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# APPLICATIONS OF BOLZANO-WEIERSTRASS METHOD

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## Abstract

In most of proofs of the Bolzano-Weierstrass theorem stating that: *Every bounded sequence of real numbers has at least one point of accumulation*, a method of dividing intervals into parts is used. We show how some modification of this method, which we call the Bolzano-Weierstrass method, can be adopted for infinitary combinatorics.

In 1817 Bernard Bolzano, the outstanding Czech thinker, philosopher, and mathematician, published his intermediate value theorem with the proof, which uses the method of dividing intervals. In many textbooks with the mathematical analysis this method is called the Bolzano-Weierstrass method. We shall apply it for infinitary combinatorics. Our proofs show that the Bolzano-Weierstrass method seems to be stronger than methods which work with the Erdös-Rado theorem, see [1] or [2]. It is possible to use this method to verify Turzański's results, which were published in [6] and [7]. The Bolzano-Weierstrass method yields a Ramsey-type theorem, see Theorem 2. We give another proofs of Malyhin's theorem on families of pairwise disjoint open subsets of a hyperspace, see [4]; and Kurepa's theorem on the same type families of a product space, see [3].

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*Key words*: the Bolzano-Weierstrass method; well ordered sets; a centered family; a base set; a family of pairwise disjoint sets.

Greek letters denote ordinal or cardinal numbers. Let $<$ denote the natural order between ordinal numbers, let $\lambda^+$ denote the first cardinal number greater than $\lambda$, and by $\kappa^\lambda$ we denote the cardinality of the family of all functions from $\lambda$ into $\kappa$. If $1 < \kappa$ and $\lambda$ is an infinite cardinal, then we used the equalities $\lambda^+ \times \kappa^\lambda = \kappa^\lambda$ and $(2^\lambda)^\lambda = 2^\lambda$, only.

Let us consider a sequence $\{a_n\}_{n\in\omega}$ of different real numbers. How can we construct a monotone subsequence? One can do this following Bolzano. For each real number $r$ denote the half line $(-\infty, r)$ as $F_r(0)$; and the half line $(r, +\infty)$ as $F_r(1)$. We shall construct an increasing sequence of natural numbers $n_0 < n_1 < n_2 < \ldots$ and a zero-one sequence $\varepsilon_0, \varepsilon_1, \varepsilon_2, \ldots$ by induction as follows. Put $n_0 = 0$ and choose an $\varepsilon_0$ such that the half line $F_{a_{n_0}}(\varepsilon_0)$ contains infinitely many elements of the sequence $\{a_n\}_{n\in\omega}$. Assume that we have defined natural numbers $n_0 < n_1 < n_2 < \ldots < n_k$ and a zero-one sequence $\varepsilon_0, \varepsilon_1, \varepsilon_2, \ldots \varepsilon_k$ such that the intersection

$$F_{a_{n_0}}(\varepsilon_0) \cap F_{a_{n_1}}(\varepsilon_1) \cap F_{a_{n_2}}(\varepsilon_2) \cap \ldots \cap F_{a_{n_k}}(\varepsilon_k)$$

contains infinitely many elements of the sequence $\{a_n\}_{n\in\omega}$. Let us choose a natural number $n_{k+1} > n_k$ such that $a_{n_{k+1}}$ belongs to the above intersection. Then choose $\varepsilon_{k+1}$ such that the intersection

$$F_{a_{n_0}}(\varepsilon_0) \cap F_{a_{n_1}}(\varepsilon_1) \cap F_{a_{n_2}}(\varepsilon_2) \cap \ldots \cap F_{a_{n_{k+1}}}(\varepsilon_{k+1})$$

contains infinitely many elements of the sequence $\{a_n\}_{n\in\omega}$. If the zero-one sequence $\varepsilon_0, \varepsilon_1, \varepsilon_2, \ldots$ is such that infinitely many $\varepsilon_k$ equal 0, then the subsequence $\{a_{n_k}\}_{\varepsilon_k=0}$ is decreasing. If the zero-one sequence $\varepsilon_0, \varepsilon_1, \varepsilon_2, \ldots$ is such that infinitely many $\varepsilon_k$ equal 1, then the subsequence $\{a_{n_k}\}_{\varepsilon_k=1}$ is increasing.

