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SOME SEPARATION AXIOMS IN TOPOLOGICAL INVERSE SEMIGROUPS

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Abstract

In topological groups the separation axioms T_0 and T_2 are equivalent. This equivalence disappears if we consider more general algebraic structures like inverse semigroups or weaken the connection between the topological and algebraic structures.

A. Conte gave sufficient conditions for topological inverse semigroups which ensure the validity of separation axioms T_0, T_1, T_2 and those falling between T_0 and T_1 ([4], [5]).

The aim of this presentation is to study the separation axioms between T_1 and T_2, the axiom T_3, and also some order separation axioms introduced by McCartan [8] in semitopological groups and semitopological and topological inverse semigroups.

The given conditions show the importance of the set of idempotents for the separation of inverse semigroups.

0. Introduction

It is well-known that in topological groups the separation axioms $\mathbf{T}_0$ and $\mathbf{T}_2$ are equivalent. This equivalence disappears if we consider more general algebraic structures or weaken the connection between the topological and the algebraic structures ([4], [5], [7]).

A. Conte gave sufficient conditions for topological inverse semigroups which ensure the validity of the separation axioms $\mathbf{T}_0$, $\mathbf{T}_1$, $\mathbf{T}_2$ and those falling between $\mathbf{T}_0$ and $\mathbf{T}_1$ ([4], [5]). He also gave examples of topological inverse semigroups where the mentioned separation axioms are not equivalent. His idea was to require separation-like conditions related to the set of idempotents, or in the relation between an idempotent and another element.

The aim of this presentation is to study in semitopological and topological inverse semigroups separation axioms between $\mathbf{T}_1$ and $\mathbf{T}_2$, $\mathbf{T}_3$ and also those which satisfy certain order conditions.

The separation axioms under study here were introduced by C.E. Aull [1], [2] , H.F. Cullen [6] , McCartan [8] , W.J. Pervin, H.J. Biesterfeldt [9] , and A.Wilansky [10] .

For more properties of inverse semigroups one can see [3], [11].

1. Terminology

Definition 1. [3] *If a semigroup* $\boldsymbol{S}$ *satisfies the following conditions*

(i) $e_1e_2 = e_2e_1$ *for all idempotents* $e_1, e_2 \in \mathbf{S}$,

(ii) $\forall x \in \mathbf{S} \quad \exists y \in \mathbf{S} \quad xyx = x$ *and* $yxy = y$;

$\mathbf{S}$ *is said to be an* ***inverse semigroup.*** *We put* $\mathbf{E}_S$ *for the set of idempotents in* $\boldsymbol{S}$.

Remarks. [3] Let $\mathbf{S}$ be an inverse semigroup.

1. For $x \in \mathbf{S}$, the element y given by condition (ii) above is unique, we denote it by x^{-1}.
2. For all $x \in \mathbf{S}$ we have xx^{-1}, $x^{-1}x \in E_S$.
3. $(x^{-1})^{-1} = x$; $(xy)^{-1} = y^{-1}x^{-1}$; $x^3 = x \Leftrightarrow x = x^{-1}$; $x \in E_S \Longrightarrow x^{-1} \in E_S$; $e \in E_S \Longrightarrow e = e^{-1}$.

4. For a set $\mathbf{H}$ the collection $F(\mathbf{H})$ of all injective maps $f : \mathbf{A} \longrightarrow \mathbf{H}$ from subsets $\mathbf{A}$ of $\mathbf{H}$ into $\mathbf{H}$ is an inverse semigroup under partial composition. It is called the **symmetric inverse semigroup** of $\mathbf{H}$.

5. A partial order on $\mathbf{S}$ may be defined as follows $x \leq y$ if and only if the following equivalent conditions hold $(\alpha) xy^{-1} = xx^{-1}$; $(\beta) x^{-1}y = x^{-1}x$; $(\gamma) yx^{-1} = xx^{-1}$; $(\delta) y^{-1}x = x^{-1}x$; $(\varepsilon) xy^{-1}x = x$; $(\zeta) x^{-1}yx^{-1} = x^{-1}$.

