \partial A_j) = \partial B_i' \cap \partial B_j'$ for every i and j. By the proof of Theorem 2.2 extend $f|_{\partial A_i}$ to a homeomorphism $f|_{A_i} : A_i \longrightarrow B_i'$. Thus we have constructed an extension $f : X_1 \longrightarrow X_2$ and it is not difficult to verify that f is a homeomorphism.

Remark 5.2. *Menger curve.*

By a minor modification the proofs presented in this paper can be used to obtain a characterization of the Menger curve. Since the Menger curve is compact it suffices to use finite partitions. Note that for finite brick partitions Proposition 2.6 can be easily derived from Proposition 3.7. Hence, the work in Section 4 is unnecessary to obtain a characterization of the Menger curve.

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