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ON THE DARBOUX PROPERTY OF APPROXIMATELY CONTINUOUS MULTIVALUED FUNCTIONS

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Abstract

In this paper we study the Darboux property of multivalued functions from a measurable metric space $(X, d, \mathcal{M}(X), \mu)$ to a uniform space $(Y, \mathcal{U}(Y))$. We introduce the density tpopology $\mathcal{T}_d(X)$ and show that the open connected sets are connected in this topology. The notion of approximate continuity with respect to a differentiation basis $\mathcal{F}$ is defined. The approximately continuous multivalued functions with respect to $\mathcal{F}$ are continuous in the new topology. We introduce $\mathcal{T}_d(X)$-regular sets and prove that approximately continuous multivalued functions with respect to $\mathcal{F}$ take $\mathcal{T}_d(X)$-regular sets into connected sets, i.e., they have a Darboux property.

1. Introduction

A. Denjoy in his work [3] on derivatives introduced an interesting class of real functions, the approximately continuous functions. One of the facts discovered by him in this work was that these functions have the Darboux property. In this paper we

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discus the above property in the case of multivalued functions from a measurable metric space into arbitrary uniform space. The proof of Denjoy depends on properties of derivatives and cannot be extended to our case.

2. Notations and Definitions

Let X and Y be two non-empty sets and let us assume that for every point $x \in X$ a non-empty subset $F(x)$ of Y is given. In this case we say that F is a multivalued function from X to Y and we write $F : X \to Y$.

For $F : X \to Y$ and any set $B \subset Y$ we denote

$$F^+(B) = \{x \in X : F(x) \subset B\}$$

and

$$F^-(B) = \{x \in X : F(x) \cap B \neq \emptyset\}$$

Let $(X, \mathcal{T}(X))$ and $(Y, \mathcal{T}(Y))$ be topological spaces. A multivalued function F is called upper (resp. lower) semicontinuous at a point $x \in X$ if the following condition is valid :

$$\text{(1)}\quad \forall G \in \mathcal{T}(Y)\ (F(x) \subset G \Longrightarrow x \in \operatorname{Int} F^+(G))$$
$$\text{(resp. } \forall G \in \mathcal{T}(Y)\ (F(x) \cap G \neq \emptyset \Longrightarrow x \in \operatorname{Int} F^-(G))).$$

F is called continuous at the point $x \in X$ if it is simultaneously upper and lower semicontinuous at this point.

Let $(Y, \mathcal{U}(Y))$ be a uniform space and let $\mathcal{P}$ be the family of pseudometrics ϱ on the space Y such that $\{\{(x,y) \in X \times Y : \varrho(x,y) < 2^{-n}\} : \varrho \in \mathcal{P}, n \in N\}$ (N denotes the set of all positive integers) is a base for the uniformity $\mathcal{U}(Y)$ (see [7]).

The symbol $\mathcal{C}(Y)$ is used to denote the class of all nonempty compact subsets of Y with respect to the topology induced by $\mathcal{U}(Y)$.

For each $\varrho \in \mathcal{P}$, a number $r > 0$ and each $y \in Y$ and $A \subset Y$ we will write

$$K(y, \varrho, r) = \{z \in Y : \varrho(y, z) < r\}$$

and

$$K(A, \varrho, r) = \bigcup\{K(y, \varrho, r) : y \in A\}.$$

A multivalued function $F : X \to Y$ is called h-upper (resp. h-lower) semicontinuous at a point $x \in X$ if the following condition holds true:

(2) for each $\varepsilon > 0$ and $\varrho \in \mathcal{P}$ there exists a neighbourhood $\mathcal{U}(x_0)$ of x_0 such that $F(x) \subset K(F(x_0), \varrho, \varepsilon)$ (resp. $F(x_0) \subset K(F(x), \varrho, \varepsilon)$) for each $x \in \mathcal{U}(x_0)$.