6. The above order is compatible with the operation of $\mathbf{S}$. We shall make use of the following property of this order:

7. For $x, y \in \mathbf{S}$ if $x \neq y$ then either $xy^{-1} \neq yy^{-1}$ or $y^{-1}x \neq x^{-1}x$, ([4]).

Definition 2. *Let S be an inverse semigroup.*
Put $\mathbf{M_S} = \{\mathbf{m} \in \mathbf{S} \mid \forall y \in \mathbf{S}\ mm^{-1}y = y \text{ and } m^{-1}my = y\}$. *If for every $x \in \mathbf{S}$ there exists $m \in \mathbf{M}_S$ such that $x \leq m$, we shall say that S is* ***maximized.***

Definition 3. [7] $(\mathbf{S}, \cdot, \nu)$ *is a* ***semitopological inverse semigroup*** *if the following conditions are met:*

(i) $(\mathbf{S}, \cdot)$ *is an inverse semigroup,*

(ii) $(\mathbf{S}, \nu)$ *is a topological space.*

(iii) The operation is continuous in each component (equivalently: $\forall x, y \in \mathbf{S}$ and $V_{xy} \in \nu_{xy}$ $\exists V_x \in \nu_x$ and $V_y \in \nu_y$ such that $xV_y \subseteq V_{xy}$ and $V_x y \subseteq V_{xy}$).

Definition 4. [4] *A semitopological inverse semigroup is* ***topological*** *if in addition to (i), (ii), (iii)* $(\mathbf{S}, \cdot, \nu)$ *satisfies:*

(iv) The operation $\cdot$ is continuous (equivalently $\forall x, y \in \mathbf{S}$ and $\forall V_{xy} \in \nu_{xy}$ $\exists V_x \in \nu_x$ and $\exists V_y \in \nu_y$ such that $V_x V_y \subseteq V_{xy}$);

(v) *The unary operation* $(\ )^{-1}$ *(inversion) is continuous (equivalently* $\forall x \in \mathbf{S}$ *and* $\forall V_{x^{-1}} \in \nu_{x^{-1}}\ \exists V_x \in \nu_x$ *such that* $(V_x)^{-1} \subseteq V_{x^{-1}}$*).*

Definition 5. ([2],[6]) *A topological space is a **US**-space if convergent sequences have unique limits.*

Definition 6. [10] *A topological space is a **KC**-space if its compact sets are also closed.*

Remark. Obviously we have $\mathbf{T}_2 \Longrightarrow \mathbf{KC} \Longrightarrow \mathbf{US} \Longrightarrow \mathbf{T}_1$.

2. Axioms for Semitopological Groups

Remark. We can formulate and prove sufficient conditions for semitopological groups, which ensure the given separation, e.g.:

A semitopological group G (with the unit e)satisfies the separation axiom

1. $\mathbf{T}_0$ if for $\forall x \in G$ and $x \neq e$, either exists $V_x \in \nu_x$ such that $e \notin V_x$ or exists $V_e \in \nu_e$ such that $x \notin V_e$.
2. $\mathbf{T}_1$ if $\{e\}$ is closed.
3. **US** if $\forall \{x_n\}_{n\in \mathbf{N}}$, $x_n \longrightarrow x$, *and* $x_n \longrightarrow e$ we have x=e.
4. **KC** if e is a boundary point of a compact $K \subseteq G$, then $e \in K$.
5. $\mathbf{T}_2$ if $\forall \{x_\alpha\}_{\alpha\in \mathbf{A}}, x_\alpha \longrightarrow x$, and $x_\alpha \longrightarrow e$ we have $x = e$.
6. $\mathbf{T}_{2a}$ if $\forall x \in G, x \neq e, \exists V_x \in \nu_x$ and $\exists V_e \in \nu_e$ such that $\overline{V_x} \cap \overline{V_e} = \emptyset$.
7. $\mathbf{T}_3$ if $\forall V \in \nu_e\ \exists U \in \nu_e$ such that $\overline{U} \subseteq V$.
8. $\mathbf{T}_{3a}$ if $\forall F \subseteq G$, F closed and $e \notin F$, $\exists f : G \longrightarrow [0,1]$ such that $f(e) = 0$.

The proof is based on the homogeneity of semitopological groups and is left to the reader.

