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Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA
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CONVEXITY AND THE BROUWER FIXED POINT THEOREM

Władysław Kulpa

Abstract

In this paper, a class of spaces which is a generalization of topological linear spaces is introduced. The Schauder fixed point theorem and the Helly theorem on centered families of convex sets are proved. A new characterization of metric ANR and AR-spaces is given.

1. Introduction

The aim of this paper is to present a class of spaces which contains topological linear spaces, simplicial complexes and topological manifolds. Some concepts are taken from algebraic topology. However, the main tool is the Brouwer fixed point theorem, which appears here in the form of four lemmas of Sperner. A simple and short proof of the Brouwer theorem, based on combinatorial technique and Sperner's lemma, was given by Knaster, Kuratowski and Mazurkiewicz in 1929. Nowadays, there are many proofs of this theorem which omit combinatorial methods (see, e.g. [6]). And this is the justification for using the fixed point theorem in proofs of the lemmas.

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Key words: the Schauder fixed point theorem, Schauder's conjecture, the Helly theorem on convex sets, the Dugundji extension theorem, ANR and AR-spaces

A set of $(n+1)$ points $p_0, \dots, p_n \in R^m$ is said to be (affinely) *independent* if the vectors $p_1 - p_0, \dots, p_n - p_0$ are linearly independent. This is equivalent to the statement that for each real numbers $t_0, \dots, t_n$ the following implication holds

$$\sum_{i=0}^{n} t_i \cdot p_i = 0 \ \ \& \ \ \sum_{i=0}^{n} t_i = 0 \qquad \Longrightarrow \quad t_0 = \dots = t_n = 0$$

The definition of independence does not depend on the order of points $p_0, \dots, p_n$.

Let the points $p_0, \dots, p_n$ be independent. Their convex hull

$$conv\{p_0, \dots, p_n\} := \{x \in R^m : x = \sum_{i=0}^{n} t_i \cdot p_i, \ \sum_{i=0}^{n} t_i = 1, \ 0 \le t_i\}$$

with the subspace topology is said to be n-dimensional (geometric) *simplex* spanned by the vertices p_i. We shall use notation $[p_0, \dots, p_n]$ instead of $conv\{p_0, \dots, p_n\}$. From independence of the points it follows that each point $x \in [p_0, \dots, p_n]$, $x = \sum_{i=0}^{n} t_i \cdot p_i$ is uniquely determined by its *barycentric coordinates* $t_i \ge 0$.

Let $P = [p_0, \dots, p_n]$ be an n-dimensional simplex. The subset

$$[p_0, \dots, \hat{p}_i, \dots, p_n] := [p_0, \dots, p_{i-1}, p_{i+1}, \dots, p_n]$$

is called the i-th $(n-1)$-dimensional *face* of the n-dimensional simplex $[p_0, \dots, p_n]$.

First Lemma of Sperner. *Let $\{A_0, \dots, A_n\}$ be an open (or a closed) covering of n-dimensional simplex $P = [p_0, \dots, p_n]$. Then there exists a sequence $0 \le i_0 < \dots < i_k \le n$ such that*
(i) $[p_{i_0}, \dots, p_{i_k}] \cap A_{i_0} \cap \dots \cap A_{i_k} \neq \emptyset$.

Proof. (I). Assume that the sets A_i are open and let us define a continuous map $f : P \longrightarrow P$

$$f(x) = \sum_{i=0}^{n} \frac{d_i(x)}{d(x)} \cdot p_i$$

where $d_i(x) := inf\{||x - y|| : y \in P \setminus A_i\}, d(x) = \sum_{i=0}^{n} d_i(x)$.

Since the sets A_i form an open covering of the simplex P, we infer that $d(x) > 0$ for each point $x \in P$. According to the Brouwer fixed point theorem there exists a point $a \in P$ such that $f(a) = a$. This means that

$$d_i(a) = t_i(a) \cdot d(a) \quad \text{for each} \quad i = 0, \ldots, n$$

Since the sets A_i are open and $d(a) > 0$ we infer that

$$t_i(a) > 0 \quad \text{if and only if } a \in A_i \quad \text{for each} \quad i = 0, \ldots, n.$$

Now, let us put $\{i_0, \ldots, i_k\} = \{i \leq n : t_i(a) > 0\}$. Then, from the above we get

$$a \in [p_{i_0}, \ldots, p_{i_k}] \cap A_{i_o} \cap \ldots \cap A_{i_k}.$$

(II). Assume now, that the sets A_i are closed and define for each $m = 1, 2, \ldots$ open sets

$$U_i^m := \{x \in P : ||x - a|| < \frac{1}{m} \quad \text{for some } a \in A_i\}$$

From the part (I) it follows that for each $m = 1, 2, \ldots$ there exists a set $I_m \subset \{0, \ldots, n\}$ and a point $a_m \in P$ such that

$$a_m \in conv\{p_i : i \in I_m\} \ \cap \ \bigcap\{U_i^m : i \in I_m\}$$

Since the family of distinct sets I_m is finite and the simplex P is compact, there exists an infinite subset $M \subset N$ such that for each $m \in M$ we have

$$I_m = I \quad \text{and} \quad \lim_{m \in M} a_m = a \in P.$$

And this implies that $a \in conv\{p_i : i \in I\} \ \cap \ \bigcap\{A_i : i \in I\}$. □

Second Lemma of Sperner. *If open (or closed) sets* $B_0, \ldots, B_n$ *of an* n*-dimensional simplex* $P = [p_0, \ldots, p_n]$ *satisfy the following condition*
(ii) $[p_{i_0}, \ldots, p_{i_k}] \subset B_{i_0} \cup \ldots \cup B_{i_k}$ *for each* $0 \leq i_0 < \ldots < i_k \leq n$
then $B_0 \cap \ldots \cap B_n \neq \emptyset$.