It is known that

(3) If F is upper (resp. h-lower) semicontinuous at a point $x \in X$, then it is h-upper (resp. lower) semicontinuous at the point x ([6], th. 1.15 and 1.12). If moreover $F(x) \in \mathcal{C}(Y)$, then conditions (1) and (2) are equivalent ([6],th. 1.17 and 1.14).

Let $(X, \mathcal{T}(X))$ and $(Y, \mathcal{T}(Y))$ be topological spaces. In the work [4] the following definition of a Darboux property was given:

Definition 1. A multivalued function $F : X \to Y$ has the $\mathcal{D}$ property if the image $F(E) = \bigcup_{x \in E} F(x)$ is connected for any connected set $E \subset X$.

Continuous multivalued functions do not necessarily have the property $\mathcal{D}$, but the following is true.

Theorem 1. *If a multivalued function $F : X \to Y$ with closed and connected values is continuous, then it takes connected sets into connected sets.*

This theorem is established by observing, that in this case F is a function from X to the space of all closed non-empty subsets of Y.

A special case of Theorem 1, when both X and Y are the sets of real numbers, was shown already, see theorem 1 and 2 in [2].

Let $(X, d, \mathcal{M}(X), \mu)$ be a measurable metric space with a metric d, with a σ-finite G_δ-regular complete measure μ defined on a σ-field $\mathcal{M}(X)$ of subsets of X containing Borel sets. Let μ^* be the outer measure corresponding to μ.

Let $\mathcal{F} \subset \mathcal{M}(X)$ be a family of μ-measurable sets with nonempty interiors of a positive and finite measure μ, the boundaries of which are of μ-measure zero.

Let $\{I_n\}_{n\in N} \subset \mathcal{F}$ and $x \in X$. We take $I_n \to x$ to mean that $x \in \mathrm{Int}(I_n)$ for $n \in N$ and the sequence of diameters $\delta(I_n)$ converges to zero if n approaches infinity . Let us assume that for every $x \in X$ there exists a sequence $(I_n)_{n\in N}$ from $\mathcal{F}$ converging to the point x. The pair $(\mathcal{F}, \to)$ forms a differentiation basis of $(X, d, \mathcal{M}(X), \mu)$ in accordance with Bruckner's terminology ([1], p.30).

Let $A \subset X$ and $x \in X$. The upper outer density of A at the point x with respect to $\mathcal{F}$ is

$$\operatorname{limsup}_{I_n \to x} \frac{\mu^*(A \cap I_n)}{\mu(I_n)}.$$

Replacing limsup by liminf we obtain the lower outer density of A at x with respect to $\mathcal{F}$. These densities we will denote by ${D^*}_u(x, A)$ and ${D^*}_l(x, A)$ respectively. If both these densities are equal, then their common value is called the outer density of the set A at the point x with respect to $\mathcal{F}$ and is denoted by $D^*(x, A)$.

If $A \in \mathcal{M}(X)$, then outer density of the set A at the point $x \in X$ with respect to $\mathcal{F}$ is called density of A at x with respect to $\mathcal{F}$.

A point $x \in X$ is called a density point of a set $A \subset X$ with respect to $\mathcal{F}$ if there exists a set $B \in \mathcal{M}(X)$ such that $B \subset A$ and density of B at x with respect to $\mathcal{F}$ is equal to 1. We will write $D(x, A) = 1$.

In the remainder of the paper we assume that

(4) $\mathcal{F}$ has the density property, i.e. for every set $A \subset X$ $\mu(\{x \in A : D^*{}_l(x, A) < 1\}) = 0$.

A measurable set in X will be called homogeneous with respect to $\mathcal{F}$ if its density with respect to $\mathcal{F}$ is one at each of its points.

Let $\phi(A)$ denote the set of all density points of $A \in \mathcal{M}(X)$ with respect to $\mathcal{F}$. Then for $A \in \mathcal{M}(X)$ inclusion $A \subset \phi(A)$ means that A is a homogeneous set with respect to $\mathcal{F}$. It is easy to see that $\phi(A \cap B) = \phi(A) \cap \phi(B)$ for each $A \in \mathcal{M}(X)$ and $B \in \mathcal{M}(X)$. Therefore the finite intersection of homogeneous sets with respect to $\mathcal{F}$ is homogeneous with respect to $\mathcal{F}$.