3. Axioms Between $\mathbf{T}_1$ and $\mathbf{T}_2$.

Proposition 1. *A semitopological maximized inverse semigroup* $\boldsymbol{S}$ *is a* ***KC*** *-space if and only if*
*(**kc**)* *for* $K \subseteq \mathbf{S}$ *compact and* $m \in \mathbf{M}_S$, *if an element* e *of* E_S *is a boundary point of* $m\boldsymbol{K}$ *then it is in* $m\boldsymbol{K}$.

Proof. To see necessity assume that $\mathbf{S}$ is a **KC**-space, $\mathbf{K} \subseteq \mathbf{S}$ is compact, $m \in \mathbf{M}_S$ and $e \in E_S$ is a boundary point of m**K**. Then there must be a net $\{e_\nu\}_{\nu\in\Lambda} \subseteq m\mathbf{K}$ with $e_\nu \to e$. It follows that $m^{-1}e_\nu \to m^{-1}e$. Since $\{m^{-1}e_\nu\}_{\nu\in\Lambda} \subseteq \mathbf{K}$ and $\mathbf{K}$ is closed, $m^{-1}e \in \mathbf{K}$, whence $e \in m\mathbf{K}$. Thus (**kc**) holds.

To show the converse, assume now that $\mathbf{S}$ satisfies (**kc**) but is not a **KC** space, i. e. there exists a compact $\mathbf{K} \subseteq \mathbf{S}$ with a boundary point x outside $\mathbf{K}$. We have a net $\{x_\alpha\}_{\alpha\in A} \subseteq \mathbf{K}$ with $x_\alpha \to x$. Pick an $m \in \mathbf{M}_S$ such that $x \leq m$. Then $m^{-1} \in \mathbf{M}_S$ and $m^{-1}x = x^{-1}x$ is a boundary point of $m^{-1}\mathbf{K}$, since $m^{-1}x_\alpha \to m^{-1}x = x^{-1}x \in E_S$. From (**kc**) $m^{-1}x \in m^{-1}\mathbf{K}$ whence $mm^{-1}x \in mm^{-1}\mathbf{K}$, and so $x \in \mathbf{K}$, a contradiction. □

Proposition 2. *A topological inverse semigroup is a* $\mathbf{T}_2$ *-space if and only if*
(t_2) *For every net* $\{x_\alpha\}_{\alpha\in A} \subseteq \mathbf{S}$,
if $x_\alpha \to e \in E_S$ *and* $x_\alpha \to x$ *then* $e = x$.

Proof. We have to show only sufficiency. Assume that $\mathbf{S}$ satisfies ($\mathbf{t}_2$) but is not a $\mathbf{T}_2$-space, i. e. there is a net for which $x_\alpha \to x$, $x_\alpha \to y$ but $x \neq y$. Clearly $x_\alpha y^{-1} \to xy^{-1}$, $x_\alpha y^{-1} \to yy^{-1}$, and $x_\alpha^{-1}x \to x^{-1}x$, $x_\alpha^{-1}x \to y^{-1}x$. But $yy^{-1}, x^{-1}x \in E_S$, whence $xy^{-1} = yy^{-1}$ and $x^{-1}x = y^{-1}x$ at the same time, a contradiction. □

Corollary 3. *A topological inverse semigroup is a* ***US*** *-space if and only if*

(us) *Every sequence* $\{x_n\}_{n\in N} \subseteq S$*, with* $x_n \to e \in E_S$*, has a unique limit.*

Definition 7. [9] *An* ***iterated net*** *in a topological space is a net* $\{x_\mu\}_{\mu\in M}$ *such that for every* $\mu \in M$ *there is an other net* $\{x^\mu_\delta\}_{\delta\in D_\mu}$ *such that* $x^\mu_\delta \to x_\mu$*. The* ***compound net*** *of* $\{x_\mu\}_{\mu\in M}$ *is* $\{x^\mu_{\beta(\mu)}\}_{(\mu,\beta(\mu))\in T}$ *where the index set* $T = M \times \prod_{\mu\in M} D_\mu$ *has the usual cartesian order.*