Proof. Suppose that $B_0 \cap \ldots \cap B_n = \emptyset$ and let us put $A_i := P \setminus B_i$. Then the family $\{A_0, \ldots, A_n\}$ is a closed (open) covering of the simplex P and according to (ii)

$$[p_{i_0}, \ldots, p_{i_k}] \cap A_{i_0} \cap \ldots \cap A_{i_k} = \emptyset \quad \text{for each } 0 \leq i_0 < \ldots < i_k \leq n,$$

contrary to (i) of the First Lemma of Sperner. □

Third Lemma of Sperner. *If* $\{B_0, \ldots, B_n\}$ *is an open (or a closed) covering of an* n*-dimensional simplex* $P = [p_0, \ldots, p_n]$ *such that*
(iii) $B_i \cap [p_0, \ldots, \hat{p}_i, \ldots, p_n] = \emptyset$ *for each* $i = 0, \ldots, n$,
then $B_0 \cap \ldots \cap B_n \neq \emptyset$

Proof. It suffices to check that the sets B_i satisfy the condition (ii) of the Second Lemma of Sperner. Let us fix a sequence $0 \leq i_0 < \ldots < i_k \leq n$ and $i \neq i_0, \ldots, i_k$. From (iii) if follows that

$$[p_{i_0}, \ldots, p_{i_k}] \subset [p_0, \ldots, \hat{p}_i, \ldots, p_n] \subset P \setminus B_i$$

Since $P = B_0 \cup \ldots \cup B_n$, we infer that $[p_{i_0}, \ldots, p_{i_k}] \subset B_{i_0} \cup \ldots \cup B_{i_k}$. □

Fourth Lemma of Sperner. *Let* $\{A_0, \ldots, A_n\}$ *be a closed (or an open) covering of an* n*-dimensional simplex* $P = [p_0, \ldots, p_n]$. *If*
(iv) $[p_0, \ldots, \hat{p}_i, \ldots, p_n] \subset A_i$ *for each* $i = 0, \ldots, n$,
then $A_0 \cap \ldots \cap A_n \neq \emptyset$.

Proof. Suppose that $A_0 \cap \ldots \cap A_n = \emptyset$ and let us put $B_i := P \setminus A_i$. From (iv) it follows that the condition (iii) holds for the covering $\{B_0, \ldots, B_n\}$. Therefore $\emptyset \neq B_0 \cap \ldots \cap B_n = P \setminus (A_0 \cup \ldots \cup A_n)$, contrary to $P = A_0 \cup \ldots \cup A_n$. □

2. Simplicial Structures

Let $X = (X, \mathcal{T})$ be a topological space. Any continuous map $\sigma : [p_0, \ldots, p_n] \longrightarrow X$ from a geometric simplex into X, is said to be *singular simplex* contained in X. For each singular simplex $\sigma : [p_0, \ldots, p_n] \longrightarrow X$ let us introduce the following notations:

$$dom\ \sigma := [p_0, \ldots, p_n],\ im\ \sigma := \sigma[p_0, \ldots, p_n],$$
$$vert\ \sigma := \{\sigma(p_0), \ldots, \sigma(p_n)\}.$$

For a given topological space $(X, \mathcal{T})$ the family of all singular simplexes contained in X will be denoted by Σ.

A family $\mathcal{F} \subset \Sigma$ is said to be *simplicial structure* in a space X if for each singular simplex $\sigma \in \mathcal{F}$, $\sigma : [p_0, \ldots, p_n] \longrightarrow X$ and for each sequence of indexes $0 \leq i_0 < ... < i_k \leq n$ we have $\sigma|[p_{i_0}, \ldots, p_{i_k}] \in \mathcal{F}$.

A triple $(X, \mathcal{T}, \mathcal{F})$, where $\mathcal{T}$ is a topology on X and $\mathcal{F}$ is a simplicial structure in the space $(X, \mathcal{T})$ is said to be *topological simplicial space.* In the case when (X, ρ) is a metric space or $(X, ||\cdot||)$ is normed space, the triples $(X, \rho, \mathcal{F})$, $(X, ||\cdot||, \mathcal{F})$ will be called metric, or normed simplicial space.

Example 1.
1. It is clear that for a given space X the family Σ is an example of simplicial structure.
2. Also the family $\mathcal{C} \subset \Sigma$ of all the constant maps is a simplicial structure.
3. A very important example of simplicial structure is the family $\mathcal{L} \subset \Sigma$ of all affine maps , $l : [p_0, \ldots, p_n] \longrightarrow X$; $l(\sum_{i=0}^{n} t_i \cdot p_i) = \sum_{i=0}^{n} t_i \cdot l(p_i)$, where X is a convex subset of a linear topological space E.
4. A topological manifold can be described by a simplicial structure $\mathcal{I} \subset \Sigma$ consisting of all the singular simplices which

are one-to-one maps.

Now, we are going to extend the notion of convexity from linear topological spaces onto topological simplicial spaces.

A topological simplicial space $(X, \mathcal{T}, \mathcal{F})$ is said to be *convex* if for each finite set $A \subset X$ there exists a simplex $\sigma \in \mathcal{F}$ such that $A = vert\ \sigma$, and it is *locally convex at a point* $x \in X$ if for each its open neighbourhood U_x there exists an open set V_x, $x \in V_x \subset U_x$ such that (a) for each finite subset $F \subset V_x$ there exists $\sigma \subset \mathcal{F}$ with $vert\ \sigma = F$, and (b) for each $\sigma \in \mathcal{F}$; $vert\ \sigma \subset V_x \quad \Longrightarrow \quad im\ \sigma \subset U_x$

A simplicial space X which is locally convex at each point $x \in X$ is said to be *locally convex.*

Let us recall that a subset $C \subset X$ of a topological linear space X is convex if for each $n+1$ points $c_0, ..., c_n \in C$, each convex combination $\sum_{i=0}^{n} t_i \cdot c_i$ belongs to C. In our terminology it means that for each singular linear simplex $\sigma \in \mathcal{L}$; $vert\ \sigma \subset C$ implies $im\ \sigma \subset C$. Thus in the case when X is a topological linear space and $\mathcal{F} = \mathcal{L}$ is a simplicial structure consisting of all the affine simplices, then the notion of convexity in our sense coincides with the notion convexity in the classical sense.

3. A Fixed Point Theorem

In this part we shall use the First Lemmma of Sperner as a main tool for investigating fixed points. Let us state the lemma as

Theorem on Indexed Covering. *Let $\{U_o, ..., U_n\}$ be an open covering of a topological space and $\sigma : [p_0, \dots, p_n] \longrightarrow X$ a singular simplex. Then there exists a sequence $0 \le i_0 < ... < i_k \le n$ of indexes such that $\sigma[p_{i_0}, \dots, p_{i_k}] \cap U_{i_0} \cap ... \cap U_{i_k} \neq \emptyset$*

The following theorem is a sharpened version of the Schauder fixed point theorem for convex subspaces of normed spaces.

