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FINITE APPROXIMATION OF COMPACT HAUSDORFF SPACES

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Abstract

We consider the question of how to approximate Hausdorff spaces by representing them as the Hausdorff reflection of inverse limits of finite spaces. Our principal theorem states that a T_2-space is approximable in this sense if and only if it is compact. We also consider connectedness in this context and among other results we show that a space is approximable by finite connected spaces if and only if it is a continuum. A number of these results explain and extend theorems of Flachsmeyer [Fl] in a bitopological setting.

0. Motivation

How should one study the properties of a compact Hausdorff space? Over thirty years ago, Flachsmeyer [Fl] suggested one approach, namely, to approximate the compact space by means of finite T_0-spaces. His work went largely unnoticed by topologists (indeed, we were unaware of it until we had obtained

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almost all of our results). Possibly this is because compact Hausdorff spaces are the best the field has to offer, while finite T_0-spaces are T_1 only if they are discrete; thus the finite spaces seemed more difficult to understand than the compact spaces under study.

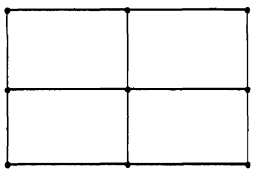
Times change however and nowadays, with a computer on almost every desk, the use of "large finite" objects is pervasive. Whenever you look at a computer screen, you are essentially looking at a finite topological space. Indeed, although the computer screen looks like a product of two intervals, much use is made of the fact that individual minimal dots (pixels) on it can be accessed as elements of the product of two finite 1-dimensional sets. In [KKM] this situation is given a topological description. An interval in the reals is an example of a **connected ordered topological space (COTS)**: a connected topological space in which, among any 3 points is one whose deletion leaves the other two in separate components of the remainder. But there are also finite COTS; except for the two point indiscrete space, these are always homeomorphic to finite intervals of the **Khalimsky line**: the integers, $\mathbb{Z}$, equipped with the topology generated by the subbase $\{\{2n-1, 2n, 2n+1\} : n \in \mathbb{Z}\}$. Thus the screen can be viewed as a product of two finite COTS, or more profitably, as the part of that product consisting of all those points both of whose coordinates are odd (that is, open). This view suggests two conventions which enable us to draw "Euclidean" pictures and interpret them as finite topological spaces:

- apparently featureless sets represent points,
- sets which 'look' open to Euclidean-trained eyes are open.

The following diagrams use these conventions; for example, in the first figure, the portion usually thought of as the infinite open $(0, 1)$ represents a single open point in the three-point COTS.

$\{0,1,2\} \subseteq \mathbb{Z}$, a 3-point COTS: an open point, two closed points in its closure.

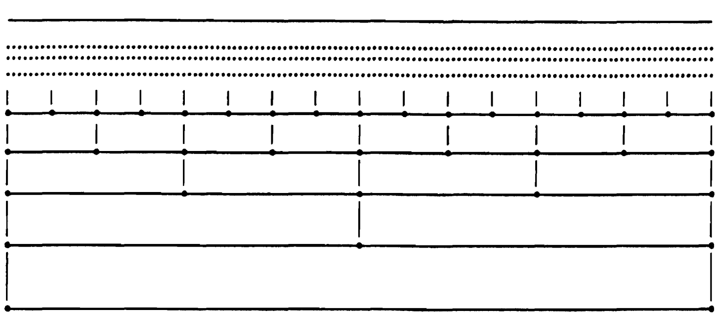
$\{1,2,3\} \subseteq \mathbb{Z}$, a 3-point COTS: a closed point in the closure of each of two open points.



The product of two 5-point COTS: $(\{0,1,2,3,4\}^2, \{0,1,2,3,4\} \subseteq \mathbb{Z})$ It has four open points, nine closed points, six (open,closed) points, and six (closed,open) points.

Another example of use of these conventions is viewing a 2-simplex (triangle) as a set with 7 points: the '2-dimensional' interior, the 3 '1-dimensional' (open segment) edges and the 3 (closed) '0-dimensional' vertices. Certainly we are not limited to use of subsets of $\mathbb{R}^2$; a 3-simplex, for example, is represented by an apparent tetrahedron in $\mathbb{R}^3$, which is seen to have 15 points: 1 open (the 'interior'), 4 faces, 6 edges and 4 vertices. It should be pointed out that the conventions do not always lead to a unique, or even always a good intuitive picture of a finite space. For example, the three-point subspace $\{(0,0),(1,1),(2,0)\} \subseteq \{0,1,2,3,4\}^2$ is homeomorphic to the 3-point COTS $\{0,1,2\}$, where both $\{0,1,2,3,4\}$ and $\{0,1,2\}$ are taken to be subsets of the Khalimsky line with the relative topology.

Our goal in this paper is not simply to view finite spaces, but to use them to approximate others. The next diagram suggests a way to approximate the unit interval by finite spaces. Its top horizontal line represents the unit interval, but those at the bottom are meant to be finite COTS: $D_n = \{\frac{i}{2^n} \mid 0 \leq i \leq 2^n\} \cup \{(\frac{i}{2^n}, \frac{i+1}{2^n}) \mid 0 \leq i < 2^n\}$, with $2^{n+1}+1$ points and the quotient topology induced from $[0,1]$. The vertical lines indicate maps going down, for which a closed point is the image of the one directly above it, while an open point is that of the three above it:



Although a picture may be worth a thousand words, it is not a mathematical theory. In order to develop such a theory of finite and other asymmetric spaces (ones in which $x \leq_\tau y \not\Rightarrow y \leq_\tau x$), we shall restrict our attention to those asymmetric spaces (X, τ), in which a second topology, τ^D (called a **dual** of τ) is defined, such that the bitopological space (X, τ, τ^D) exhibits some 'better' bitopological separation property than the original topological space. For example, in a finite space, the collection of closed sets forms such a topology, which we will call τ^G since it is a special case of the de Groot dual, discussed in the next section. As we develop the theory of finite approximations, we will see that if (X, τ) is a finite T_0-space then the bitopological space (X, τ, τ^G) mimics properties of a compact Hausdorff space in a way that will be made clear below. Unfortunately, (X, τ, τ^G) also exhibits a type of zero-dimensionality ($\tau \vee \tau^G$ is discrete) and it requires some work to reconcile this fact.

In the next section we give our basic notational conventions. Section 2 will describe the construction of finite approximations, while in section 3 we will discuss related results and equivalent constructions. Finally in section 4 we will discuss connectedness which perversely coexists with 0-dimensionality (in fact, each space pictured above is connected).

1. Notation

Asymmetric spaces: We require the following definitions: The **specialization** order of a topological space (X, τ) is defined by $x \leq_\tau y \Leftrightarrow x \in \mathsf{cl}\{y\}$. Then each closed $A \subseteq X$ is certainly a $\leq_\tau$-lower set (that is, $a \in A$ and $x \leq_\tau a \Rightarrow x \in A$). A set A is **saturated** if it is a $\leq_\tau$-upper set: that is to say, $x \in A$ and $x \leq_\tau y \Rightarrow y \in A$. The **saturation** of A is the smallest saturated set containing A, namely, $\uparrow_\tau A = \{x \mid a \leq_\tau x$ for some $a \in A\}$. It is easy to see that the saturation of any set A is the intersection of all its open neighborhoods and we will denote this saturation by $\mathsf{n}(A)$ or in the case that $A = \{x\}$ by $\mathsf{n}\{x\}$. If $x, y \in X$, then $x \in \mathsf{cl}\{y\} \Leftrightarrow x \leq_\tau y \Leftrightarrow y \in \mathsf{n}\{x\}$.