Proposition 4. *A semitopological maximized inverse semigroup* $\boldsymbol{S}$ *is a* $\mathbf{T}_3$ *-space if and only if*
($\mathbf{t}_3$) *For every iterated net* $\{x_\mu\}_{\mu\in M} \subseteq \mathbf{S}$*, if its compound* $x^\mu_{\beta(\mu)} \to e \in E_S$ *then* $x_\mu \to e$*.*

Proof. Necessity follows from a condition equivalent to $\mathbf{T}_3$ given in [9]. To verify sufficiency, let us assume that $\mathbf{S}$ satisfies ($\mathbf{t}_3$) and is not a $\mathbf{T}_3$-space, i. e. there exists an iterated net $\{x_\mu\}_{\mu\in M} \subseteq S$ which does not converge to the limit of its compound net $x^\mu_{\beta(\mu)}$; thus $x^\mu_{\beta(\mu)} \to x$ but $x_\mu \not\to x$. Further let $m \in M_S$ (hence $m^{-1} \in M_S$) such that $m \geq x$ (i. e. $mx^{-1} = xx^{-1}$ and $m^{-1}x = x^{-1}x$). Now $m^{-1}x^\mu_{\beta(\mu)} \to m^{-1}x = x^{-1}x \in E_S$ and so, by our assumption, $m^{-1}x_\mu \to x^{-1}x$. Consequently $mm^{-1}x_\mu \to mx^{-1}x = xx^{-1}x$, and thus $x_\mu \to x$, a contradiction. □

4. Order-Separation Axioms

McCartan introduced in [8] order-separation axioms for topological spaces which are at the same time ordered spaces. The study of these axioms is motivated by the fact that in semitopological and topological inverse semigroups, the algebraic structure induces a natural order (see Remarks in Terminology.).

Definition 8. [8] *An ordered topological space X is said to be*

lower (upper) $\mathbf{T}_1$ ***ordered*** *if for every net* $\{x_\alpha\}_{\alpha\in A}$
$x_\alpha \longrightarrow x$ *and* $x_\alpha \leq a \quad \forall\alpha \in A \Longrightarrow x \leq a$,
$(x_\alpha \longrightarrow x$ *and* $x_\alpha \geq a \quad \forall\alpha \in A \Longrightarrow x \geq a)$.

Example. Take the symmetric inverse semigroup $F(\mathbf{H})$ of the set **H,** with the topology induced by order. Multiplication and inversion are both continuous, thus the resulting topological inverse semigroup is upper $\mathbf{T}_1$-ordered, but not $\mathbf{T}_1$.

Proposition 5. *A semitopological inverse semigroup* $\boldsymbol{S}$ *is* ***lower*** $\mathbf{T}_1$ ***ordered*** *if and only if*
$(\mathbf{LT}_1)$ *For every net* $\{x_\alpha\}_{\alpha\in A} \subseteq S$ *and* $a \in S$ *if* $x_\alpha \rightarrow e \in E_S$ *and* $x_\alpha \leq a \quad \forall\alpha \in A$ *then* $e \leq a$.

Proof. We only need to show sufficiency. Assume that **S** satisfies $(\mathbf{LT}_1)$ and let $\{x_\alpha\}_{\alpha\in A} \subseteq S$ be a net with $x_\alpha \rightarrow a$ and further, let $b \in S$ be such that $x_\alpha \leq b \quad \forall\alpha \in A$. It follows that $x_\alpha a^{-1} \rightarrow aa^{-1}$ and $x_\alpha a^{-1} \leq ba^{-1} \quad \forall\alpha \in A$. Since $aa^{-1} \in E_S$, our assumption implies $aa^{-1} \leq ba^{-1}$, i. e. $ba^{-1}aa^{-1} = aa^{-1}aa^{-1}$.

In consequence $ba^{-1} = aa^{-1}$ $(a \leq b)$, and so **S** is lower $\mathbf{T}_1$-ordered. The almost dual statement is as follows: □

Proposition 6. *A semitopological inverse semigroup* $\boldsymbol{S}$ *is* ***upper*** $\mathbf{T}_1$***-ordered*** *if and only if*
$(\mathbf{UT}_1)$ *For every net* $\{x_\alpha\}_{\alpha\in A} \subseteq S$ and $e \in E_S$
if $x_\alpha \longrightarrow a \in S$ *and* $e \leq x_\alpha$ $\forall\alpha\in A$, *then* $e \leq a$.
The proof is left to the reader.