The Schauder Fixed Point Theorem. *Let $(X, \mathcal{T}, \mathcal{F})$ be a convex Hausdorff topological simplicial space, and let $g : X \longrightarrow X$ be a continuous map such that $\overline{g(X)}$ is compact and X is locally convex at each point $x \in \overline{g(X)}$. Then g has a fixed point.*

Proof. Let us put $Y := \overline{g(X)}$ and suppose, contrary to our claim, that $g(x) \neq x$ for each $x \in X$. Since X is a Hausdorff space hence for each $x \in X$ there exists an open neighbourhood W_x of x such that
(1) $W_x \cap g(W_x) = \emptyset$
Let us put $\mathcal{W} = \{W_x : x \in Y\}$ and let $\mathcal{V}$ be an open covering of Y satisfying the condition of local convexity:
(2) for each $V \in \mathcal{V}$ there exists $W(V) \in \mathcal{W}$, $V \subset W(V)$, such that for each $\sigma \in \mathcal{F}$;
$\quad vert\, \sigma \subset V \implies im\, \sigma \subset W(V)$.
The family $\mathcal{V}$ is an open covering of Y, which is a Hausdorff compact space and therefore there exists a relatively open in Y finite covering $\mathcal{U} = \{U_0, ..., U_n\}$ which is a star-refinement of $\mathcal{V}$ (cf. Engelking [4],p. 377) i.e., $Y = U_0 \cup ... \cup U_n$ and for each $y \in Y$ there exists $V \in \mathcal{V}$ such that
(3) $st(y, \mathcal{U}) := \bigcup\{U \in \mathcal{U} : y \in U\} \subset V$
Convexity of X implies that there exists a singular simplex $\sigma \in \mathcal{F}$, $\sigma : [p_0, ..., p_n] \longrightarrow X$, such that $\sigma(p_i) \in U_i$ for each $i = 0, \ldots, n$. The family $\{g^{-1}(U_i) : i = 0, \ldots, n\}$ is an open covering of X and according to the theorem on indexed covering there exist a sequence $0 \leq i_0 < ... < i_k \leq n$ and a point $w \in X$ such that
(4) $w \in \sigma[p_{i_0}, ..., p_{i_k}] \cap g^{-1}(U_{i_0}) \cap .... \cap g^{-1}(U_{i_k})$
From the above we get that $g(w) \in U_{i_0} \cap ... \cap U_{i_k}$ and since $\sigma(p_i) \in U_i$, we infer from (3) that there exists $V \in \mathcal{V}$ such that
(5)$\sigma(p_{i_0}), ..., \sigma(p_{i_k}) \in st(g(w), \mathcal{U}) \subset V$
But the condition (2) of local convexity implies that $w, g(w) \in W(V)$, contrary to (1). □

We shall show that the assumption of local convexity is essential.

Example 2. Fix $n > 0$ and let us define $X := Q \cup \{p\}$, where $Q := \{x \in R^n : ||x|| < 1\}$ and $p := (1, 0, \ldots, 0) \in R^n$. The set X is a convex subset of R^n. Let $\mathcal{L}$ be the affine simplicial structure consisting of all the affine simplices in X. Now, describe a new topology $\mathcal{T}$ on X generated by a base of open neighbourhoods; for every $x \in Q$ and $\epsilon > 0$ define neighbourhoods $U_x(\epsilon) := \{y \in Q : ||x - y|| < \epsilon\}$ the same as in the Euclidean topology, and for p, put $U_p(\epsilon) := \{p\} \cup \{x \in Q : ||x|| > \epsilon\}$, $0 < \epsilon < 1$, $\epsilon \longrightarrow 1$.

The topology $\mathcal{T}$ is weaker than the Euclidean topology on X and therefore the triple $(X, \mathcal{T}, \mathcal{L})$ is a topological simplicial space. The space X is locally convex at each point $x \neq p$ because the neighbourhoods $U_x(\epsilon)$ are linearly convex. It is easy to see that X is not locally convex at p because for each point $x \in U_p(\epsilon)$, $x \neq p$, the 1-dimensional linear simplex with vertexes x and $-x$ must contain $0 = (0, \ldots, 0)$.

Let $B := \{x \in R^n : ||x|| \leq 1\}$. It is known that the quotient space $B/\partial B$ is homeomorphic to n-dimensional sphere $S^n := \{x \in R^n : ||x|| = 1\}$, and therefore the space $(X, \mathcal{T})$ has not fixed point property. Thus the Schauder Theorem does not hold for the space $(X, \mathcal{T}, \mathcal{L})$ though it is convex and locally convex at each point but one.

This example is related to still unsolved problem of Schauder from the *Scottish Book* [8, Problem 54], which can be expressed in our terminology as the following question; Is the assumption of local convexity, in the Schauder fixed point theorem, essential for linear topological spaces (with the simplicial linear structure)? Our example shows that the answer is affirnative for spaces with simplicial affine structure and non-linear topology.

The example shows also that the sphere S^n has a simplicial structure consisting of the all maps $h \circ l,\ l \in \mathcal{L}$, where $h : X \longrightarrow S^n$ is a fixed homeomorphism. This structure is convex but, as it was shown, cannot be locally convex. On the other hand one can find a simplicial locally convex structure on S^n but in view of the Schauder Theorem it cannot be convex.

Some new results related to the Schauder problem can be found in [9].

4. Centered Families and Convexity

Let us recall the definition of covering dimension, *dim* X, of a topological space X; *dim* $X < n$ provided that for each open covering $\mathcal{W}$ there exists an open covering $\mathcal{U}$ such that $\mathcal{U}$ is a refinement of $\mathcal{W}$ (i.e., for each $U \in \mathcal{U}$ there is $W \in \mathcal{W}$ such that $U \subset W$) and for each $x \in X$; $|\{U \in \mathcal{U} : x \in U\}| \leq n$.

Lemma on Collapse of a Singular Simplex.
If $\sigma : [p_0, ..., p_n] \longrightarrow X$ is a singular simplex in a Hausdorff space X and dim $X < n$, then

$$\bigcap\{\sigma[p_0, ..., \hat{p}_i, ..., p_n] : i = 0, ..., n\} \neq \emptyset.$$

Proof. Suppose, contrary to our claim, that $\sigma(P_0) \cap ... \cap \sigma(P_n) = \emptyset$, where $P := [p_0, \ldots, p_n]$ and $P_i := [p_0, \ldots, \hat{p}_i, \ldots, p_n]$. Then $\mathcal{W} = \{W_0, \ldots, W_n\}$, where $W_i := X \setminus \sigma(P_i)$, is an open covering of X. Let $\mathcal{U} = \{U_s : s \in S\}$ be an open covering of X, which is a refinement of $\mathcal{W}$. Since $\mathcal{U}$ is a refinement of $\mathcal{W}$ hence there exists a function $\phi : S \longrightarrow \{0, \ldots, n\}$ such that $U_s \subset W_{\phi(s)}$ for each $s \in S$. Letting $B_i := \bigcup\{\sigma^{-1}(U_s) : \phi(s) = i\}$ we obtain an open covering $\{B_0, \ldots, B_n\}$ of the simplex P, which satisfies the condition (iii) of the Third Lemma of Sperner and therefore there exists a point $a \in P$ such that $a \in B_0 \cap ... \cap B_n$. Hence $|\{s \in S : \sigma(a) \in U_s\}| > n$. But this means that $dim X \geq n$, a contradiction. □

Observe, that from the above lemma it follows that $dim\ R^n \geq n$.