Let us note that if $r$ is an accumlation point of the sequence $\{a_n\}_{n\in\omega}$, then for every induction step one could choose the intersection which contains $r$: any such intersection has to contain infinitely many elements of the sequence $\{a_n\}_{n\in\omega}$ for all but at most one choice, when $a_{n_k} = r$.

Any sequence can be considered as a well ordered set, ordered by indexes. So, we are proposing the following generalization of the above method. We call it the Bolzano-Weierstrass method. The proof in our generalization imitates a hiker's track. Each hiker's step corresponds to dividing. It is uniquely determined by his previous steps and by his destination point.

**Theorem 1.** *Suppose $\lambda$ and $\kappa$ are cardinal numbers such that $\kappa > 1$ and $\lambda$ is infinite. Assume that $X = \{x_\alpha : \alpha < (\kappa^\lambda)^+\}$ is a set of distinct indexed points. If for any $\alpha < (\kappa^\lambda)^+$ the family $F_{x_\alpha} = \{F_{x_\alpha}(\beta) : \beta < \kappa\}$ consists of pairwise disjoint subsets of $X$ such that*

$$(*) \qquad X \setminus \{x_\beta : \beta < \alpha\} \subseteq \cup F_{x_\alpha},$$

*then there exist a function $f : \lambda^+ \to \kappa$ and an indexed subset $\{p_\gamma : \gamma < \lambda^+\} \subseteq X$ such that $p_\tau \in F_{p_\beta}(f(\beta))$ for each $\beta < \tau < \lambda^+$; moreover the intersection $\cap\{F_{p_\beta}(f(\beta)) : \beta < \lambda^+\}$ is non-empty.*

*Proof.* For any point $p \in X$, where $p \neq x_0$, we shall define the function $f_p : \lambda_p \to \kappa$ and the subset $\{p_\gamma : \gamma < \lambda_p\} \subseteq X$ with the properties: For any $\beta < \lambda_p$ the point $p$ belongs to $F_{p_\beta}(f_p(\beta))$ and the point $p_\beta$ is the first element (with respect to indexes in $\{x_\alpha : \alpha < (\kappa^\lambda)^+\} = X$) in the intersection $\cap\{F_{p_\gamma}(f_p(\gamma)) : \gamma < \beta\}$, as follows.

Put $p_0 = x_0$. Using $(*)$, choose $f_p(0)$ such that $p \in F_{p_0}(f_p(0))$. If points $\{p_\beta : \beta < \tau\}$ and the function $f_p : \tau \to \kappa$ have been defined, then let $p_\tau$ be the first element, with respect to indexes in $\{x_\alpha : \alpha < (\kappa^\lambda)^+\} = X$, from the set

$$\cap\{F_{p_\beta}(f_p(\beta)) : \beta < \tau\} \setminus \{p_\beta : \beta < \tau\}.$$

When $p = p_\tau$, then put $\lambda_p = \tau$ and stop the construction. In the other case, using $(*)$, choose $f_p(\tau)$ such that $p \in F_{p_\tau}(f_p(\tau))$. Thus $p_\tau \in F_{p_\beta}(f_p(\beta))$ for any $\beta < \tau$ .

From the above construction it follows that for any function $f : \lambda_p \to \kappa$ there exists at most one point $p \in X$ such that $f = f_p$ and if $p$ and $q$ are different points from $X$, then functions $f_p$ and $f_q$ are different. Since there are $\kappa^\lambda$ many functions of the form $f : \lambda \to \kappa$ and $X$ has the cardinality $(\kappa^\lambda)^+$, there exists a point $p \in X$ such that $f_p : \lambda^+ \to \kappa$ and $\{p_\gamma : \gamma < \lambda^+\}$ are as desired. □

We give another proof the following modification of Turzański's theorems, which were stated in [6] and [7]. Fix a base of open sets for some topological space. Each element of this base is called a *base set.* Note that a base is an equivalent name for a family closed under finite intersection, because every base is a subset of some base that is closed under finite intersections. A family of sets is said to be *centered*, if any of its non-empty and finite subfamily has non-empty intersection. The collection of all finite intersections of members of a centered family is centered.