Let $\{A_t\}_{t \in T}$ be the family of homogeneous sets with respect to $\mathcal{F}$. The measure μ is σ-finite, so we can suppose that $\mu(X) < \infty$. Let b denote the upper bound of all measures of finished sums of the sets of family $\{A_t\}_{t \in T}$. Then there exists $\{t_n\} \subset T$ such that $\mu(\bigcup_{n \in N} A_{t_n}) = b$. Let $A = \bigcup_{n \in N} A_{t_n}$. Then $A \in \mathcal{M}(X)$ and $\mu(A_t \setminus A) = 0$ for each $t \in T$. Moreover $\phi(A_t) \subset \phi(A)$ because $A_t \setminus (A_t \setminus A) \subset A$ for each $t \in T$. The sets A_t are homogeneous with respect to $\mathcal{F}$, so $A_t \subset \phi(A_t)$ for each $t \in T$. Therefore $A \subset \bigcup_{t \in T} A_t \subset \bigcup_{t \in T} \phi(A_t) \subset \phi(A)$ and $\bigcup_{t \in T} A_t \in \mathcal{M}(X)$ because $\mu(A \setminus \phi(A)) = 0$. Finally $\bigcup_{t \in T} A_t \subset \phi(\bigcup_{t \in T} A_t)$, i.e. the union of homogeneous sets with respect to $\mathcal{F}$ is homogeneous with respect to $\mathcal{F}$. Since the empty set and the whole space are homogeneous sets with respect to $\mathcal{F}$, the space X can be topologized by taking the homogeneous sets with respect to $\mathcal{F}$ as open sets. This topology we will denote by $\mathcal{T}_d(X)$ (comp. [8] or [5]).

Definition 2. A set $A \subset X$ is called $\mathcal{T}_d(X)$-connected if it is connected in the density topology $\mathcal{T}_d(X)$.

Let $(Y, \mathcal{T}(Y))$ be a topological space.

Definition 3. A multivalued function $F : X \to Y$ is called approximately upper (resp. lower) semicontinuous at a point $x_0 \in X$ with respect to $\mathcal{F}$ if there exists a set $A \in \mathcal{M}(X)$ such that $D(x_0, A) = 1$ and the restriction $F|_A$ is upper (resp. lower) semicontinuous at x_0.

F is called approximately upper (resp. lower) semicontinuous with respect to $\mathcal{F}$ if it is approximately upper (resp. lower) semicontinuous with respect to $\mathcal{F}$ at each point $x \in X$.

If F is simultaneously approximately upper semicontinuous and approximately lower semicontinuous with respect to $\mathcal{F}$, then it is called approximately continuous with respect to $\mathcal{F}$.

3. Main Results

Let $(X, d, \mathcal{M}(X), \mu)$ be a space defined as before, let $(Y, \mathcal{U}(Y))$ be a uniform space and let $F : X \to Y$ be a multivalued function. For fixed $x_0 \in X$, $\varrho \in \mathcal{P}$ and $\varepsilon > 0$ let us put

$$A_u(x_0, \varrho, \varepsilon) = \{x \in X : F(x) \subset K(F(x_0), \varrho, \varepsilon)\}$$

and

$$A_l(x_0, \varrho, \varepsilon) = \{x \in X : F(x_0) \subset K(F(x), \varrho, \varepsilon)\}.$$

Let us consider the following conditions:

(a_1) F is approximately upper semicontinuous at x_0 with respect to $\mathcal{F}$,

(a_2) F is approximately lower semicontinuous at x_0 with respect to $\mathcal{F}$,

(b_1) $\forall \varrho \in \mathcal{P}\ \forall \varepsilon > 0\ D(x_0, A_u(x_0, \varrho, \varepsilon)) = 1$,