A subset C of a convex topological simplicial space $(X, \mathcal{T}, \mathcal{F})$ is said to be *convex* if for each $\sigma \in \mathcal{F}$; *vert* $\sigma \subset C$ implies im $\sigma \subset C$.

A family $\mathcal{C}$ of sets is said to be *centered* if each finite subfamily $\mathcal{D} \subset \mathcal{C}$ has a non-empty intersection.

The Helly Theorem. *Let $(X, \mathcal{T}, \mathcal{F})$ be a convex topological space with dim $X < n$. If $\mathcal{C} \subset 2^X$ is a family of convex subsets such that each n element subfamily $\mathcal{D} \subset \mathcal{C}$ has a non-empty intersection, then $\mathcal{C}$ is centered.*

Proof. There is no loss of generality in assuming $\mathcal{C} = \{C_0, \ldots, C_m\}$ and $m \geq n$. By induction we may assume also that for each $i = 0, \ldots, m$ there exists $x_i \in C_0 \cap ....C_{i-1} \cap C_{i+1}, \ldots, C_m$. Since X is convex, hence there exists $\sigma \subset \mathcal{F}$, $\sigma : [p_0, \ldots, p_m] \longrightarrow X$ such that $\sigma(p_i) = x_i$. By Lemma on a collapse of a singular simplex there is $c \in \bigcap\{\sigma[p_0, \ldots, \hat{p}_i, \ldots, p_m] : i = 0, \ldots, m\}$. From the choice of the points x_i it follows that $x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_m \in C_i$. Hence, since C is convex, we have; $c \in \sigma[p_0, \ldots, \hat{p}_i, \ldots, p_m] \subset C_i$ for each i, and this yields $c \in C_0 \cap ... \cap C_m$. □

Helly's theorem which was first published in 1921 and proved for $X = R^n$ plays an important role in the geometry of convex sets. For a recent account of results related to the Helly theorem and its applications we refer the reader to [5].

Numerous applications of the Second Lemma of Sperner, which is known as the Kuratowski-Knaster-Mazurkiewicz Theorem, were developed by Ky Fan, and presented as the theory of KKM-maps. We shall only show how to extend the definition of KKM-maps onto topological simplicial spaces.

Let $(E, \mathcal{T}, \mathcal{F})$ be a convex topological simplicial space and $X \subset E$ a given subset. A map $G : X \longrightarrow 2^E$ is called a

KKM-map if for each $\sigma \in \mathcal{F}$;

$$vert\ \sigma \subset X \quad \Longrightarrow \quad im\ \sigma \subset \bigcup\{G(x) : x \in vert\ \sigma\}.$$

It is clear that, the following statement is equivalent to the Second Lemma of Sperner

KKM-map Principle. *If $G : X \longrightarrow 2^E$ is a KKM-map and $G(x)$ is closed for each $x \in X$, then the family $\{G(x) : x \in X\}$ is centered.*

For informations on KKM-maps the reader is referred to Dugundji-Granas book [3].

For each subset $A \subset X \times Y$ let us define

$$A_x := \{y \in Y : (x,y) \in A\}, \qquad A^y : \{x \in X : (x,y) \in A\}$$

Now, assume that X is a topological space and $(Y, \mathcal{T}, \mathcal{F})$ is a convex topological simplicial space. A set $A \subset X \times Y$ is said to be a *river* if for each $x \in X$, A_x is convex and non-empty. A river $A \subset X \times Y$ is *parallel to* X if there exists $y \in Y$ such that $X \times \{y\} \subset A$ (i.e., $X = A^y$) and A is *locally parallel to* X if for each $x \in X$ there exists $y \in Y$ such that $x \in Int\ A^y$.

Theorem on a Parallel River. *Assume that dim $Y \leq n$ and let $A \subset X \times Y$ be a river such that; 1. A_x is compact and closed for each $x \in X$, and 2. for each $(n+1)$ points $x_0, \ldots, x_n \in X$ there exists $y \in Y$ such that $x_0, \ldots, x_n \in A^y$. Then A is parallel to X.*

Proof. Fix $x_0, \ldots, x_n \in X$ and choose $y \in Y$ such that $x_0, \ldots, x_n \in A^y$. But this is equivalent to $y \in A_{x_0} \cap \ldots \cap A_{x_n}$. In view of the Helly Theorem we infer that $\{A_x : x \in X\}$ is a centered family and therefore there exists $z \in \bigcap\{A_x : x \in X\}$. This finishes the proof that $X \times \{z\} \subset A$. $\square$

Theorem on a River and a Stream. *Let $A \subset X \times Y$ be a river locally parallel to X and let $g : Y \longrightarrow X$ be a continuous map such that $\overline{g(X)}$ is compact. Then there exists $w \in Y$ such that $(g(w), w) \in A$.*

Proof. From the assumption it follows that $\{Int\ A^y : y \in Y\}$ is an open covering of X. Since $\overline{g(X)}$ is compact, there exist points $y_0, \ldots, y_n \in Y$ such that $\overline{g(X)} \subset U_0 \cup \ldots \cup U_n$, where $U_i := Int\ A^{y_i}$. Choose $\sigma \in \mathcal{F}$, $\sigma : [p_0, \ldots, p_n] \longrightarrow Y$ such that $\sigma(p_i) = y_i$ for $i = 0, \ldots, n$. The family $\{g^{-1}(U_0), \ldots, g^{-1}(U_n)\}$ is an open covering of Y and according to Theorem on indexed covering there exist a sequence $0 \leq i_0 < \ldots < i_k \leq n$ and a point $w \in Y$ such that $w \in \sigma[p_{i_0}, \ldots, p_{i_k}] \cap g^{-1}(U_{i_0}) \cap \ldots \cap g^{-1}(U_{i_k})$. Hence $g(w) \in U_{i_0} \cap \ldots \cap U_{i_k} \subset A^{y_{i_0}} \cap \ldots \cap A^{y_{i_k}}$. This implies that $y_{i_0}, \ldots, y_{i_k} \in A_{g(w)}$ i.e., $vert\ \sigma|[p_{i_0}, \ldots, p_{i_k}] \subset A_{g(w)}$. This and convexity of $A_{g(w)}$ yields $w \in A_{g(w)}$, which completes the proof. □

The theorem has the following interpretation; A river locally parallel to X and a stream running along Y (= the graph of $g : Y \longrightarrow X$) must meet themselves.