Definition 9. [8] *An ordered topological space is said to be* $\mathbf{T}_2$***-ordered*** *if for any pair* $\{x_\alpha\}_{\alpha\in A}$ *and* $\{y_\alpha\}_{\alpha\in A}$ *of nets with* $x_\alpha \longrightarrow x, y_\alpha \longrightarrow y$ *and* $x_\alpha \leq y_\alpha \quad \forall\alpha \in A$, *we have* $x \leq y$.

Proposition 7. *A semitopological inverse semigroup* $\boldsymbol{S}$ *is* $\mathbf{T}_2$***-ordered*** *if and only if*
$(\mathbf{OT}_2)$ *For any net* $\{e_\alpha\}_{\alpha\in A}$ *and* $\{x_\alpha\}_{\alpha\in A}$,

if $e_\alpha \to e \in E_S$, $x_\alpha \to x \in S$ *and* $e_\alpha \leq x_\alpha$ $\forall \alpha \in A$ *then* $e \leq x$.

Proof. We only need to show sufficiency. Assume that **S** satisfies $(\mathbf{OT}_2)$ and let $\{x_\alpha\}_{\alpha \in A}$, $\{y_\alpha\}_{\alpha \in A}$ be nets with $x_\alpha \leq y_\alpha$ $\forall \alpha \in A$. Assuming $x_\alpha \to x$ and $y_\alpha \to y$, we have $x_\alpha x^{-1} \to xx^{-1} \in E_S$, $y_\alpha x^{-1} \to yx^{-1}$ and $x_\alpha x^{-1} \leq y_\alpha x^{-1}$ $\forall \alpha \in A$. By $(\mathbf{OT}_2)$ we deduce that $xx^{-1} \leq yx^{-1}$, whence $yx^{-1}xx^{-1} = xx^{-1}$ and so $yx^{-1} = xx^{-1}$. Thus $x \leq y$. □

Remark.

These examples show the importance of the set of idempotents of a topological inverse semigroup when we need necessary conditions to satisfy separation axioms. In the case of semitopological groups one can formulate, as shown, similar, but more simple conditions for the neutral element of the group. In the case of order-separation axioms we formulated again conditions concerning an idempotent and another element.

Finally, let us introduce some "separation like" properties, and call them shadow separation axioms $\mathbf{ST}_i (i = 0, 1, 2, 2_a)$ to reflect their similarity to the usual separation axioms $\mathbf{T}_i (i = 0, 1, 2, 2_a)$.

If the definition of the separation axiom $\mathbf{T}_i$ in a topological space X has got the following form:

$(\mathbf{T}_i)$ *For all* $x \in X$ *and all* $y \neq x$ *property* $\mathbf{P}_i$ *holds.*

then the shadow separation axiom $\mathbf{ST}_i$ will have the following:

Definition 10. *A topological space* X *is said to be* $\mathbf{ST}_i$ ***space*** *if*

$(\mathbf{ST}_i)$ *For all* $x \in X$ *and some* $y \neq x$ *property* $\mathbf{P}_i$ *holds.*

Obviously an $\mathbf{ST}_i$ shadow axiom is weaker than the respective $\mathbf{T}_i$ separation axiom.

Example. Take a topological space X satisfying the separation axiom $\mathbf{T}_i$, and an "outside" $y \notin X$. The set $X \cup y$ will be an $\mathbf{ST}_i$ topological space, if the open sets G_y of $X \cup \{y\}$ are $G \cup y$, where G is open in X.

Example. Every upper(-lower) $\mathbf{T}_1$-ordered topological space is at the same time a $\mathbf{ST}_1$ space, and every $\mathbf{T}_2$ -ordered topological space is $\mathbf{ST}_2$.

Closing Remark. The majority of the conditions given in [4], [5] and the present paper for separation axioms $\mathbf{T}_i$ will remain true for similar conditions for the shadow separation axioms $\mathbf{ST}_i$ introduced here.

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