The specialization completely determines a finite space, if X is finite then arbitrary unions of closed sets are closed, and therefore $A \subseteq X$ is closed if and only if $A = \bigcup_{a \in A} \mathsf{cl}\{a\} = \downarrow_{\leq_\tau} A$, that is, if and only if A is a $\leq_\tau$-lower set. Thus A is open if and only if A is the complement of a $\leq_\tau$-lower set, i.e., saturated. This holds more generally for **Alexandroff** spaces: those in which arbitrary intersections of open sets are open (or equivalently, in which $\mathsf{n}\{x\}$ is open for each x). It is then easy to show that the preorders on a set X are in 1-1 correspondence with the Alexandroff topologies on it, and the partial orders correspond to the T_0 Alexandroff topologies on it, via the map $\preceq \rightarrow \alpha(\preceq) = \{T \subseteq X \mid (y \in T \text{ and } y \preceq x) \Rightarrow x \in T\}$.

Bitopological notation: We will be interested in the relationships between one topology on a set and another, so we work below with the idea of **bitopological space**, a set with two topologies[1]: (X, τ, τ^*). For such, we use the notation $\tau^S = \tau \vee \tau^*$. Our maps are the **pairwise continuous** functions from (X, τ_X, τ_X^*) to (Y, τ_Y, τ_Y^*), that is, those $f : X \rightarrow Y$ which are continuous from τ_X to τ_Y and from τ_X^* to τ_Y^*; unless oth-

[1] Unless otherwise noted, our bitopological notation and results are from [Ko].

erwise indicated, functions considered below are pairwise continuous. Notice that each pairwise continuous function from X to Y is **symmetrically continuous** (**^{S}-continuous**), that is, continuous from τ_X^S to τ_Y^S.

The last sentence above illustrates a common problem in dealing with sets with a basic topology and some other topologies which are viewed as subservient to study of the former. When a topological concept is mentioned, one may wonder to which topology it refers. We use the following convention: topological terms such as open, closed, dense, continuous, when undecorated, refer to the basic topology; if the terms are intended to refer to another topology, then they are prefixed by an adjective or a symbol. For example, given (X, τ_X, τ_X^*), (Y, τ_Y, τ_Y^*), $T \subseteq X$ is ^{S}-open (or symmetrically open) if $T \in \tau_X^S$, $f : X \to Y$ is *-continuous if f is continuous from τ_X^* to τ_Y^*.

Bitopological definitions: A few definitions for bitopological spaces refer to the first topology only; for example, (X, τ, τ^*) is **connected** or **compact** if (X, τ) is. But most bitopological definitions, and particularly those of separation properties, refer to both topologies. We use the convention that for any property Q, a bitopological space (X, τ, τ^*) is **pairwise** Q if it and (X, τ^*, τ) both satisfy Q. A bitopological space (X, τ, τ^*), is:

weakly symmetric (ws)[2] if $x \in \mathrm{cl}_{\tau^*}\{y\} \Rightarrow y \in \mathrm{cl}_{\tau}\{x\}$.

pseudoHausdorff (pH) if $x \notin \mathrm{cl}_{\tau}\{y\} \Rightarrow (\exists T \in \tau, T^* \in \tau^*) x \in T, y \in T^*, T \cap T^* = \emptyset$,

0-dimensional if τ has a base of τ^*-closed τ-open sets,

joincompact if pairwise pH, and τ^S is compact and T_2.

Skew compactness: A topological space (X, τ) is **skew compact** if there is a second topology so that (X, τ, τ^*) is joincompact (see [Ko]); it is **spectral**, (or **Priestley**) (see [Pr]) if

[2] This is the bitopological analogue of the R_0-separation property of Davis, [Da].

there is topology τ^* on X such that (X, τ, τ^*) is joincompact and pairwise 0-dimensional. Thus skew compact and spectral are topological notions, and can be characterized in terms of the original topology, τ:

A topological space is skew compact iff it is T_0, and

(i) If $\mathcal{S}$ is a family of sets, each of which is either closed or compact saturated, and $\mathcal{S}$ has the finite intersection property, then $\bigcap \mathcal{S} \neq \emptyset$.

(ii) if $x \notin \mathsf{cl}\{y\}$ then for some open T and compact K, $x \in T \subseteq K$ and $K \cap \mathsf{cl}\{y\} = \emptyset$.

A topological space is spectral iff it is compact and T_0, and it has a base of open sets, arbitrary intersections of which are compact. Thus all finite T_0-spaces are spectral.

We now consider inverse limits of finite spaces, which will be used below. Products and symmetrically closed subspaces of joincompact spaces are joincompact, thus so too are inverse limits with respect to pairwise continuous maps. It is useful to notice that the specialization on the product is the product of the specializations, and the specialization on a subspace is the restriction of that on the entire space; as a result, the specialization on an inverse limit is the restriction of the product specialization to the inverse limit. This yields corresponding results for skew compact and spectral spaces.

De Groot dual: The **de Groot dual** of a topology τ, is the topology τ^G generated by the collection of complements of compact, saturated sets or equivalently (since the saturation of a compact set is compact), the topology generated by all complements of saturations of compact sets. For a skew compact topology τ, τ^G is the unique topology τ^* so that (X, τ, τ^*) is joincompact (see [Ko]); it is also true that if (X, τ, τ^*) is joincompact, then so is (X, τ^*, τ), and as a result, for each skew compact topology τ, τ^G is skew compact and $(\tau^G)^G = \tau$. A function from X to Y is **de Groot** if it is pairwise continuous from (X, τ_X, τ_X^G) to (Y, τ_Y, τ_Y^G). If we denote $\tau \vee \tau^G$ by τ^{SG} then clearly the de Groot maps are ^{SG}-continuous.

Two examples of the de Groot dual are particularly important in this paper:

If X is finite, then each of its subsets is compact, so $A \subseteq X$ is a de Groot basic closed set if and only if A is saturated, that is open, since X is Alexandroff. Thus the de Groot basic closed sets are the open sets in τ, and so as indicated above, for finite spaces, τ^G is the topology of closed sets. It follows that for finite X, (X, τ, τ^G) is pairwise 0-dimensional. Also as a result of the same argument, if X, Y are finite, and $f : X \to Y$ is continuous, then f must be continuous from τ_X^G to τ_Y^G, since a function is continuous if and only if the inverse image of a closed set is closed. For the same reason, quotients of finite spaces are pairwise quotients. Finally, if X is a finite T_0-space then X is skew compact, and for each $x \in X$, $\{x\} = \mathsf{cl}\{x\} \cap \mathsf{n}\{x\} \in \tau^{SG}$, so τ^{SG} is discrete.

If X is T_1, then $\leq_\tau$ is equality, so each set is saturated[3]. If further, X is compact Hausdorff, then a set is basic de Groot closed if and only if it is compact, thus if and only if it is closed. This shows that $\tau = \tau^G$, and thus $\tau = \tau^{SG}$ as well.

Topological ordered spaces: We also use a characterization of skew compact spaces in terms of compact topological ordered spaces (see [Na] for further information): (X, τ) is skew compact if and only if there is a compact Hausdorff topology θ on X, together with a partial order, $\leq$, closed in $(X, \theta) \times (X, \theta)$, such that $\tau = \theta^U = \{T \in \theta \mid T \text{ is a } \leq\text{-upper set}\}$ (also $\tau^G = \theta^L = \{T \in \theta \mid T \text{ is a } \leq\text{-lower set}\}$). In fact, the connection is even stronger:

The topology and partial order are determined by $\leq = \leq_\tau$

[3] De Groot originally limited his definition of the dual to T_1-spaces in [dG].

and $\theta = \tau^{SG}$, and whenever $(X, \theta, \leq)$ is such a space, then $\tau = \theta^U$ is a skew compact topology.