**Theorem.** (compare M. Turzański [6], Th. 2). *Suppose $\{H_\alpha : \alpha < (\kappa^\lambda)^+\}$ is a transfinite sequence of centered families of base sets such that any condition $\beta < \alpha < (\kappa^\lambda)^+$ implies that for each $h \in H_\alpha$ the family $H_\beta \cup \{h\}$ is not centered. If each family $H_\alpha$ is non-empty and has the cardinality not greater than $\kappa$, then there exists a family of cardinality $\lambda^+$ consisting of pairwise disjoint base sets.*

*Proof.* For any $\alpha < (\kappa^\lambda)^+$ choose an element $x_\alpha \in H_\alpha$; well order the family $H_\alpha = \{h_\alpha(\beta) : \beta < \kappa\}$; and put $X = \{x_\alpha : \alpha < (\kappa^\lambda)^+\}$. Then for any $\beta < \kappa$ put

$$F_\alpha(\beta) = \{x \in X : x \cap h_\alpha(\beta) = \emptyset\}.$$

For families $F_{x_\alpha}(\beta) = F_\alpha(\beta) \setminus \cup\{F_\alpha(\gamma) : \gamma < \beta\}$ Theorem 1 works. It yields a transfinite sequence $\{x_{\alpha_\beta} : \beta < \lambda^+\}$ and a function $f : \lambda^+ \to \kappa$ such that any condition $\gamma < \beta < \lambda^+$

implies

$$x_{\alpha_\beta} \in F_{\alpha_\gamma}(f(\gamma)), \quad \text{in fact} \quad x_{\alpha_\beta} \cap h_{\alpha_\gamma}(f(\gamma)) = \emptyset.$$

Thus the set $\{x_{\alpha_\beta} \cap h_{\alpha_\beta}(f(\beta)) : \beta < \lambda^+\}$ consists of non-empty pairwise disjoint base sets. □

Let us consider the following Ramsey-type theorem.

**Theorem 2.** *Any family of sets of the cardinality $(2^\lambda)^+$ closed under finite intersections contains some subfamily of $\lambda^+$ pairwise disjoint subsets or contains a centered subfamily of the cardinality $(2^\lambda)^+$.*

*Proof.* Let us consider a family of sets of the cardinality $(2^\lambda)^+$ closed under finite intersections. We can assume that this family is a base for the topology. Let $H_0^*$ be some maximal centered family of base subsets. If centered families $\{H_\beta^* : \beta < \alpha\}$ of base subsets are defined, then let $V_\alpha$ be a base set such that for each $\beta < \alpha$ the family $H_\beta^* \cup \{V_\alpha\}$ is not centered. Choose $H_\alpha^*$ to be a maximal centered family of base subsets such that $V_\alpha \in H_\alpha^*$. The family $H_\alpha^*$, being maximal, is closed under finite intersections. If the family $H_\alpha^*$ is defined, then we put

$$H_\alpha = \{h \cap V_\alpha : h \in H_\alpha^*\}.$$

If one assumes that any centered family of base subsets does not have the cardinality $(2^\lambda)^+$, then some transfinite sequence $\{H_\alpha : \alpha < (2^\lambda)^+\}$ of centered families, such that if $\beta < \alpha$, then the family $H_\beta \cup \{h\}$ is not centered for each $h \in H_\alpha$, could be defined. For $\kappa = 2^\lambda$ we have $\kappa^\lambda = (2^\lambda)^\lambda = 2^\lambda$ and the previous theorem works. By this theorem there exists a family of the cardinality $\lambda^+$, which consists of pairwise disjoint base subsets. □

The above theorem, for bases consisting of regular open sets, can be deduced from Shelah's result on independent families, see [5]. Let us note that cardinality of each independent family of base sets is not greater than the weight of the space. There exist topological spaces (for instance any generalized Cantor cube: any compact space with cardinality greater than its weight) with centered families of base sets of the cardinality greater than its weight. Thus Theorem 2 differs somewhat from Shelah's result [5].