5. Simplex-Like Families

In this part we shall discuss some applications of the Fourth Lemma of Sperner.

A finite family $\{C_0, \ldots, C_n\}$ of subsets of a topological simplicial space $(X, \mathcal{T}, \mathcal{F})$ is said to be simplex-like if for each i; $C_0 \cap \ldots \cap C_{i-1} \cap C_{i+1} \cap \ldots. \cap C_n \neq \emptyset$.

Define $|x| := \sum_{i=0}^{n} |x_i|$ if $x = (x_0, \ldots, x_n) \in R^{n+1}$

Equilibrium Theorem. *Let $\{C_0, \ldots, C_n\}$ be a simplex-like family of convex subsets in a convex topological space $(X, \mathcal{T}, \mathcal{F})$ and let $f : X \longrightarrow [0, \infty)^{n+1}$, $f = (f_0, \ldots, f_n)$, be a continuous map such that for each i; $f_i(C_i) = \{0\}$.*

Then for each continuous map $g : X \longrightarrow [0, \infty)^{n+1}$ *there exists a point* $x \in X$ *such that;* $f(x) \cdot |g(x)| = |f(x)| \cdot g(x)$.

Proof. Choose points $x_i \in C_0 \cap \ldots \cap C_{i-1} \cap C_{i+1} \cap \ldots \cap C_n$ and a singular simplex $\sigma \in \mathcal{F}$, $\sigma : [p_0, \ldots, p_n] \longrightarrow X$ such that $\sigma(p_i) = x_i$ for each $i = 0, \ldots, n$, and let us put $F := f \circ \sigma$, $G := g \circ \sigma$, $P := [p_0, \ldots, p_n]$, *and* $P_i := [p_0, \ldots, \hat{p}_i, \ldots, p_n]$. Convexity of the sets C_i implies that $\sigma(P_i) \subset C_i \subset f_i^{-1}(0)$. Letting

$$A_i := \{x \in P : F_i(x) \cdot |G(x)| \leq G_i(x) \cdot |F(x)|\},$$

we obtain a family of closed sets satisfying the condition (iv) of the Fourth Lemma of Sperner.

Observe that $P = A_0 \cup \ldots \cup A_n$, because, if $a \in P \setminus (A_0 \cup \ldots \cup A_n)$, then for $x := \sigma(a)$; $f_i(x) \cdot |g(x)| > g_i(x) \cdot |f(x)|$, for each $i = 0, \ldots, n$, and in consequence

$$|f(x)| \cdot |g(x)| = \sum_{i=0}^{n} f_i(x) \cdot |g(x)| > \sum_{i=0}^{n} g_i(x) \cdot |f(x)| = |g(x)| \cdot |f(x)|,$$

a contradiction.

According to the Fourth Lemma of Sperner there exist $a \in A_0 \cap \ldots \cap A_n$. Letting $x = \sigma(a)$, the same reasoning as above yields; $f(x) \cdot |g(x)| = g(x) \cdot |f(x)|$. □

For any point $x \in X$, where (X, ρ) is a metric space, let $d(x, A) = inf\ \{\rho(x, a) : a \in A\}$ denotes as usual the distance between the point x and the set A.

Corollary 5.1. *Let* $\{C_0, \ldots, C_n\}$ *be a simplex-like family in a convex metric simplicial space* (X, ρ) *and let be given sets* $A_i \subset X$ *with* $C_i \subset A_i$ *for each* $i = 0, \ldots, n$. *Then there exists* $x \in X$ *such that* $d(x, A_0) = \ldots = d(x, A_n)$.

Proof. Apply the Equilibrium Theorem to the constant map $g : X \longrightarrow \{t\}$, $t = (1/(n+1), \dots, 1/(n+1))$ and $f : X \longrightarrow [0,\infty)^{n+1}$, where $f_i(x) := d(x, A_i)$. It is clear that $x \in X$, where $f(x) = t \cdot |f(x)|$ is the point as we asserted. □

We turn now to some applications of the Equilibrium Theorem for the case where X is equal to an n-dimensional simplex $D := [d_0, \dots, d_n]$ and the simple-like family $\{C_0, \dots, C_n\}$ consists of the i-th $(n-1)$-dimensional faces $D_i = [d_0, \dots, \hat{d}_i, \dots, d_n]$, $C_i = D_i$, of the simplex D.

Fix $n > 1$. Let $T := \{x \in [0,\infty)^{n+1} : |x| = 1\}$ be n-dimensional standard simplex. In our notation $T = [e_0, \dots, e_n]$, where $e_0 = (1, 0, \dots, 0)$, $e_1 = (0, 1, 0, \dots, 0), ..., e_n = (0, \dots, 0, 1)$. Then $T_i = \{t \in T : t_i = 0\}$ is the i-th face of T.

1. Assume that $f : T \longrightarrow T$ is the identity map $g : T \longrightarrow [0,\infty)^{n+1}$ is continuous. The Equilibrium Theorem implies that there exists $x \in T$ such that $g(x) = |g(x)| \cdot x$. This observation gives a generalization of the Perron-Frobenius Theorem which states that every square matrix $\{a_{ij}\}$ with $a_{ij} \geq 0$ has at least one non-negative real eigenvalue. This theorem plays very important role in economics models (cf. H. Nikaido [10]).
2. If, in addition, $g(T) \subset T$, then $|g(x)| = 1$, and as a consequence we obtain the Brouwer Fixed Point Theorem.
3. Throughout the remaing part of this paper we shall deal with constant maps $g : D \longrightarrow T$ and f arbitrary satisfying the assumptions of the Equilibrium Theorem. If $g(x) = t$, then we immediately obtain

Corollary 5.2. *Let $f : D \longrightarrow [0,\infty)^{n+1}, f = (f_0, \dots, f_n)$, be a continuous map such that $f_i(D_i) = \{0\}$ for each $i = 0, \dots, n$. Then for each point $t \in T$, there exists $x \in D$ such that $f(x) = |f(x)| \cdot t$.*

Using the above corollary one can obtain direct proofs of the Kuratowski-Steinhaus Sandwich Theorem [7, 1] and a theorem on dividing of curves due to Urbanik [12].

Let $\mu(A)$ means the n-dimensional Lebesgue measure of the set $A \subset R^n$. For any point $x \in D$ let us denote

$$D_i(x) := [d_0, \dots, d_{i-1}, x, d_{i+1}, \dots, d_n]$$

the convex hull of the set $\{d_0, \dots, d_{i-1}, x, d_{i+1}, \dots, d_n\}$.