(b_2) $\forall \varrho \in \mathcal{P}\ \forall \varepsilon > 0\ D(x_0, A_l(x_0, \varrho, \varepsilon)) = 1$,

(c_1) $\forall G \in \mathcal{T}(Y)$ $(F(x_0) \subset G \Rightarrow D(x_0, F^+(G)) = 1)$,

(c_2) $\forall G \in \mathcal{T}(Y)$ $(F(x_0) \cap G \neq \emptyset \Rightarrow D(x_0, F^-(G)) = 1)$.

Theorem 2. *Let $(X, d, \mathcal{M}(X), \mu)$ and $(Y, \mathcal{U}(Y))$ be as it was stated above and let $F : X \to Y$ be a multivalued function. Then the following implications are valid:*

(i) $(a_1) \Rightarrow (b_1), (b_2) \Rightarrow (a_2), (c_1) \Rightarrow (b_1)$ *and* $(b_2) \Rightarrow (c_2)$.

(ii) *If F has compact values, then* $(b_1) \Rightarrow (a_1), (a_2) \Rightarrow (b_2), (b_1) \Rightarrow (c_1)$ *and* $(c_2) \Rightarrow (b_2)$.

Proof. Fix $\varepsilon > 0$ and $\varrho \in \mathcal{P}$. Suppose F is approximately upper semicontinuous with respect to $\mathcal{F}$ at the point x_0. Then there exists a set $A \in \mathcal{M}(X)$ such that $D(x_0, A) = 1$ and $F|_A$ is upper semicontinuous at x_0. From (3) it follows that there exists an open set $\mathcal{U}(x_0)$ containing x_0 such that $M = A \cap \mathcal{U}(x_0) \subset A_u(x_0, \varrho, \varepsilon)$. Moreover $D(x_0, M) = 1$, which establishes the implication $(a_1) \Rightarrow (b_1)$.

In the same manner we can see that $(a_2)\Rightarrow (b_2)$ holds true.

Let us assume that (b_2) is valid. Let $\varrho \in \mathcal{P}$ and let $(\varepsilon_n)_{n\in N}$ be a sequence of positive numbers decreasing to zero. Then

(5) $\forall n \in N\ \forall k \in N\ A_l(x_0, \varrho, \varepsilon_{n+k}) \subset A_l(x_0, \varrho, \varepsilon_n)$

From (b_2) it follows that for each $n \in N$ there exists a set $B_n \in \mathcal{M}(X)$ such that $B_n \subset A_l(x_0, \varrho, \varepsilon_n)$ and $D(x_0, B_n) = 1$. Therefore

(6) $\forall n \in N\ \exists k_n \in N\ \forall k \in N$ (if $x_0 \in I_k$, $I_k \subset I_{k_n}$ and $\delta(I_k) < \delta(I_{k_n})$, then

$$\frac{\mu(B_n \cap I_k)}{\mu(I_k)} > 1 - \varepsilon_n)$$

We may suppose that the sequence $(I_{k_n})_{n\in N}$ is decreasing. Let us put

(7) $A = \bigcup_{n\in N} C_n \cup \{x_0\}$,

where $C_n = B_n \cap (I_{k_n} \setminus I_{k_{n+1}})$.

We will show that $D(x_0, A) = 1$. Suppose $I_k \to x_0$. For each $k \in N$ there exists $n \in N$ such that $I_{k_{n+1}} \subset I_k \subset I_{k_n}$ and $\mu(I_{k_{n+1}}) < \varepsilon_n \mu(I_k)$. From (6) we have

(8) $\mu(B_n \cap I_k) > (1 - \varepsilon_n)\mu(I_k)$.

On the other hand we have the inequalities:

(9) $\mu(B_n \cap I_{k_{n+1}}) \leq \mu(I_{k_{n+1}}) < \varepsilon_n \mu(I_k)$.