The ^{SG}-continuous, specialization-preserving maps between skew compact spaces are precisely the de Groot maps. To see this, suppose that $f : (X, \tau_X^{SG}) \to (Y, \tau_Y^{SG})$ is continuous and specialization preserving and let C be a basic τ_Y^G-closed set. To prove that f is continuous as a function from (X, τ_X^G) to (Y, τ_Y^G), we need to show that $f^{-1}[C]$ is compact and saturated. However, $f^{-1}[C]$ is a τ_X^{SG}-closed subset of the compact Hausdorff space (X, τ_X^{SG}), hence $f^{-1}[C]$ is τ_X^{SG}-compact and so τ_X-compact. Furthermore, $f^{-1}[C]$ is saturated, since if $x \in f^{-1}[C]$ and $x \leq_\tau y$, then $f(x) \leq_\tau f(y) \in C$ since C is saturated and so $y \in f^{-1}[C]$ and we are done. τ-continuity results from the fact that $\tau = (\tau^G)^G$.

Reflections: We also need the notion of reflection from categorical topology [HS]. Recall that if X is any topological space and T any separation property, then a **T-reflection** of X (if it exists), is a T-space X_T together with a continuous map $t : X \to X_T$ such that whenever $f : X \to Z$ is continuous and Z is a T-space, then for some unique $\hat{f} : X_T \to Z$, $f = \hat{f}t$. (A standard universality argument shows that for any two T-reflections, $t : X \to X_T$ and $s : X \to X'_T$ of X, there is a homeomorphism $h : X_T \to X'_T$ such that $s = ht$, that is, the T-reflection is unique up to homeomorphism.) The T-reflection of a space exists if $T \in \{T_0, T_1, T_2, Urysohn, T_3, T_{3.5}\}$. For a space X, let $h : X \to X_H$ denote its Hausdorff reflection; h is known to be surjective.

2. Finite Approximation

Below, we exhibit compact Hausdorff spaces as certain quotients of inverse limits of finite T_0-spaces. But it should first be noticed that inverse limits themselves are not enough. For suppose that we have an inverse spectrum of finite T_0-spaces and continuous maps, $((X_\gamma, \tau_\gamma), f_{\gamma\delta})_{\gamma,\delta\in\Gamma}$, on the directed set $(\Gamma, \leq)$, and that (X, τ) is the inverse limit of this system. As pointed out in the discussion of the de Groot dual in Section 1, each $f_{\gamma\delta}$ is also continuous from τ_γ^{SG} to τ_δ^{SG}. As a result each $f_\gamma : X \to X_\gamma$ is continuous from (X, τ^{SG}) to $(X_\gamma, \tau_\gamma^{SG})$. But then (X, τ^{SG}) is an inverse limit of discrete spaces, and so is 0-dimensional. But for compact Hausdorff spaces, $\tau = \tau^{SG}$, and so τ must be 0-dimensional. As is well-known, each compact, 0-dimensional space is an inverse limit of finite discrete spaces, and so a compact Hausdorff space is an inverse limit of finite T_0-spaces and continuous maps if and only if it is 0-dimensional.

Suppose A is a collection of open sets in a topological space (X, τ). $\mathcal{B}(A)$ will denote the set of finite Boolean combinations of elements of A and $\mathcal{O}(A)$ will denote those finite Boolean combinations of elements of A which are open sets. Define a preorder $\leq_F$ on X by

$$x \leq_F y \Leftrightarrow \{T \in \mathcal{O}(F) \mid x \in T\} \subseteq \{T \in \mathcal{O}(F) \mid y \in T\},$$

and the equivalence relation $\sim_F = \leq_F \cap \leq_F^{-1}$. For $x \in X$ let x_F denote the equivalence class of x, X_F the quotient set modulo $\sim_F$, $\pi_F : X \to X_F$ the natural map, and τ_F the quotient topology arising from π_F. Then, of course, $A \in \mathcal{B}(F)$ if and only if $A = \pi_F^{-1}[\pi_F[A]]$, and so $A \in \mathcal{O}(F)$ if and only if $\pi_F[A] \in \tau_F$ and $A \in \mathcal{B}(F)$. As a result, the minimal elements of $\mathcal{B}(F)$ can be viewed as the singletons of X_F, and $\mathcal{O}(F)$ can be viewed as τ_F. Also let $\preceq_F = \{(x_F, y_F) \mid x \leq_F y\}$ (it is easy to see that $\preceq_F$ is well-defined since if $x \leq_F y$ and $x \sim_F x'$ and $y \sim_F y'$, then $x' \leq_F y'$).

Lemma 2.1. *Let F be a finite collection of open sets in a topological space (X, τ). Then (X_F, τ_F) is a finite T_0-space and τ_F is $\alpha(\preceq_F)$.*

Proof. Clearly X_F is finite. To show that (X_F, τ_F) is T_0, suppose that the typical elements $\pi_F(x), \pi_F(y)$ of X_F are distinct. Then $x \not\sim_F y$, so either $x \not\leq_F y$ or $y \not\leq_F x$, and with no loss of generality assume the first. Then by definition of $\leq_F$, there must exist some $T \in \mathcal{O}(F)$ such that $x \in T$ and $y \notin T$. But then $\pi_F(x) \in \pi_F[T] \in \tau_F$ and $\pi_F(y) \notin \pi_F[T]$, so $\pi_F(x) \notin \mathrm{cl}_{\tau_F}\{\pi_F(y)\}$.

To show that $\tau_F = \alpha(\preceq_F)$, it suffices by finiteness to show that for each $x, y \in X$, $\pi_F(x) \notin \mathrm{cl}_{\tau_F}\{\pi_F(y)\}$ if and only if $x \not\leq_F y$. But both of these conditions are equivalent to the assertion that there is some open Boolean combination, T, of elements of F, such that $x \in T$ and $y \notin T$. □

Definition 2.2. *A T_2 space X, is* **approximable by finite spaces** *if it is the Hausdorff reflection of an inverse limit of finite T_0-spaces and quotient maps.*

Proposition 2.3. *If (X, τ) is a skew compact space, then its T_2-reflection X_H is its quotient by the smallest symmetrically closed equivalence relation containing the specialization order of τ, $\leq_\tau$ (i.e., the intersection of all symmetrically closed equivalence relations containing $\leq_\tau$). Furthermore, X_H is the T_i-reflection of X for $2 \leq i \leq 4$.*

Proof. Let $\sim$ be the smallest ^{SG}-closed equivalence relation which contains the specialization order $\leq_\tau$, and $p : (X, \tau^{SG}) \to X/\sim$ be the quotient map and θ the quotient topology on $X/\sim$ which henceforth we will denote by $X_\sim$. For $A \subseteq X$, let $\sim[A] = \{x \mid (\exists a \in A)(a \sim x)\}$. We now show that p is a closed map by proving that for each closed subset C, $\sim[C] = p^{-1}[p[C]]$ is ^{SG}-closed, implying that $p[C]$ is closed. To this end, if $z \in \mathrm{cl}_{\tau^{SG}}(\sim[C])$ then there is a net $x_n \to z$, with each $x_n \in \sim[C]$. As a result, for each x_n there is a $c_n \sim x_n$. Since X is ^{SG}-compact, c_n has an ^{SG}-convergent subnet, $c_b \to c$; necessarily $c \in C$. Since $\sim$ is ^{SG}-closed and each $(z_b, c_b) \in \sim$,

$(z, c) \in\sim$, so $z \in\sim [C]$.