Sandwich Theorem. *Let $A \subset D$ be a measurable set. Then for any point $t \in T$ there exists a point $x \in D$ such that for each $i = 0, \dots, n$*

$$\mu[A \cap D_i(x)] = t_i \cdot \mu(A)$$

Proof. Define a continuous map $f : D \longrightarrow [0, \infty)^{n+1}$, $f = (f_0, \dots, f_n)$,

$$f_i(x) := \mu[A \cap D_i(x)] \; i = 0, \dots, n$$

It is clear that for each $x \in D$; $|f(x)| = \mu(A)$.

According to Corollary 5.2. for each point $t \in T$ there is a point $x \in D$ such that $f(x) = |f(x)| \cdot t$. Since $|f(x)| = \mu(A)$ hence for each i = 0, ... ,n $f_i(x) = \mu(A) \cdot t_i$. □

For a given set $A \subset R^n$ and a point $x \in R^n$ let

$$A - x := \{a - x : a \in A\}$$

means a translation of the set A.

Assume that $P := [p_0, \dots, p_n]$ is an n-dimensional simplex such that $0 \in IntP$. Let for each $i = 0, \dots, n$ M_i be the cone consisting of the union of all the rays joining 0 to the points of $(n-1)$-dimensional face $P_i := [p_0, \dots, \hat{p}_i, \dots, p_n]$.

The Kuratowski-Steinhaus Theorem. *Let $A \subset R^n$ be a bounded Lebesgue measurable set. Then for each point $t \in T$ there exist a point $x \in R^n$ such that for each $i = 0, \ldots, n$*

$$\mu[(A - x) \cap M_i] = \mu(A) \cdot t_i$$

Proof. Since the set A is bounded there exist a number $s > 0$ such that

$$A \subset D \ \ and \ \ (A - x) \cap M_i = \emptyset, \quad for\ each\ i\ and\ x \in D_i,$$

where $D := [d_0, \ldots, d_n]$, $d_i = s \cdot p_i$. Define a continuous map $f : D \longrightarrow [0, \infty)^{n+1}$, $f = (f_0, \ldots, f_n)$,

$$f_i(x) := \mu[(A - x) \cap M_i] \quad \text{for each} \quad i = 0, \ldots, n$$

It is clear that for each $x \in D$ and $i = 0, \ldots, n$; $|f(x)| = \mu(A)$ and $f_i(D_i) = \{0\}$. Then for a given point $t \in T$ there is a point $x \in D$ such that $f(x) = \mu(A) \cdot t$. And this means that $\mu[(A - x) \cap M_i] = \mu(A) \cdot t_i$ for each $i = 0, \ldots, n$. □

In this part we shall consider some results related to the Urbanik's paper [12].

Lemma 5.3. *Let $g : [0, 1] \times [0, 1] \longrightarrow [0, \infty)$ be a continuous function with the following properties;*
(a) $g(u, u) = 0$ and $g(0, 1) > 0$,
(b) $g(u, v) = 0$ and $g(v, w) = 0$ implies $g(u, w) = 0$.

Then for each natural number $n > 0$ there exist a real number $d > 0$ and a sequence $0 = u_0 < \ldots < u_n < u_{n+1} = 1$ such that $g(u_i, u_{i+1}) = d$ for each $i = 0, \ldots, n$.

Proof. Let us define a continuous functions $u_i : D \longrightarrow [0, 1]$ for $i = 0, \ldots, n + 1$,

$$u_0(x) = 0, \quad u_i(x) = t_0(x) + \cdots + t_{i-1}(x)$$

and functions $f_i : D \longrightarrow [0, \infty)$; $f_i(x) = g[u_i(x), u_{i+1}(x)]$, for $i = 0, \ldots, n$. Observe that if $x \in D_i$ then $t_i(x) = 0$, and in consequence $u_i(x) = u_{i+1}(x)$, but this implies that $f_i(x) = 0$. From Corollary 5.2., for $t = (1/(n+1), \ldots, 1/(n+1))$, it follows that there is a point $x \in D$ such that $f_0(x) = ... = f_n(x)$. Let us put for each i = 0, ... ,n

$$u_i = u_i(x) \quad and \quad d = f_i(x).$$

Then $d = g(u_i, u_{i+1})$ for each i. We shall show that $d > 0$. Suppose that $d = 0$. Then according to the assumption (b) we get

$$g(u_0, u_1) = ... = g(u_n, u_{n+1}) = 0.$$

And this implies that $g(0, 1) = 0$, contrary to (a). □

The Urbanik Theorem. *Let $f : [0, 1] \longrightarrow X$ be a continuos map into a metric space (X, ρ) such that $f(0) \neq f(1)$. Then for each natural number $n > 0$ there exist a real number $d > 0$ and a sequence*

$$0 = u_0 < u_1 < ... < u_n < u_{n+1} = 1$$

such that for each i = 0, ... ,n

$$d = \rho[f(u_i), f(u_{i+1})].$$

Proof. Indeed, the function $g(u, v) := \rho[f(u), f(v)]$ satisfies the assumptions (a), (b) of the Lemma . □

Corollary 5.4. *Let $f : S \longrightarrow [0, \infty)$ be a continuous function defined on a triangle $S := \triangle ABC$ such that*

$$(1) \qquad f(x) = 0 \text{ iff } x \in sideAB.$$

Then for each natural number $n > 1$ there exists a sequence of points belonging to the side AB,

$$A = P_0 < P_1 < ... < P_n < P_{n+1} = B$$

such that

$$f(Q_0) = ... = f(Q_n)$$

where the points $Q_0, \ldots, Q_n \in S$ *are vertices of the triangles* $\triangle P_i Q_i P_{i+1}$, $i = 0, \ldots, n$ *which are similar to the triangle* S.

Proof. Consider a coordinate system such that the side AB is contained in the diagonal and $A = (0,0)$ and the product $[0,1] \times [0,1]$ is equal to the parallelogram $ABCD$. Now, extend the function f to a continuous function g defined by

$g(u,v) = f(u,v)$ if $u \leq v$, and $g(u,v) = f(v,u)$ if $v \leq u$.