From (8) and (9) it follows immediately that $\mu(A \cap I_k) \geq \mu(C_n \cap I_k) = \mu(B_n \cap (I_{k_n} \setminus I_{k_{n+1}}) \cap I_k) = \mu(B_n \cap I_k) - \mu(B_n \cap I_{k_{n+1}}) > (1 - \varepsilon_n)\mu(I_k) - \varepsilon_n \mu(I_k) = (1 - 2\varepsilon_n)\mu(I_k)$.

Thus we have $\dfrac{\mu(A \cap I_k)}{\mu(I_k)} > (1 - 2\varepsilon_n)$.

If $I_k \to x_0$, then $\varepsilon_n \to 0$ and this establishes $D(x_0, A) = 1$.

Now we show that $F|_A$ is lower semicontinuous at the point x_0. Let $\varepsilon > 0$. There exists $n \in N$ such that $\varepsilon_n < \varepsilon$. Let $r = \delta(I_{k_{n+1}})$ and let $x \in A \cap K(x_0, \varrho, r)$. Then $x \in \bigcup_{i\in N} C_{n+i} \cup \{x_0\}$. Moreover for each $i \in N$ we have $C_{n+i} \subset A_l(x_0, \varrho, \varepsilon_{n+i}) \subset A_l(x_0, \varrho, \varepsilon)$. Therefore $F(x_0) \subset K(F(x), \varrho, \varepsilon)$ for each $x \in A \cap K(x_0, r)$. By virtue of (3) the multifunction $F|_A$ is lower semicontinuous at x_0, which establishes the implication $(b_2) \Rightarrow (a_2)$.

Applying this argument again, with (b_2) replaced by (b_1) and with assumed compactness of values of F, we obtain $(b_1) \Rightarrow (a_1)$.

To prove the implication $(c_1) \Rightarrow (b_1)$ it is sufficient to observe that the equality $A_u(x_0, \varrho, \varepsilon) = F^+(K(F(x_0), \varrho, \varepsilon))$ holds true.

Now suppose that F has compact values and (b_1) is valid. Let G be an open set such that $F(x_0) \subset G$. Let $\varrho \in \mathcal{P}$ and let $(\varepsilon_n)_{n\in N}$ be a decreasing sequence of positive numbers convergent to zero such that $K(F(x_0), \varrho, \varepsilon_n) \subset G$ for each $n \in N$.

By virtue of (b_1) in the same fashion as before we can construct the set A (see (7)) such that $D(x_0, A) = 1$. We have $F(x) \subset K(F(x_0), \varrho, \varepsilon_n) \subset G$ for each $x \in A$. Thus we have $A \subset F^+(G)$ and $D(x_0, F^+(G)) = 1$, which establishes the implication $(b_1) \Rightarrow (c_1)$.

Suppose that F has compact values and (c_2) is valid. Let $\varrho \in \mathcal{P}$ and $\varepsilon > 0$. There exists a sequence $(y_i)_{i=1,2,...,n}$ such that $F(x_0) \subset \bigcup_{i=1,2,...,n} K(y_i, \varrho, \frac{\varepsilon}{2})$ and $F(x_0) \cap K(y_i, \varrho, \frac{\varepsilon}{2}) \neq \emptyset$ for each $i = 1, 2, ..., n$. Let $U_i = F^-(K(y_i, \varrho, \frac{\varepsilon}{2}))$ for each $i = 1, 2, .., n$ and let $U(x_0) = \bigcap_{i=1,2,...,n} U_i$. By (c_2) we have $D(x_0, U_i) = 1$ for each $i = 1, 2, ..., n$. Then $x_0 \in U(x_0)$ and $D(x_0, U(x_0)) = 1$. It suffices to show that $U(x_0) \subset A_l(x_0, \varrho, \varepsilon)$. Let $x \in U(x_0)$. Then for each $i = 1, 2, ..., n$ we have $F(x) \cap K(y_i, \varrho, \frac{\varepsilon}{2}) \neq \emptyset$. Thus

$$(10)\ \forall i \in \{1, 2, ..., n\}\ \exists z_i \in F(x)\ y_i \in K(z_i, \varrho, \frac{\varepsilon}{2}).$$