Thus $(X_\sim, \theta)$, being the continuous, closed image of a normal T_1-space, is Hausdorff. Since $x \leq_\tau y \Rightarrow p(x) = p(y)$, p is specialization-preserving, and since it is also ^{SG}-continuous, it follows from remarks in Section 1 that p is continuous from (X, τ) to $(X_\sim, \theta)$; further, since p is closed from (X, τ^{SG}) to $(X_\sim, \theta)$, it is closed, thus a quotient map, from (X, τ) to $(X_\sim, \theta)$. Now, suppose $f : X \to Y$ is any continuous map of (X, τ) to a T_2-space; then $\sim_f = \{(x, y) \mid f(x) = f(y)\}$ is a closed equivalence relation on X. Thus by the definition of $\sim$, $x \sim y \Rightarrow x \sim_f y \Rightarrow f(x) = f(y)$, so f factors through p as a function; let $f = gp$. If $C \subseteq Y$ is closed, then so is $f^{-1}[C] = p^{-1}[g^{-1}[C]]$; since p is a closed map, $g^{-1}[C] = pf^{-1}[C]$, is also closed, so g is continuous. Since X_H is a T_4 space and each map to a T_2-space factors through p, we are done. $\square$

Theorem 2.4. *A T_2 space is approximable by finite spaces if and only if it is compact.*

Proof. Given any compact T_2-space (X, τ), let $\mathcal{F}$ be a collection of finite sets of open subsets of X such that:

(i) $\mathcal{F}$ is directed by $\subseteq$, and

(ii) $\bigcup \mathcal{F}$ is a base for τ.

If $\mathcal{C}$ is a base for τ, then the collection of finite subsets of $\mathcal{C}$ forms a set satisfying (i) and (ii), so there is such a set. If $F \subseteq G$, $F, G \in \mathcal{F}$, then $\leq_G$ refines $\leq_F$ ($x \leq_G y \Rightarrow x \leq_F y$), so $\sim_G$ refines $\sim_F$. The natural map $p_{GF} : X_G \to X_F$ for which $\pi_F = p_{GF}\pi_G$, is a quotient map since the π_F are: for if $A \subseteq X_F$, $\pi_F^{-1}[A] = \pi_G^{-1}[p_{GF}^{-1}[A]]$, so $A \in \tau_F \Leftrightarrow p_{GF}^{-1}[A] \in \tau_G$.

The following construction is used below: If $z \in T \in \tau$, since (X, τ) is regular and $\bigcup \mathcal{F}$ is a base for τ, we can choose $V \in \bigcup \mathcal{F}$, so that $z \in V$ and $\mathrm{cl}(V) \subseteq T$. Additionally, since

(X, τ) is compact, we can find a finite cover $V_1, \dots, V_n$ of $X \setminus T$ by elements of $\bigcup \mathcal{F}$ which are subsets of $X \setminus \mathrm{cl}(V)$; and let $C = X \setminus (V_1 \cup \dots \cup V_n)$. By (i) let $F \in \mathcal{F}$ be so that $V, V_1, \dots, V_n \in F$. Thus: $F \in \mathcal{F}$, $C = \pi_F^{-1}[\pi_F[C]]$, $V = \pi_F^{-1}[\pi_F[V]]$, $z \in V \subseteq C \subseteq T$.

By (i), $(X_F, p_{GF})_{F,G \in \mathcal{F}}$ is an inverse system; let (Y, θ, θ^*) be its inverse limit (with respect to the original topologies and their de Groot duals) and $p_F : Y \to X_F$ the projections. Since Y is joincompact, $\theta^* = \theta^G$. Let $\pi : X \to Y$ denote the (continuous) map which exists by definition of inverse limit, such that for each $F \in \mathcal{F}$, $p_F \pi = \pi_F$. Further, $\pi[X]$ is ^{SG}-dense in Y, because θ^{SG} is generated by sets of the form $p_F^{-1}[V]$, where $V \in \tau_F^{SG}$, is nonempty, and since each π_F is onto, $p_F \pi[X] = \pi_F[X]$ contains each $V \subseteq X_F$, so for each such V, $\pi[X] \cap p_F^{-1}[V] \neq \emptyset$.

We now claim that:

(*) For each $y \in Y$ there is a unique $x \in X$ such that $\pi(x) \in \mathrm{cl}\{y\}$.

To establish (*), first notice that if $y, z \in Y$ then $z \in \mathrm{cl}\{y\} \Leftrightarrow$ for each $F \in \mathcal{F}, p_F(z) \in \mathrm{cl}\{p_F(y)\}\}$. Thus $\pi(x) \in \mathrm{cl}\{y\}$ if and only if $x \in \bigcap S_y$, where $S_y = \{\mathrm{cl}(T) \mid T \in F \in \mathcal{F}, p_F(y) \in \pi_F[T]\}$. Thus it will do to show that $\bigcap S_y$ consists of a single point. But there cannot be more than one: let $u \neq t, u \in \bigcap S_y$. So there is a $T \in \bigcup \mathcal{F}$ such that $u \in T, t \notin \mathrm{cl}(T)$; find $F, V_1, \dots, V_n$ by the construction. Since F is in the directed $\mathcal{F}$ and $T \in \bigcup \mathcal{F}$ let $F \cup \{T\} \subseteq G \in \mathcal{F}$. $X = T \cup \bigcup_1^n V_i$, and since $u \notin \mathrm{cl}(V_i)$, for any i, $p_G(u) \notin \pi_G[V_i]$, for any i, by definition of S_y. Hence $p_G(u) \in \pi_G[T]$. But $t \notin \mathrm{cl}(T)$, thus $p_G(t) \notin \pi_G[T]$, and so $t \notin \bigcap S_y$.

Next, $\bigcap S_y$ is nonempty: S_y is a collection of closed sets in a compact space, and it has the finite intersection property: if $\mathrm{cl}(T_1), \dots, \mathrm{cl}(T_n) \in S_y$, $T_i \in F_i \in \mathcal{F}$, $p_{F_i}(y) \in \pi_{F_i}[T_i]$, there is by (i), $G \in \mathcal{F}$ such that each $F_i \subseteq G$, and so for each

i, $p_G(y) = p_{F_iG}p_{F_i}(y) \in p_{F_iG}[\pi_{F_i}[T_i]] = \pi_G[T_i]$, but this implies that $\emptyset \neq T_1 \cap \ldots \cap T_n \subseteq \mathrm{cl}(T_1) \cap \ldots \cap \mathrm{cl}(T_n)$. This completes the proof of (*).

We now use (*) to establish our theorem. For $y \in Y$ let $p(y)$ be the unique point x defined by (*). Note that $p : (Y, \theta) \to (X, \tau)$ is continuous: If $p(y) \in T \in \tau$, find F, V, C as in the construction. Then $p(y) \in \pi_F^{-1}[\pi_F[V]] = \pi^{-1}[p_F^{-1}[\pi_F[V]]]$, so $\pi(p(y)) \in p_F^{-1}[\pi_F[V]] \in \theta$, thus $\mathrm{cl}\{y\}$ meets the open $p_F^{-1}[\pi_F[V]]$, and so $y \in p_F^{-1}[\pi_F[V]]$. If $z \in p_F^{-1}[\pi_F[V]]$ then $z \in p_F^{-1}[\pi_F[C]]$, a θ-closed set, so $\pi(p(z)) \in \mathrm{cl}\{z\} \subseteq p_F^{-1}[\pi_F[C]] = \pi[\pi_F^{-1}[\pi_F[C]]] = \pi[C]$, that is, $p(z) \in C \subseteq T$. Also, $p\pi = 1_X$ since by definition of p, for each $x \in X$, both $p(\pi(x))$ and x are the unique $z \in X$ such that $\pi(z) \in \mathrm{cl}\{\pi(x)\}$. This also shows that $p : (Y, \theta) \to (X, \tau)$ is a retract.