If $X$ is a topological space, then let $\exp(X)$ denote the hyperspace of all closed subsets of $X$ equipped with the Vietoris topology. Recall that the base defining the Vietoris topology consists of sets $< U_1, U_2, \ldots, U_n >$, where $U_1, U_2, \ldots, U_n$ are non-empty open subsets of $X$, but $< U_1, U_2, \ldots, U_n >$ denotes the family of all closed subsets contained in $U_1 \cup U_2 \cup \ldots \cup U_n$ and meeting every $U_k$.

**Theorem.** (V. I. Malyhin [4]). *Suppose $\lambda$ is an infinite cardinal number and $X$ is a $T_1$ topological space. If no family of pairwise disjoint open subsets of $X$ has cardinality greater than $\lambda$, then no family of pairwise disjoint open subsets of the hyperspace* $\exp(X)$ *has cardinality greater than* $2^\lambda$.

*Proof.* Suppose $\{U_\alpha : \alpha < (2^\lambda)^+\}$ is a family of pairwise disjoint open subsets of the hyperspace $\exp(X)$. For each $\alpha < (2^\lambda)^+$ let

$$U_\alpha^* =< U_1^\alpha, U_2^\alpha, \ldots, U_{n_\alpha}^\alpha >\subseteq U_\alpha$$

be some non-empty base set. Thus, for some fixed natural number $n$ there exists a subset $Y \subseteq (2^\lambda)^+$ such that for $n = n_\alpha$ the family $\{U_\alpha^* : \alpha \in Y\}$ is of the cardinality $(2^\lambda)^+$ and consists of distinct base subsets. Let $Y_1 \subseteq Y$ be a subset of the cardinality $(2^\lambda)^+$ such that the family $\{U_1^\alpha : \alpha \in Y_1\}$ is centered, but when a suitable choice is impossible, then let $Y = Y_1$. If subsets $Y_{k-1} \subseteq Y_{k-2} \subseteq \ldots \subseteq Y_1$ are defined, then let $Y_k \subseteq Y_{k-1}$ be a subsets of the cardinality $(2^\lambda)^+$ such that the family $\{U_k^\alpha : \alpha \in Y_k\}$ is centered, but when a suitable choice is impossible, then let $Y_{k-1} = Y_k$.

Define the sets $A$ and $B$, where $A \cup B = \{1, 2, \ldots, n\}$, such that for any $k \in A$ the family $\{U_k^\alpha : \alpha \in Y_n\}$ is centered and for any $k \in B$ the family $\{U_k^\alpha : \alpha \in Y_n\}$ contains no centered family of the cardinality $(2^\lambda)^+$. By Theorem 2, for any $k \in B$ the family $\{U_k^\alpha : \alpha \in Y_n\}$ is of the cardinality not greater than $2^\lambda$. Choose a subset $Y^* \subseteq Y_n$ of the cardinality $(2^\lambda)^+$ such that for any $k \in B$ the family $\{U_k^\alpha : \alpha \in Y^*\}$ has exactly one element. For any two numbers $\alpha$ and $\beta$ from $Y^*$ choose points $x_k \in U_k^\alpha \cap U_k^\beta$, where $1 \leqslant k \leqslant n$. The set $\{x_1, x_2, \ldots, x_n\}$ is closed, since $X$ is a $T_1$ space. It belongs to $\exp(X)$, and it belongs to $U_\alpha^* \cap U_\beta^*$. A contradiction, because the sets $\{U_\alpha^* : \alpha \in Y^*\}$ are pairwise disjoint. □

Note that what we did for $\alpha$ and $\beta$ can be done for any finite subset of $Y^*$. Thus we have:

**Corollary 3.** *If no family of pairwise disjoint open subsets of a $T_1$-space $X$ has cardinality greater than $\lambda$, then any family of open subsets in* $\exp(X)$ *of the cardinality* $(2^\lambda)^+$ *contains a centered subfamily of the cardinality* $(2^\lambda)^+$.