Let $y \in F(x_0)$. Then there exists $j \in \{1, 2, ..., n\}$ such that $\varrho(y, y_j) < \frac{\varepsilon}{2}$. Let z_j be chosen for y_j according to (10). Then $\varrho(z_j, y) < \varepsilon$. Thence $F(x_0) \subset K(F(x), \varrho, \varepsilon)$ and $x \in A_l(x_0, \varrho, \varepsilon)$. The proof of $(c_2) \Rightarrow (b_2)$ is complete. □

Now let us suppose that (b_2) is valid and let $G \subset Y$ be an open set such that $F(x_0) \cap G \neq \emptyset$. Let $y \in F(x_0) \cap G$ and let $(\varepsilon_n)_{n \in N}$ be a decreasing to zero sequence of positive numbers such that $K(y, \varrho, \varepsilon_n) \subset G$ for each $n \in N$. Again likewise as before we can construct a set A (see (7)) such that $D(x_0, A) = 1$. It remains to prove that $A \subset F^-(G)$. Let $x \in A$. Then there exists $n \in N$ such that $x \in C_n \subset A_l(x_0, \varrho, \varepsilon_n)$. Thus we conclude that $F(x_0) \subset K(F(x), \varrho, \varepsilon_n)$. Because $y \in F(x_0)$, then there exists $z \in F(x)$ such that $y \in K(z, \varrho, \varepsilon_n)$. According to assumption $K(y, \varrho, \varepsilon_n) \subset G$, thus we conclude that $z \in G$, hence $F(x) \cap G \neq \emptyset$, i.e. $x \in F^-(G)$, which establishes the implication $(b_2) \Rightarrow (c_2)$ and completes the proof of theorem 2.

Summarizing above results, we have

Conclusion 1. *Let $F : X \to Y$ be a multivalued function with compact values. Then F is approximately continuous if and only if F is continuous with respect to the density topology $\mathcal{T}_d(X)$.*

Let $A \subset X$ and $x \in X$. Let us note that

(11) x is a limit point of the set A in the topology $\mathcal{T}_d(X)$ if and only if $D_u{}^*(x, A) > 0$.

A point with this property we will call a $\mathcal{T}_d(X)$-limit point.

A $\mathcal{T}_d(X)$-connected set is the one which is not the union of two nonempty subsets of X neither of which contains a $\mathcal{T}_d(X)$-limit point of the other.

The next theorem is established by the Conclusion 1.

Theorem 3. *Let $F : X \to Y$ be an approximately continuous with respect to $\mathcal{F}$ multivalued function with compact and connected values. Then F takes $\mathcal{T}_d(X)$-connected sets into connected sets.*

Theorem 4. *If a set $A \subset X$ is open and connected, then it is $\mathcal{T}_d(X)$-connected subset of X.*

Proof. Suppose A is open and connected but not $\mathcal{T}_d(X)$-connected. Then there exist sets B and C nonempty, disjoint, homogeneous with respect to $\mathcal{F}$ and such that $A = B \cup C$. Let $b \in B$ and $c \in C$. Let $B(a, \varepsilon)$ be an open ball including b and c. For any set $P \in \mathcal{M}(X)$ the function $f(x) = \frac{\mu(P \cap B(x,r))}{\mu(B(x,r))}$ is continuous for fixed $r > 0$ (the number $\frac{\mu(P \cap B(x,r))}{\mu(B(x,r))}$ is called the relative measure of P in $B(x, r)$).