The above shows that $p : Y \to X$ is a continuous map of Y onto a Hausdorff space. To show that X is the T_2 reflection of Y, we need only show that each $f : Y \to Z$, Z Hausdorff, factors through p. But $p(y) \in \mathrm{cl}\{y\}$, so $f(p(y)) \in \mathrm{cl}\{f(y)\} = \{f(y)\}$, so if $p(y) = p(z)$ then $f(y) = f(p(y)) = f(p(z)) = f(z)$. Thus there is a function g so that $f = gp$. To see that g is continuous, simply notice that if $C \subseteq Z$ is closed, then $g^{-1}[C] = p[f^{-1}[C]]$, and $f^{-1}[C]$ is closed in Y, thus compact, and therefore $p[f^{-1}[C]]$ is compact in X, so closed.

We have now shown that if a T_2-space is compact then it is approximable by finite spaces. But since inverse limits and continuous images preserve compactness, the converse is clear. $\square$

Notes: (a) The p in the proof is also continuous from θ^* to τ: Referring to the portion of that proof which shows p continuous from θ to τ, simply repeat the construction, obtaining D, W, G so that $W, X \setminus D \in \tau$ and $D = \pi_G^{-1}[\pi_G[D]]$, $W = \pi_G^{-1}[\pi_G[W]]$, $p(y) \in W \subseteq D \subseteq V$. Since $\mathcal{F}$ is directed, we may assume $F \subseteq G$. Therefore, if $z \in p_G^{-1}[\pi_G[D]] \in \theta^*$, then $z \in p_G^{-1}[\pi_G[V]] \subseteq$

$p_F^{-1}[\pi_F[V]]$ as well, so as before, $p(z) \in C \subseteq T$. But $\pi : (X,\tau) \to (Y,\theta^*)$ need not be continuous, so $p : (Y,\theta^*) \to (X,\tau)$ may not be a retract.

(b) It follows from the definition of p that if $p(y) = x$ then $\pi(x) \leq_\theta y$. Thus $\pi(x)$ must be the $\leq_\theta$-minimum element of $p^{-1}(x)$, so $\pi[X]$ is the set of $\leq_\theta$-minimal elements of Y.

Definition 2.5. *Any collection $\mathcal{F}$ which satisfies conditions (i) and (ii) of the first paragraph of the proof of Theorem 2.4 is called an* **approximating family**. *The inverse system constructed in its proof is called the* $\mathcal{F}$-**finitary spectrum**.

The inverse limit Y constructed in Theorem 2.4, depends on $\mathcal{F}$; and we will often use the subscript $_\mathcal{F}$ to denote that fact. For any set R, let $\mathcal{P}^{<\omega}(R)$ denote the collection of finite subsets of R; then notice that $\mathcal{F}$ is a cofinal subset of $\mathcal{P}^{<\omega}(\bigcup\mathcal{F})$ and so $Y_\mathcal{F}$ is homeomorphic to $Y_{\mathcal{P}^{<\omega}(\bigcup\mathcal{F})}$ in a natural way. As a result, we could restrict our attention to those $\mathcal{F}$ which are of the form $\mathcal{P}^{<\omega}(\mathcal{C})$ for some base $\mathcal{C}$ of our topology.

Examples and Questions 2.6. (a) First we will formalize the approximation of $X = [0,1]$ by the D_j as pictured in the Motivation section. For i,j nonnegative integers, $i \leq 2^j$, let $T_{ij} = (\frac{i-1}{2^j}, \frac{i+1}{2^j}) \cap [0,1]$, which is clearly a nonempty open connected subset of $[0,1]$. The collection $\mathcal{FS}$ of sets of the form $F_j = \{T_{ij} \mid 0 \leq i \leq 2^j\}$ are an approximating family, and each $F_j \subseteq \mathcal{O}(F_{j+1})$. Notice now that $D_j = X_{F_j}$:

For $x \in [0,1]$, if for some $i < 2^j$, $x \in (\frac{i}{2^j}, \frac{i+1}{2^j})$, then $\mathsf{n}\{x_{F_j}\} = \{x_{F_j}\} = \{(\frac{i-1}{2^j}, \frac{i+1}{2^j}) \cap (\frac{i}{2^j}, \frac{i+2}{2^j})\} = \{(\frac{i}{2^j}, \frac{i+1}{2^j})\}$, otherwise, for some $i \leq 2^j$, $x = \frac{i}{2^j}$, and the equivalence class x_{F_j} is $[0,1] \cap (\frac{i-1}{2^j}, \frac{i+1}{2^j}) \setminus [(\frac{i-2}{2^j}, \frac{i}{2^j}) \cup (\frac{i}{2^j}, \frac{i+2}{2^j})] = \{x\}$ and $\mathsf{n}\{x_{F_j}\} = \{(\frac{i-1}{2^j}, \frac{i}{2^j}), \{x\}, (\frac{i}{2^j}, \frac{i+1}{2^j})\}$.

Denote the inverse limit of the system by $(\mathbf{S}, \theta, \theta^*)$, and let $p : \mathbf{S} \to [0,1]$ be as constructed in the proof of Proposition 2.3. Notice that for each non-dyadic $x \in [0,1]$, $p^{-1}(x)$ consists of a unique point, $\hat{x}$, defined by $\hat{x}(j) = (\frac{i}{2^j}, \frac{i+1}{2^j})$, where $x \in (\frac{i}{2^j}, \frac{i+1}{2^j})$. For each dyadic (given in lowest terms) $x = \frac{m}{2^n} \in [0,1]$, there are three $y \in p^{-1}(x)$, which we call $\hat{x}, \hat{x}^-, \hat{x}^+$; for each of them if $j < n$, $y(j) = (\frac{i}{2^j}, \frac{i+1}{2^j})$ where $x \in (\frac{i}{2^j}, \frac{i+1}{2^j})$; if $j \geq n$ then $\hat{x}(j) = (\frac{2^{j-n}m-1}{2^n}, \frac{2^{j-n}m+1}{2^n})$, $\hat{x}^-(j) = (\frac{2^{j-n}m-1}{2^n}, \frac{2^{j-n}m}{2^n})$ and $\hat{x}^+(j) = (\frac{2^{j-n}m}{2^n}, \frac{2^{j-n}m+1}{2^n})$ (except that $\hat{0}^-, \hat{1}^+$ do not exist).

The D_m inherit a quotient order from $[0,1]$, and we view $\mathbf{S}$ as ordered as well; this order is the usual on $\{\hat{x} \mid x \in [0,1]\}$, but $\hat{x}^-$ is always the immediate predecessor of $\hat{x}$, and $\hat{x}^+$ is always the immediate successor of $\hat{x}$. Now a base of θ is $\{(\hat{x}, \hat{y}) \mid x < y, x, y \text{ dyadics in } [0,1]\}$, and one of θ^* is $\{[\hat{x}, \hat{y}] \mid x \leq y, x, y \text{ dyadics in } [0,1]\}$.

This space is called the Smyth interval (see [Sm], [KKW]) and we note that its subspace, $\{\hat{x} \mid x \in [0,1]\}$ with the relative topology τ is homeomorphic to $[0,1]$. It is unsettling but important to notice that although $(\mathbf{S}, \theta)$ and this subspace are both skew compact, $\theta^G|[0,1] \neq \tau^G$.

(b) If (X, τ) is 0-dimensional then $\mathcal{F}$ can be chosen so that $\bigcup \mathcal{F}$ consists only of clopen sets. In this case, each X_F is discrete, and $(Y, \theta, \theta^*) = (X, \tau, \tau)$. On the other hand, even if (X, τ) is zero-dimensional but $\mathcal{F}$ is chosen so that $\bigcup \mathcal{F}$ contains sets which are not clopen, then in general, $(Y, \theta, \theta^*) \neq (X, \tau, \tau)$ as the following example shows.