According to the previous Lemma there exist a real number $d > 0$ and a sequence $0 = u_0 < u_1 < ... < u_n < u_{n+1} = 1$ such that $d = g(u_i, u_{i+1})$ for each $i = 0, \ldots, n$. Now, the Corollary becomes obvious when we put $Q_i := (u_i, u_{i+1})$. □

6. Axiom of Uniqueness

A simplicial structure $\mathcal{F}$ on X satisfies the *axiom of uniqueness* if for each pair of simplices $\sigma_i : [p_0^i, ..., p_n^i]$ for $i = 1, 2$, the following equality holds

$$\sigma_1 \circ l_1 = \sigma_2 \circ l_2$$

whenever

$$(\sigma_1 \circ l_1)(e_j) = (\sigma_2 \circ l_2)(e_j) \text{ for each } j = 0, ..., n,$$

where
$e_0 = (1, 0, \ldots, 0)$, $e_1 = (0, 1, 0, \ldots, 0), \ldots, e_n = (0, \ldots, 0, 1)$ and $l_i : [e_0, \ldots, e_n] \longrightarrow [p_0^i, \ldots, p_n^i]$ is the unique linear homeomorphism induced by the vertex maps $l_i(e_j) = p_j^i$; $i = 1, 2$; $j = 0, \ldots, n$.

Example 3.
1. Let $\mathcal{L}$ be the affine simplicial structure, described in Example 2, on a convex subset of linear topological space. It is

obvious that $\mathcal{L}$ satisfies the axiom of uniqueness.
2. Now, let $(X, \mathcal{T}, \mathcal{F})$ be a topological simplicial structure for which the axiom of uniqueness holds and let $f : X \longrightarrow Y$ be a continuous one-to-one map onto Y. The family of maps of the form $f \circ \sigma$, where $\sigma \in \mathcal{F}$, is a simplicial structure on Y which preserves the axiom of uniqueness.

The following theorem is a first step to provide a natural and intrinsic characterization of metric spaces with simplicial structures where the axiom of uniqueness holds.

Theorem 6.1. *Let Y be a subspace of a space X with simplicial structure $\mathcal{F}$ being locally convex and satisfying the axiom of uniqueness and let $r : U \longrightarrow Y$ be a retraction from an open set U, $Y \subset U \subset X$, (i.e., r is continuous and $r(x) = x$ for each $x \in Y$). Then the family $\mathcal{G}$ of all maps of the form $r \circ \sigma$, where $\sigma \in \mathcal{F}$ and vert $\sigma \subset Y$, is a simplicial locally convex structure with the axiom of uniqueness.*

Moreover, if in addition, $U = X$ and $\mathcal{F}$ is convex then $\mathcal{G}$ is convex, too.

Proof. First let us check that $\mathcal{G}$ satisfies the axiom of uniqueness. Fix $\sigma_i : [p_0^i, \ldots, p_n^i] \longrightarrow X$, $\sigma_i \in \mathcal{F}$ for $i = 1, 2$, with *vert* $\sigma_i \subset Y$ and let $l_i : [e_0, \ldots, e_n] \longrightarrow [p_0^i, \ldots, p_n^i]$ be affine maps such that $(r \circ \sigma_1 \circ l_1)(e_j) = (r \circ \sigma_2 \circ l_2)(e_j)$ for each $j = 0, \ldots, n$. Since $r(x) = x$ for each $x \in Y$, we infer that $(\sigma_1 \circ l_1)(e_j) = (\sigma_2 \circ l_2)(e_j)$. By the axiom of uniqueness we have; $\sigma_1 \circ l_1 = \sigma_2 \circ l_2$ and hence $r \circ \sigma_1 \circ l_1 = r \circ \sigma_2 \circ l_2$.

Next we shall show that $\mathcal{G}$ is locally convex. Fix $x \in Y$ and its open neighbourhood $W_x \subset X$. Since r is continuous there is an open set $U_x \subset X$; $x \in U_x \subset W_x$, such that $r(U_x) \subset W_x$. Choose an open set $V_x \subset X$; $x \in V_x \subset U_x$, which satisfies the conditions (a) and (b) of definition of local convexity at x. Now, let $\sigma \in \mathcal{F}$ with *vert* $\sigma \subset Y$ be such that *vert*$(r \circ \sigma) \subset V_x$.

It is clear that *vert* $\sigma \subset V_x$ and according to the condition (b); *im* $\sigma \subset U_x$, and in consequence we have $im(r \circ \sigma) \subset W_x$.

If we assume that $\mathcal{F}$ is convex and $U = X$ then it is obvious that G must be convex. □

Remarks.

1. From the above lemma it follows that each each retract of a normed space has a convex locally convex simplicial structure satisfying the axiom of uniqueness.
2. In fact, we have proved that a continuous closed map f preserves local convexity at each point y such that $|f^{-1}(y) \cap Y| = 1$.

Let us put $B(x, \epsilon) := \{y \in X : \rho(x, y) < \epsilon\}$, where (X, ρ) is a metric space, $A \subset X$ and $\epsilon > 0$.

For a singular simplex $\sigma : [p_0, \dots, p_n] \subset Y$ denote; *vert dom* $\sigma := \{p_0, \dots, p_n\}$.

Lemma 6.2. *Let A be a non-empty closed subset of a metric space (X, ρ). Then there exists a family $\{U_s, a_s : s \in S\}$ (called a Dugundji system for A) such that $\{U_s : s \in S\}$ is locally finite open covering of $X \setminus A$, $a_s \in A$, and for each $a \in A$, $s \in S$, $\epsilon > 0$ the following implication holds;*

(D) $\qquad U_s \cap B(a, \epsilon) \neq \emptyset$ *implies* $a_s \in B(a, 5\epsilon)$.

Proof. Let $\{U_s : s \in S\}$ be a locally finite open covering of $X \setminus A$ which refines the covering $\{B(x, r(x)) : x \in X \setminus A\}$, where $r(x) = \frac{1}{2}d(x, A)$. For each $s \in S$ choose $x_s \in X \setminus A$ and $a_s \in A$ such that $U_s \subset B(x_s, r(x_s))$ and $\rho(x_s, a_s) < 3r(x_s)$.

To show that (D) holds, fix $b \in U_s \cap B(a, \epsilon)$. Then $2r(x_s) \leq \rho(x_s, a) \leq \rho(x_s, b) + \rho(b, a) < r(x_s) + \epsilon$, and hence $r(x_s) < \epsilon$.

On the other hand

$\rho(a_s, a) \leq \rho(a_s, x_s) + \rho(x_s, b) + \rho(b, a) < 3r(x_s) + r(x_s) + \epsilon \leq 5\epsilon$.

□

The Dugundji Extension Theorem. *Let A be a non-empty closed subset of a metric space (X,ρ) and let $(Y,\mathcal{T},\mathcal{F})$ be a topological simplicial space satisfying the axiom of uniqueness. Assume that $f : A \longrightarrow Y$ is a continuous map such that Y is locally convex at each point $y \in f(\partial A)$. Then there exist an open set $U \subset X$ and a continuous map $F : U \longrightarrow Y$ such that $A \subset U$ and $F|A = f$.*

Moreover, if in addition, Y is convex, then $U = X$.