Let $\varepsilon_1 = \min(\varepsilon - d(a, b), \varepsilon - d(a, c))$. Then $B(x, \varepsilon_1) \subset B(a, \varepsilon)$ for each $x \in B(a, \varepsilon - \varepsilon_1)$. Let us choose $r_1 \leq \varepsilon_1$ so that the relative measure of B in $B(b, r_1)$ exceeds $\frac{1}{2}$ and the relative measure of B in $B(c, r_1)$ is less than $\frac{1}{2}$. This is possible since B has at b density one with respect to $\mathcal{F}$ and at c density zero with respect to $\mathcal{F}$. There must then be a point $a_1 \in B(a, \varepsilon)$ such that the relative measure of B in $B(a_1, r_1)$is exactly $\frac{1}{2}$. The ball $B(a_1, r_1)$ must have interior points $b_1 \in B$ and $c_1 \in C$. Let $\varepsilon_2 = \min(r_1 - d(a_1, b_1), r_1 - d(a_1, c_1))$. Analogously as before we may choose $r_2 < r_1$ so that $B(x, r_2) \subset B(a_1, r_1)$ for every $x \in B(a_1, r_1 - \varepsilon_2)$ and so that the relative measure of B in $B(b_1, r_2)$ exceeds $\frac{1}{2}$ and is less than $\frac{1}{2}$ in $B(c_1, r_2)$. There is then a point $a_2 \in B(a_1, r_1)$ such that the relative measure of B in $B(a_2, r_2)$ is exactly $\frac{1}{2}$. Continuing this process we construct a sequence $B(a_n, r_n)$ of open balls such that for each $n \in N$ we have $B(a_{n+1}, r_{n+1}) \subset Cl(B(a_n, r_n)) \subset B(a_{n-1}, r_{n-1})$ ($Cl(M)$ denotes the closure of M) and the relative measure of B in the ball $B(a_n, r_n)$ is $\frac{1}{2}$. This sequence converges to a point $a_0 \in A$ but neither B nor C have density one at a_0 with respect to $\mathcal{F}$. Since B and C are homogeneous with respect to $\mathcal{F}$ $a_0 \notin B$ and $a_0 \notin C$. This contradiction establishes the theorem. □

Let us observe that a line segment in euclidean space R^2 is not $\mathcal{T}_d(R^2)$- connected. This is an example showing the necessity of the opennes of A in the above theorem. However Theorem 4 is true if a connected set A has only the property: $\text{Int}(A) \subset A \subset \text{Cl}(\text{Int}(A))$. This follows by the same method as in the proof of Theorem 4, because we may assume $\text{Int}(A) \neq \emptyset$.

Definition 4. Let $A \subset X$ be a closed set with connected interior. The set A is called $\mathcal{T}_d(X)$-regular if its boundary points are $\mathcal{T}_d(X)$-limit points of the interior of A.

Theorem 5. *If a set $A \subset X$ is $\mathcal{T}_d(X)$-regular, then it is $\mathcal{T}_d(X)$-connected.*

Proof. Suppose that A is $\mathcal{T}_d(X)$-regular and is not $\mathcal{T}_d(X)$-connected. The set A is $\mathcal{T}_d(X)$-regular, so interior of A is a connected subset of A. Moreover A is not $\mathcal{T}_d(X)$-connected. Then there exist two homogeneous with respect to $\mathcal{F}$ sets B and C such that B and C are disjoint, non-empty and each contains none of the other's $\mathcal{T}_d(X)$-limit points. One of the sets, say B, must then be contained entirely in the boundary of A, for otherwise this separation would induce a separation of the interior of A which is not possible (see th. 4) because interior of A is open and connected. Then the set B contains a $\mathcal{T}_d(X)$-limit point of the interior of A and hence of C, which is contrary to our assumption. This finishes the proof of theorem. □

Let A be an open sphere in R^2 and let us delete an open radius. The resulting set is $\mathcal{T}_d(R^2)$-connected. This example shows that the converse of Theorem 4 and Theorem 5 is not true.

Conclusion 2. *Let $F : X \to Y$ be an approximately continuous with respect to $\mathcal{F}$ multivalued function with compact and connected values. Then F takes $\mathcal{T}_d(X)$-regular sets into connected sets.*

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