Let $(\mathbb{K},\tau)$ denote the Cantor set with the usual Euclidean topology. It is possible to repeat the construction defined in (a) using the triadic rationals $\mathbb{Q}_3$ (which are dense in $\mathbb{K}$) instead of the dyadic rationals; thus the intervals T_{ij} will have the form $(q, r) \cap \mathbb{K}$, where q is a right-hand end-point of one, and r is a left-hand end-point of another of the open intervals defining $[0,1] \setminus \mathbb{K}$ $(q < r)$ and hence both q and r are triadic rationals. It is not hard to see that if $p : Y \to \mathbb{K}$ is the function constructed in Theorem 2.4, then the resulting inverse limit (Y, θ) has the following properties:

(i) For each $x \in \mathbb{K}$ which is not a triadic rational, $|p^{-1}(x)| = 1$.

(ii) If x is a triadic rational which is a right-hand endpoint of one of the open intervals in $[0,1] \setminus \mathbb{K}$, then $p^{-1}(x) = \{\hat{x}, \hat{x}^+\}$ and $\hat{x} \in \mathrm{cl}_\theta\{\hat{x}^+\}$.

(iii) If x is a triadic rational which is a left-hand endpoint of one of the open intervals in $[0,1] \setminus \mathbb{K}$, then $p^{-1}(x) = \{\hat{x}^-, \hat{x}\}$ and $\hat{x} \in \mathrm{cl}_\theta\{\hat{x}^-\}$.

Thus (Y, θ) is a spectral compactification of $(\mathbb{K}, \tau)$ distinct from $(\mathbb{K}, \tau)$.

Briefly we describe three more examples:

(c) Let $\mathcal{FS}$ be as in (a), $W_1 = \bigcup_{n=2}^{\infty}(\frac{3}{2^i}, \frac{4}{2^i})$ and $W_2 = \bigcup_{n=2}^{\infty}(\frac{2}{2^i}, \frac{3}{2^i})$, and let $\mathcal{G} = \{F_n \cup \{W_1, W_2\} \mid F_n \in \mathcal{FS}\}$. Then $Y_\mathcal{G}$ is the Smyth line except that 0^+ is replaced by the two points $w_1, w_2 \in \mathsf{n}\{0\}$ where w_1 and w_2 are incomparable in the specialization order of $Y_\mathcal{G}$.

(d) Barycentric approximation: Let X be an m-simplex and for each n let X_n be the abstract n-th barycentric subdivision (treated as a finite set, with the Alexandroff quotient topology). Let $G_n = \bigcup_{j \le n}\{\pi_j^{-1}[\mathsf{n}\{x\}] \mid x \text{ closed (i.e., a vertex) in } X_j\}$, $\mathcal{F} = \{G_n \mid n \in \omega\}$. This generalizes (a).

(e) Approximation of Products: If for each $\gamma \in \Gamma$, X_γ is a compact T_2 space and $\mathcal{F}_\gamma$ is an approximating family for (X_γ, τ_γ), then $\mathcal{F} = \Pi_\Gamma \mathcal{F}_\gamma$ is an approximating family for $\Pi\{X_\gamma \mid \gamma \in \Gamma\}$, where $\Pi_\Gamma \mathcal{F}_\gamma$ is the collection of all sets of the form $\times_\Phi F_\gamma$ for $F_\gamma \in \mathcal{F}_\gamma$ and where for each finite set $\Phi \subset \Gamma$, $\times_\Phi F_\gamma = \{\Pi_\Gamma T_\gamma \mid T_\gamma \in F_\gamma \text{ for each } \gamma \in \Phi \text{ and } T_\gamma = X_\gamma \text{ otherwise}\}$.

3. Equivalent Methods of Finite Approximation

Given an approximating family, $\mathcal{F}$, for the compact Hausdorff space (X, τ), let τ^* be the topology generated by $\{X \setminus T \mid T \in \bigcup \mathcal{F}\}$. τ^* is a refinement of τ. To see this, suppose that $x \in V \in \bigcup \mathcal{F}$, then there is a $T \in \tau$ so that $x \in T$ and $\mathrm{cl}(T) \subseteq V$. There is a finite cover, $V_1, \ldots, V_n$ of $X \setminus V$ by elements of $\bigcup \mathcal{F}$ which are subsets of $X \setminus \mathrm{cl}(T)$, and $x \in X \setminus \bigcup_{i=1}^n V_i \subseteq V$, showing that each τ-neighborhood of an arbitrary x contains a τ^*-neighborhood of x and hence that $\tau \subseteq \tau^*$ as required. Furthermore, if $X \in T \in \tau$, then we can find $U \in \tau$ so that $x \in U \in \bigcup \mathcal{F}$ and $U \subset T$; then $X \setminus U \in \tau^*$, so (X, τ, τ^*) is 0-dimensional. To see that (X, τ^*, τ) is also 0-dimensional, suppose that $x \in T \in \tau^*$; then for some $V_1, \ldots, V_n \in \bigcup \mathcal{F}$, $x \in \bigcap_1^n (X \setminus V_i) \subseteq T$, and $\bigcap_1^n (X \setminus V_i)$ is closed and *-open as required.

If $T \in F \in \mathcal{F}$, then $X \setminus T = \pi_F^{-1}[X_F \setminus \pi_F[T]] = \pi^{-1}[p_F^{-1}[X_F \setminus \pi_F[T]]$ and so τ^* is the topology generated by $\{\pi^{-1}[W] \mid W \in \theta^*\}$ (essentially $\theta^*|X$). Thus (Y, θ, θ^*) is a pairwise 0-dimensional joincompactification of of the (necessarily) pairwise 0-dimensional (X, τ, τ^*). (Thus we have shown also that $\tau^* = \tau^S$ is a 0-dimensional topology.)

We now show that all such 0-dimensional joincompactifications arise in this way. First we make give some notation: If (X, τ, τ^*) is a bitopological space, then $\mathcal{D}(X) = \{T \in \tau \mid X \setminus T \in \tau^*\}$ and if (X, τ) is a topological space, then X^G will denote the bitopological space (X, τ, τ^G).

Theorem 3.1. *The following are equivalent for a topological space (X, τ):*

(i) It is spectral.

(ii) It is the inverse limit of a spectrum of finite T_0 spaces and onto quotients.

(iii) It is the inverse limit of its $\mathcal{P}^{<\omega}(\mathcal{D}(X^G))$-finitary spectrum.

Proof. Finite T_0 spaces are spectral and continuous maps between them are de Groot. It follows that their inverse limits are also spectral and so $(ii) \Rightarrow (i)$. Since $(iii) \Rightarrow (ii)$ trivially, it suffices to show that $(i) \Rightarrow (iii)$. To this end suppose that (X, τ) is spectral; then $\mathcal{D}(X^G)$ is a base for τ. If we let $\mathcal{F} = \mathcal{P}^{<\omega}(\mathcal{D}(X^G))$ then, using the notation of the paragraph preceding Lemma 2.1 and the proof of Theorem 2.4, for $F \in \mathcal{F}$, we define X_F, π_F and if $F \subseteq G \in \mathcal{F}$, p_{GF} and finally the inverse limit Y, and $\pi : X \to Y$, $p : Y \to X$. Although τ is not assumed to be T_2, the facts that each p_F is a quotient, and that $\pi[X]$ is ^{SG}-dense in Y, hold as in that proof.