**Theorem.** (D. Kurepa [3]). *Suppose $\lambda$ is an infinite cardinal number and $X_1, X_2, \ldots, X_n$ are topological spaces. If for every natural number $k \leqslant n$ no family of pairwise disjoint open subsets of $X_k$ has cardinality greater than $\lambda$, then no family of pairwise disjoint open subsets of the product space $X_1 \times X_2 \times \ldots \times X_n$ has cardinality greater than $2^\lambda$.*

*Proof.* If one consider products of the form $U_1 \times U_2 \times \ldots \times U_n$ instead of families of the form $< U_1, U_2, \ldots, U_n >$, the arguments from the proof of the previous theorem work. □

We give another proof of the following theorem of P. Erdös and R. Rado, compare [1] or [2]. By $[X]^2$ denote the family of all exactly two points subsets of $X$.

**Theorem.** (P. Erdös and R. Rado [1]). *Suppose $\lambda$ is an infinite cardinal number and $F$ is a partition of $[X]^2$ of the cardinality not greater than $\lambda$. If the cardinality of the set $X$ is greater than $2^\lambda$, then there exists a subset $Y \subseteq X$ of the cardinality greater than $\lambda$ such that the family $[Y]^2$ is contained in some element of $F$.*

*Proof.* Well order elements of $F$ into the size $\lambda$, i.e. $F = \{F(\beta) : \beta < \lambda\}$. For each point $x \in X$ and any ordinal number $\beta < \lambda$ put

$$F_x(\beta) = \{y \in X \setminus \{x\} : \{x, y\} \in F(\beta)\}.$$

Any family $F_x = \{F_x(\beta) : \beta < \lambda\}$ is a partition of $X \setminus \{x\}$, since $F$ has been assumed to be a partition of $[X]^2$. There exists some set $\{p_\alpha : \alpha < \lambda^+\}$ of the cardinality $\lambda^+$, with properties as in Theorem 1. Since this set has the cardinality $\lambda^+$ one can fix an ordinal number $\beta < \lambda$ such that the set

$$Y = \{p_\alpha : \alpha < \lambda^+ \text{ and } p \in F_{p_\alpha}(\beta)\}$$

has the cardinality $\lambda^+$. The set $Y$ is as we desired, i.e. $[Y]^2 \subseteq F(\beta)$. Indeed, if $\gamma < \alpha$ and $\{p_\alpha, p_\gamma\} \in Y$, then $p \in F_{p_\gamma}(\beta)$ by the definition of $Y$. Points $p$ and $p_\alpha$ belong to the same element of the partition $F_{p_\gamma}$, i.e. those elements belong to $F_{p_\gamma}(\beta)$. But $p_\alpha \in F_{p_\gamma}(\beta)$ means that $\{p_\gamma, p_\alpha\} \in F(\beta)$. Since ordinal numbers $\alpha$ and $\gamma$ has been taken arbitrary the proof is completed. □

Note that in the definition of $Y$ we put $p \in F_{p_\alpha}(\beta)$. This means, in terms of Theorem 1, that the function $f_p : \lambda_p \to \kappa$ satisfies $f_p(\alpha) = \beta$, i.e. the function $f_p$ is constant on $Y$. If $\lambda < \kappa$, then the Erdös-Rado theorem suggests no idea for a proof of the modification of Turzański's theorems, since one has to consider the cases $2^\lambda = \kappa$, too. But the Bolzano-Weierstrass method works: in this sense it looks stronger. Note that, under the assumption $\kappa \leqslant \lambda$, by the method from [2] one could produce some proof of the above theorem directly from the Erdös-Rado theorem.

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