Proof. Let $\{U_s, a_s : s \in S\}$ be a Dugundji system for A. Define a partition of unity $\{\phi_s : s \in S\}$ on $X \setminus A$ subordinated to $\{U_s : s \in S\}$;

$$\phi_s(x) := \frac{d(x, X \setminus U_s)}{\sum_{t \in S} d(x_t, X \setminus U_t)} \quad x \in X \setminus A$$

Let Φ be the family of all pairs (W,σ) with the following properties:
(1) $W \subset X \setminus A$ is an open set such that $S_W := \{s \in S : W \cap U_s \neq \emptyset\}$ is finite,
(2) $\sigma \in \mathcal{F}$ is a singular simplex with *vert* $\sigma = \{f(a_s) : s \in S_W\}$.
Next for each pair $\alpha = (W,\sigma) \in \Phi$ define a continuous map $F_\alpha : X \setminus A \longrightarrow Y$ as a composition of two maps $X \setminus A \longrightarrow \textit{dom}\ \sigma \longrightarrow Y$:

$$F_\alpha := \sigma\Big(\sum_{s \in S_W} \phi_s(x) \cdot p_s\Big)$$

where $p_s \in \textit{vert dom}\ \sigma$ and $\sigma(p_s) = f(a_s)$.

Fix $\alpha_i = (W_i, \sigma_i) \in \Phi$ for $i = 1, 2$, and let us put $W := W_1 \cap W_2$. We shall verify that

$$F_{\alpha_1}|W = F_{\alpha_2}|W.$$

It is clear that $S_W \subset S_{W_1} \cap S_{W_2}$. Arrange S_W into the sequence; $S_W = \{s_0, ..., s_k\}$, and define linear maps $l_i : [e_0, ..., e_k] \longrightarrow [p^i_{s_0}, ..., p^i_{s_k}] \subset dom\ \sigma_i$, where $p^i_{s_j} \in vert\ dom\ \sigma_i$ and $\sigma_i(p^i_{s_j}) = f(a_{s_j})$ for $i = 1, 2$ and $j = 0, ..., k$.

The axiom of uniqueness; $\sigma_1 \circ l_1 = \sigma_2 \circ l_2$ and $\phi_s^{-1}(0, 1] \subset U_s$ imply that for each $x \in W$ we have;

$$\begin{aligned}
F_{\alpha_1}(x) &= \sigma_1(\sum_{s_j \in S_W} \phi_{s_j}(x) \cdot p^1_{s_j}) \\
&= \sigma_1(\sum_{s_j \in S_W} \phi_{s_j} \cdot l_1(e_j)) \\
&= (\sigma_1 \circ l_1)(\sum_{s_j \in S_W} \phi_{s_j}(x) \cdot e_j) \\
&= (\sigma_2 \circ l_2)(\sum_{s_j \in S_W} \phi_{s_j}(x) \cdot e_j) \\
&= \sigma_2(\sum_{s_j \in S_W} \phi_{s_j}(x) \cdot l_2(e_j)) \\
&= \sigma_2(\sum_{s_j \in S_W} \phi_{s_j}(x) \cdot p^2_{s_j}) \\
&= F_{\alpha_2}(x).
\end{aligned}$$

Now, define $U := \bigcup\{W : (W, \sigma) \in \Phi\} \cup A$ and $F : U \longrightarrow Y$;

$$F(x) := \begin{cases} f(x) & \text{if} \quad x \in A \\ F_\alpha(x) & \text{if} \quad x \in U \setminus A \end{cases}$$

where $\alpha = (W, \sigma) \in \Phi$ and $x \in W$.

The map F is well-defined because it does not depend on the choice of α.

To complete the proof it suffices to check that U is an open neighbourhood of A and that F is continuous at each point

$a \in \partial A$ belonging to the boundary of A.

Fix $a \in \partial A$ and an open neighbourhood $G \subset Y$ of $f(a)$. Since Y is locally convex at $f(a)$ hence there exists an open set H; $f(a) \in H \subset G$, satisfying the conditions of local convexity;
(a) for each finite subset $F \subset H$ there exists $\sigma \in \mathcal{F}$ such that $vert\ \sigma = F$, and
(b) for each $\sigma \in \mathcal{F}$; $vert\ \sigma \subset H \implies im\ \sigma \subset G$.

Choose $\epsilon > 0$ such that $f(B(a, 5\epsilon)) \subset H$. Now, fix $x \in B(a, \epsilon)$ and let W be an open set; $x \in W \subset B(a, \epsilon)$, such that $S_W = \{s \in S : W \cap U_s \neq \emptyset\}$ is finite. Since $\{U_s, a_s : s \in S\}$ is a Dugundji system we infer that for each $s \in S$;

$$U_s \cap B(a, \epsilon) \neq \emptyset \textit{ implies } f(a_s) \in H$$

From the above it follows that $F := \{f(a_s) : s \in S\} \subset H$. In view of the condition (a) there exists $\sigma \in \mathcal{F}$ such that $vert\ \sigma = F \subset H$ and by (b); $im\ \sigma \subset G$. Thus $\alpha = (W, \sigma)$ is an element of Φ and $F(x) = F_\alpha(x) \in im\ \sigma \subset G$.

In fact, we have proved that $x \in B(a, \epsilon) \subset U$ and $F(B(a, \epsilon)) \subset G$. This proves that U is an open neighbourhood of A and that F is continuous. By definition of U it is clear that in the case when X is convex the equality $X = U$ holds. □

According to the Arens-Eells theorem each metric space (X, ρ) can be isometrically embedded as a closed subset in a normed linear space. In view of this theorem, our Theorem and the Dugundji Extension Theorem just proved, give together, a characterization of ANR and AR-spaces (cf. Borsuk [2]) in the class of metric spaces in terms of notions of simplicial structures;

ANR = locally convex with the axiom of uniqueness metric simplicial space,

AR = convex, locally convex with the axiom of uniquenss metric simplicial space.

Problem. Find an example of a compact metric space which has a convex locally convex simplicial structure, but which has no convex locally convex structure satisfying the axiom of uniqueness.

In 1975 Roberts [11] constructed an interesting class of metric compact convex sets which are non-locally convex. Some of them have the AR-property (cf.[9]). Thus from our results it follows that there exists a metric compact convex and non-locally convex set which has convex and and locally convex simplicial structure.

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University of Silesia, Department of Mathematics, ul. Bankowa 14, **40 007 Katowice**, Poland

E-mail address: `kulpa@ux2.math.us.edu.pl`