Now observe that since (X, τ) is the inverse limit of joincompact (finite) spaces, it too is joincompact, and so (X, τ^{SG}) is a compact Hausdorff space. Thus with the sets and functions, X_F, Y, p_{GF}, p_F, π and p as in Theorem 2.4, because the π_F are de Groot quotients, they are quotients from (X, τ^{SG}) to X_F, equipped with its symmetric topology, τ_F, which is discrete. As a result, the same proof goes through in this situation and since τ^{SG} is T_2, all conclusions about the sets and maps of the proof of Theorem 2.4 apply. In addition, unlike the situation in that proof, each τ_F is now Hausdorff, and thus so is the limit space. Therefore, the fact that $\pi(p(y)) \in \mathrm{cl}\{y\}$, which still applies (for τ^{SG}) now means that $\pi(p(y)) = y$, so that p and π are inverse homeomorphisms in this situation. $\square$

Definition 3.2. *A* **spectral compactification** *of a topological space (X, τ) is a spectral space (Y, θ) containing X as an ^{SG}-dense subspace. An* **approximator** *for (X, τ) is an inverse limit of finite spaces and quotient maps, in which X is an ^{SG}-dense retract.*

Theorem 3.3. *Let (X, τ) be a compact T_2 space. A topological space (Y, θ) is an approximator for (X, τ) if and only if it is a spectral compactification of (X, τ).*

Proof. By Theorem 3.1, each approximator for (X, τ) is spectral, and since X is ^{SG}-dense in (Y, θ), the latter is a spectral compactification for the former.

Conversely, let (Y, θ) be a spectral compactification of (X, τ). Then, as in the proof of Theorem 3.1, Y is the inverse limit of its $\mathcal{P}^{<\omega}(\mathcal{D}(X^G))$-finitary spectrum. But this is the same as the $\mathcal{P}^{<\omega}(\mathcal{D}(X^G))|X$-finitary spectrum: since X is ^{SG}-dense in Y, and so the image of X under each of the quotient maps is the entire space Y_F. □

3.4. Ultrafilter Characterization. We now show that approximators of compact Hausdorff spaces, defined as inverse limits of approximating families arise also in a familiar way as Stone spaces of Boolean algebras.

Suppose that (X, τ) is a compact T_2-space and $\mathcal{F} \subseteq \mathcal{P}^{<\omega}(\tau)$ is directed by inclusion, with the property that $\bigcup \mathcal{F}$ is a base for (X, τ). Let (Y, θ) be the inverse limit of the approximating family $((X_F, \tau_F), p_{GF})_{F,G \in \mathcal{F}}$ for (X, τ); then by Theorem 3.3, (Y, θ) is a spectral compactification of (X, τ), so (Y, θ, θ^G) is joincompact and pairwise 0-dimensional. For convenience we denote $\theta^G|\, X$ by τ^*.

Now let $\mathcal{B} = \mathcal{B}(\bigcup \mathcal{F})$ be the family of all Boolean combinations of elements of $\bigcup \mathcal{F}$, that is, the Boolean algebra generated by the elements of $\bigcup \mathcal{F}$. $\mathcal{B}$ is a base for τ^* since $((X_F, \tau_F), p_{GF})_{F,G \in \mathcal{F}}$ is an approximating family for X. A $\mathcal{B}$-*filter* is a subset of $\mathcal{B}$ which does not contain the empty set and which is closed under finite intersections and supersets (in $\mathcal{B}$). A $\mathcal{B}$-ultrafilter is a maximal $\mathcal{B}$-filter and the set of $\mathcal{B}$-ultrafilters will be denoted by $\mathsf{S}(\mathcal{B})$. Finally, let σ be the Stone topology on $\mathsf{S}(\mathcal{B})$; that is, σ has a subbase of sets of the form $\{q \in \mathsf{S}(\mathcal{B}) \mid U \in q\}$, where $U \in \mathcal{B}$ (see Chapter 2 of [CN] for details).

Theorem 3.5. *$(\mathsf{S}(\mathcal{B}), \sigma)$ is homeomorphic to (Y, θ^{SG}).*

Proof. Fix $y \in Y$ and let $y_F = p_F(y)$ for each $F \in \mathcal{F}$; we claim first that $\mathcal{H}_y = \{y_F \mid F \in \mathcal{F}\}$ is a base for a $\mathcal{B}$-ultrafilter on X. $\mathcal{H}_y$ is a base for a $\mathcal{B}$-filter on X because $\pi_{FG}(y_F) = y_G$ implies that $y_F \subseteq y_G$ and hence $\mathcal{H}_y$ has the finite intersection property. We claim that this filter, which we call $\mathcal{U}_y$, is a $\mathcal{B}$-ultrafilter. It suffices to show that for all $V \in \mathcal{B}$, either $V \in \mathcal{U}_y$ or $X \setminus V \in \mathcal{U}_y$. However, for some $F \in \mathcal{F}$, $V, X \setminus V \in \mathcal{B}(F)$ (the set of Boolean combinations of the finite family F) and hence $y_F \subseteq V$ or $y_F \subseteq X \setminus V$ and our claim is proved. Furthermore, the correspondence $y \to \mathcal{U}_y$ is one-to-one; to see this, suppose that $z \in Y$ and $z \neq y$, then, since $\bigcup \mathcal{F}$ is a base for (X, τ), it follows that for some $F \in \mathcal{F}$, $z_F \neq y_F$ implying that $\mathcal{H}_y$ and $\mathcal{H}_z$ have disjoint elements and the result is clear.

Conversely, each $\mathcal{B}$-ultrafilter $\mathcal{U}$ in $\mathcal{B}$, determines an element of the inverse limit $y_{\mathcal{U}} = \{y_F \mid F \in \mathcal{F}\}$. To see this, note that since each X_F is a finite T_0-space, the smallest element of $\mathcal{U}$ in X_F exists and is a singleton, whose member we call y_F. The element $y_{\mathcal{U}}$ thus determined is unique, since for distinct $x, y \in Y$, there is some $F \in \mathcal{F}$ such that $x_F \neq y_F$ and hence $\{x_F\}, \{y_F\}$ cannot both be in a filter. As before, it is easy to see that the correspondence $\mathcal{U} \to y_{\mathcal{U}}$ is one-to-one and if $\mathcal{U} \in \mathsf{S}(\mathcal{B})$, then $\mathcal{U} = \mathcal{U}_{y_{\mathcal{U}}}$ and hence there is a canonical bijection which we denote by f, between $\mathsf{S}(\mathcal{B})$ and Y.

We claim that the map $f : (\mathsf{S}(\mathcal{B}), \sigma) \to (Y, \theta^{SG})$ is a homeomorphism. Since both (Y, θ^{SG}) and $(\mathsf{S}(\mathcal{B}), \sigma)$ are compact Hausdorff spaces and f is a bijection, it is sufficient to show that f is continuous. Thus, we need to show that if W is a subbasic open set of (Y, θ^{SG}) then $f^{-1}(W)$ is open in the Stone topology. To this end, we note that (Y, θ^{SG}) is the inverse limit of the spaces X_F with the discrete topology and hence W may be taken to be of the form ${p_F}^{-1}(y_F)$ where $F \in \mathcal{F}$ and $y_F \in X_F$. But then, $f^{-1}(W) = \{q \in \mathsf{S}(\mathcal{B}) \mid y_F \in q\}$, which is open in the Stone topology and we are done. $\square$

4. Skew Continua

In this section we look at connectedness, and in particular, continua.

Definition 4.1. *A* **skew continuum** *is a connected skew compact space.*

4.2. Comments. (a) The de Groot dual of a skew compact topology is connected if and only if the original topology is connected. To see this, we first note that the de Groot dual of a disconnected skew compact space (X, τ) is disconnected: If T is a proper τ-clopen subset, then, since T is closed, it is compact and since it is open, it is saturated; thus T is τ^G-closed, as (similarly) is $X \backslash T$. Conversely, if τ^G is disconnected, then so is $(\tau^G)^G = \tau$. In particular: the de Groot dual of a skew continuum is a skew continuum.

(b) A de Groot map f on a skew compact space (X, τ) is closed if (and only if) $f[\mathrm{cl}\{x\}]$ is closed (and hence equal to $\mathrm{cl}\{f(x)\}$) for each $x \in X$. To see this, note first that a set $C \subset X$ is closed if and only if it is ^{G}-compact and ^{G}-saturated and so $f[C]$ is ^{G}-compact since f is ^{G}-continuous. The fact that $f[C]$ is closed now follows from Theorem 3.1 of [Ko] which implies immediately that $\mathrm{cl}(f[C]) = \bigcup\{\mathrm{cl}\{f(x)\} \mid x \in C\} = \bigcup\{f[\mathrm{cl}\{x\}] \mid x \in C\} \subseteq f[C]$.

Theorem 4.3. *An inverse limit of connected joincompact spaces under pairwise relative quotient maps is connected.*

Proof. Suppose, by way of contradiction, that the limit, (X, τ, τ^G), of the inverse system $((X_\gamma, \tau_\gamma, \tau_\gamma^G), \pi_{\gamma\delta})_{\gamma,\delta \in \Gamma}$ is not connected, although the X_γ are connected, and let $\pi_\gamma : X \to$

X_γ be the projections. Then there are disjoint, nonempty, closed sets C, D whose union is X and for each γ, we claim that $\pi_\gamma[C] \cap \pi_\gamma[D] \neq \emptyset$. To see this, suppose to the contrary that $\pi_\gamma[C]$ and $\pi_\gamma[D]$ are disjoint subsets of $\pi_\gamma[X]$; since their union is $\pi_\gamma[X]$ it follows that $\pi_\gamma^{-1}[\pi_\gamma[C]] = C$ and $\pi_\gamma^{-1}[\pi_\gamma[D] = D$ which, since π_γ is a quotient map, implies that $\pi_\gamma[C]$ and $\pi_\gamma[D]$ are disjoint relatively closed sets whose union is $\pi_\gamma[X]$. This contradicts the fact that $\pi_\gamma[X]$ is connected.

Since X is joincompact, and Γ is directed, $Z = \bigcap_{\gamma\in\Gamma} \pi_\gamma^{-1}[\pi_\gamma[C] \cap \pi_\gamma[D]]$ is nonempty, so let $z \in Z$. Then $\pi_\gamma(z) \in \pi_\gamma[Z] \subseteq \pi_\gamma[C] \cap \pi_\gamma[D]$ for each γ. Thus again for each γ, the symmetrically closed $\pi_\gamma^{-1}[\pi_\gamma(z)]$ meets both C and D, and therefore $\bigcap_\Gamma \pi_\gamma^{-1}[\pi_\gamma(z)]$ meets both of these sets as well. But if $y \in \bigcap_\Gamma \pi_\gamma^{-1}[\pi_\gamma(z)]$ then for each γ, $\pi_\gamma(y) \in \{\pi_\gamma(z)\}$, showing $y = z$, and as a result $\{z\} = \bigcap_\Gamma \pi_\gamma^{-1}[\pi_\gamma(z)]$ meets both C and D, contradicting the fact that C and D are disjoint. □

Lemma 4.4. *If X is skew compact and $Y \subseteq X$ is $\tau \vee \tau^G$-closed (^{SG}-closed) in X, then Y is a skew compact subspace of X, and $\tau^G|Y = (\tau|Y)^G$.*

Proof. $(Y, \tau|Y, \tau^G|Y)$, as a subspace of a pairwise pH bitopological space (X, τ, τ^G), is pairwise pH (see Comment 2.3(d) of [Ko]); furthermore, $(Y, \tau|Y \vee \tau^G|Y)$ is a compact T_2-space since it is a closed subspace of the compact T_2-space $(X, \tau \vee \tau^G)$. This shows that $(Y, \tau|Y, \tau^G|Y)$ is joincompact and hence $(Y, \tau|Y)$ is skew compact. Furthermore, by the comments in the subsection on the de Groot dual, $\tau^G|Y = (\tau|Y)^G$, as required. □

Theorem 4.5. *Inverse limits of skew continua under de Groot relative quotient maps (that is, maps which are relative quotients with respect to both the topology and the de Groot dual topology), are skew continua.*

Proof. Suppose that (X, τ) is the limit of the inverse system $((X_\gamma, \tau_\gamma), \pi_{\gamma\delta})_{\gamma,\delta\in\Gamma}$. Then each $\pi_{\gamma\delta}[X_\gamma]$ is ^{SG}-closed in X_δ, so that the limit (X, τ, τ^*) of the inverse system of connected joincompact spaces $((X_\gamma, \tau_\gamma, \tau_\gamma^G), \pi_{\gamma\delta})_{\gamma,\delta\in\Gamma}$ is the limit of an inverse system whose bonding maps are, by Lemma 4.4, pairwise relative quotients. As a result, by Theorem 4.3, the inverse limit (X, τ, τ^*) is a connected joincompact space and so $\tau^* = \tau^G$ which implies that (X, τ) is a skew continuum. □

Corollary 4.6. *(a) Intersections of chains of ^{SG}-closed connected subspaces of skew continua are connected.*
(b) Inverse limits of continua under continuous maps, are continua.

Proof. These are special cases of Theorem 4.5, since embeddings and continuous maps on compact Hausdorff spaces, are de Groot relative quotient maps. □

Question 4.7. *Unlike compact Hausdorff topologies, a skew compact topology can be strictly stronger than another; for example, consider $\{0, 1\}$ with the Sierpinski and the discrete topologies. As a result, not every de Groot map on a skew compact space is a de Groot relative quotient. If the maps are surjective de Groot maps, must the inverse limit be a skew continuum? For example, can the Cantor space topology be obtained as an inverse limit of skew continuum topologies on it?*

Theorem 4.8. *A skew compact space is a skew continuum if and only if its Hausdorff reflection is connected.*

Proof. Let (X, τ) be a skew continuum and $h : X \to X_H$ its Hausdorff reflection. Since h is surjective X_H is a continuum. Conversely, if (X, τ) is skew compact but not connected, then there is a nonempty proper clopen subset $C \subseteq X$;

as pointed out in Comment 4.2(a), C is also ^{G}-clopen, so $\chi_C : X \to \{0,1\}$ is continuous onto the two-point Hausdorff space. Thus, there is an $f : X_H \to \{0,1\}$ so that $\chi_C = fh$. Now since $C = \chi^{-1}[\{1\}]$ it follows that $f^{-1}[\{1\}] = h[C]$ and hence X_H is disconnected. □

Theorem 4.9. *Suppose (X,τ) is Hausdorff. Then the following are equivalent:*

(a) (X,τ) is a continuum.

(b) (X,τ) is approximable by finite connected spaces.

(c) (X,τ) has a connected spectral compactification.

Proof. (a) $\Rightarrow$ (b) By Theorem 2.4, each compact Hausdorff space (X,τ) has an approximating family of finite T_0-spaces which are quotients of the connected space (X,τ). Thus these finite spaces are connected.

(b) $\Rightarrow$ (c) It is a consequence of Theorem 4.3 that (X,τ) is connected. The result now follows from Theorem 3.3 and the fact that X is ^{SG}-dense, and hence dense, in a spectral compactification.

(c) $\Rightarrow$ (a) This is immediate since (X,τ) is a continuous image of its spectral compactification. □

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